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A comment on Blum's signal functions

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SUMMARY

In the present note we show that Blum's notion of "signal function" coincides with the intuitively simple notion of "step counting function", which can be made precise by using a recent representation theorem for partial recursive functions.

1. Blum's signal functions

We start from a Gödel numbering $\{\varphi_i\}$ of the unary partial recursive functions (Rogers[5]). Following Blum[1] a set $\{\Phi_i\}$ of partial recursive functions is a set of signal functions (with respect to the Gödel numbering $\{\varphi_i\}$) if the following axioms are satisfied:

(C1) $\varphi_i(n)$ defined $\iff$ $\Phi_i(n)$ defined

(C2) the function

$$M(i,n,m) = \begin{cases} 1, & \text{if } \Phi_i(n) = m \\ 0 & \text{otherwise} \end{cases}$$

is recursive (i.e. partial recursive and total).

In [1] various examples for signal functions are given. Partly, they are quite abstract constructions. However, some of them have an intuitive interpretation. For instance, define $\Phi_i(n) = m$ to hold if and only if the multiple tape machine Z_i stops exactly after m steps when placed on input n. Or let $\Phi_i(n) = m$ if and only if Z_i uses exactly m cells of the tape when working on input n.

Intuitively speaking, in the present note we shall show that for every set $\{\Phi_i\}$ of signal functions (with respect to $\{\varphi_i\}$), there exists a certain class of "machine tables" ("machine descriptions", "programs") $\{M_i\}$, such that for all i: M_i computes φ_i, and $\Phi_i(n)$ is exactly the number of steps of a computation according to the description M_i with input n. On the other hand, it is easy to verify that, given a class of machine tables $\{M_i\}$ defining a Gödel numbering $\{\varphi_i\}$ of the partial recursive functions, the function $\Phi_i(n)$ = number of steps which M_i needs for the computation of

$\varphi_i(n)$, satisfies (C1) and (C2). Thus, the notions "signal function" and "function giving the number of steps necessary for the computation of the partial recursive functions on a certain class of machines" coincide.

2. Step counting functions

Given any Gödel numbering $\{\varphi_i\}$ of the partial recursive functions we can define the binary partial recursive function

$$\psi(i,n) = \varphi_i(n) , \tag{1}$$

which "describes the numbering" (Rogers [5]). Using the results of [2] (Theorem 2.1 and Section 3) we know that there exist recursive (total!) functions ρ, $\overline{\Psi}$, κ, γ such that

$$\psi(i,n) = \rho([\overline{\Psi},\kappa]^P(\gamma(i,n))). \tag{2}$$

Here we used the following definition

$$[\overline{\Psi},\kappa]^P(\xi) = \begin{cases} \xi , & \text{if } \kappa(\xi) = 0, \\ [\overline{\Psi},\kappa]^P(\overline{\Psi}(\xi)) & \text{otherwise.} \end{cases} \tag{3}$$

The operator P is a generalized version of the "operator of conditioned iteration" introduced in [2]. It has the following intuitive significance: given a total function $\overline{\Psi}$ (considered as the transition function of an automaton with a countable number of states) and a total function κ (with $\kappa(\xi) = 0$ interpreted as "ξ is a final state") then $[\overline{\Psi},\kappa]^P(\xi)$ gives the final state which the automaton eventually reaches when starting from state ξ.

$\gamma(i,n)$ can be viewed as defining an initial state of the automaton for every pair of numbers (i,n), where i may be called a "machine table" ("machine description", "program") and n an "input".

The function $\bar{\Psi}$ plays the role of an interpreter for the "programs" i, similar to the function Λ in the Vienna-method for defining the semantics of programming languages (see [4], p.4-1) or the transition function of a computer (given by its hardware for interpreting the machine language instructions).

Thus, decomposition (2) can be read as follows: any Gödel numbering of the partial recursive functions can be thought of as being defined by an automaton (given by its (total!) transition function $\bar{\Psi}$ and a decidable termination criterion $\kappa(\xi) = 0$) that interprets the "programs" i working on inputs n which through a certain (total) encoding function γ define the initial state of the automaton. After termination, the (total) function ρ says how the final state defines the result.

By convention,

$$\bar{\Psi}^{(o)}(\xi) = \xi,$$

(4)
$$\bar{\Psi}^{(t+1)}(\xi) = \bar{\Psi}(\bar{\Psi}^{(t)}(\xi)) \quad \text{for } t > o \,.$$

Now, given any decomposition (2) of a Gödel numbering $\Psi(i,n)$ we can define the associated "step counting functions"

(5)
$$\bar{\Phi}_i(n) = (\mu t)(\kappa(\bar{\Psi}^{(t)}(\gamma(i,n))) = o) \,,$$

whose intuitive meaning is apparent.

3. Equivalence theorem

With the definitions of Section 2. we are now in a position to formulate a theorem, which is an exact counterpart of the informal assertion stated at the end of section 1.

Theorem 1: Let $\Psi(i,n)$ be a Gödel numbering of the unary partial recursive functions. Then,

1. for every set $\{\Phi_i\}$ of signal functions with respect to $\varphi(i,n)$ there exists a decomposition (2) of $\Psi(i,n)$ such that the step counting functions $\bar{\Phi}_i(n)$ defined by (5) coincide with the Φ_i, i. e.

$$(\forall i,n)\ (\bar{\Phi}_i(n) = \Phi_i(n)).$$

2. given any decomposition (2) of $\Psi(i,n)$, the step counting functions $\bar{\Phi}_i$ defined by (5) have the properties (C1) and (C2), i. e. the $\bar{\Phi}_i$ are signal functions relative to $\Psi(i,n)$.

