A. Appendix.

A.1 A Proof by the Gröbner Bases Method

The Theorem is proved by the Groebner Bases method.

The formula in the scope of the universal quantifier is transformed into an equivalent formula that is a conjunction of disjunctions of equalities and negated equalities. The universal quantifier can then be distributed over the individual parts of the conjunction. By this, we obtain:

Independent proof problems:

(Formula (Test): B1.1)

$$\underset{x,y}{\forall} ((x^2 + (-x * y) + x^2 * y + -2 * y^2 + -2 * x * y^2 = 0) \lor (-3 * x + x^2 * y \neq 0) \lor (x + y + x * y \neq 0))$$

(Formula (Test): B1.2)

$$\underset{x,y}{\forall} ((3 * x + x^2 * y = 0) \lor (-2 * x^2 + -7 * x * y + x^2 * y + x^3 * y + -2 * y^2 + -2 * x * y^2 + 2 * x^2 * y^2 = 0) \lor (-3 * x + x^2 * y \neq 0) \lor (x + y + x * y \neq 0))$$

We now prove the above individual problems separately:

Proof of (Formula (Test): B1.1):

.... (Here comes the proof of the first partial problem. We do not show it here because it is similar and, in fact, simpler that the proof of the second partial problem, which we show in all detail. ...)

Proof of (Formula (Test): B1.2):

This proof problem has the following structure:

(Formula (Test): B1.2.structure)

$$\underset{x,y}{\forall} ((\mathrm{Poly}[1] \neq 0) \lor (\mathrm{Poly}[2] \neq 0) \lor (\mathrm{Poly}[3] = 0) \lor (\mathrm{Poly}[4] = 0)),$$

where

$$\mathrm{Poly}[1] = -3 * x + x^2 * y$$
$$\mathrm{Poly}[2] = x + y + x * y$$
$$\mathrm{Poly}[3] = 3 * x + x^2 * y$$
$$\mathrm{Poly}[4] = -