Proof:

ad 1: Let us consider the following functions

$$\gamma(i,n) = \sigma_3(i,n,o), \tag{6}$$

$$\bar{\Psi}(\xi) = \sigma_3(\xi_1, \xi_2, \xi_3 + 1), \tag{7}$$

$$\kappa(\xi) = 1 - M(\xi_1, \xi_2, \xi_3), \tag{8}$$

$$(\xi) = \tilde{\Phi}(\xi_1, \xi_2, \xi_3), \tag{9}$$

where

$$\xi_j = \sigma_{3,j}(\xi) \qquad (j = 1,2,3), \tag{10}$$

$$\tilde{\varphi}(x,y,z) = \begin{cases} \Psi(x,y), & \text{if } \Phi_x(y) = z \\ o & \text{otherwise}, \end{cases} \tag{11}$$

and $\sigma_3(x,y,z)$, $\sigma_{3,1}(\xi)$, $\sigma_{3,2}(\xi)$, $\sigma_{3,3}(\xi)$ have the usual properties of pairing functions:

(12) $$\sigma_3(\sigma_{3,1}(\xi), \sigma_{3,2}(\xi), \sigma_{3,3}(\xi)) = \xi,$$

(13) $$\sigma_{3,1}(\sigma_3(x,y,z)) = x \quad \text{etc.}$$

Note, that γ, $\bar{\Psi}$, κ, ρ are total (cf. Davis [3], 4-2.5).

Let us first assume that $\Psi(i,n)$ is defined for some i,n. Then, by (C1), $\Phi_i(n)$ is defined, say

(14) $$\Phi_i(n) = T.$$

Thus, $M(i,n,t) = 0$ for $t < T$ and $M(i,n,T) = 1$. Further, it is easy to see that $\bar{\Psi}^{(t)}(\gamma(i,n)) = \sigma_3(i,n,t)$ for all t, and

(15) $$\kappa(\bar{\Psi}^{(t)}(\gamma(i,n))) = 1 - M(i,n,t) = \begin{cases} 1 & \text{for } t < T \\ 0 & \text{for } t = T \end{cases}$$

Now,

(16) $$\rho([\bar{\Psi},\kappa]^P(\gamma(i,n))) = \rho(\bar{\Psi}^{(T)}(\gamma(i,n))) = \rho(\sigma_3(i,n,T)) =$$
$$= \tilde{\Phi}(i,n,T) = \Psi(i,n), \text{ because of (14).}$$

On the other hand, if $\Psi(i,n)$ is not defined for some i,n, then $M(i,n,t) = 0$ for all t and

(17) $$\kappa(\bar{\Psi}^{(t)}(\gamma(i,n))) = 1 \quad \text{for all } t.$$

Thus, $\rho([\bar{\Psi},\kappa]^P(\gamma(i,n)))$ will be undefined, too, i. e.

(18) $$\Psi(i,n) = \rho([\bar{\Psi},\kappa]^P(\gamma(i,n))) \quad \text{for all } i,n.$$

We now evaluate $\bar{\Phi}_1(n)$ for the case $\Phi_i(n) = T$ (using (15)):

$\bar{\Phi}_i(n) = (\mu t)\ (\kappa(\bar{\Psi}^{(t)}(\gamma(i,n))) = o) = T$

Assume $\Phi_i(n)$ not defined. Then, by (17) $\bar{\Phi}_i(n)$ is not defined either.

Thus,

(19) $\Phi_i(n) = \bar{\Phi}_i(n)$ for all i,n, q.e.d.

ad 2:

Let $\Psi(i,n)$ be defined. Because of (2) there must be a T such that $\kappa(\bar{\Psi}^{(T)}(\gamma(i,n))) = o$. Thus $\bar{\Phi}_i(n)$ is defined, too. If $\Psi(i,n)$ is not defined then there cannot exist any t such that $\kappa(\bar{\Psi}^{(t)}(\gamma(i,n))) = o$ since γ, $\bar{\Psi}$, κ, ρ are supposed to be total, i.e. $\bar{\Phi}_i(n)$ is also undefined. Thus, $\{\bar{\Phi}_i(n)\}$ satisfy (C1).

As for (C2), we note that

$$M(i,n,m) = \begin{cases} 1, & \text{if } \kappa(\bar{\Psi}^{(t)}(\gamma(i,n))) \neq o \quad \text{for } t < m \\ & \& \ \kappa(\bar{\Psi}^{(m)}(\gamma(i,n))) = o, \\ o & \text{otherwise.} \end{cases}$$

With γ, $\bar{\Psi}$, κ, $M(i,n,m)$ is recursive, too.

Since we know that there exist decompositions of the form (2) for every Gödel numbering $\Psi(i,n)$ (see [2]), and that the functions γ, $\bar{\Psi}$, κ of such a decomposition define a set of signal functions $\{\bar{\Phi}_i(n)\}$ by (5) we have an easy proof of the following

Theorem 2: For every Gödel numbering $\{\varphi_i\}$ of the partial recursive functions there exists a set $\{\Phi_i\}$ of signal functions.

References

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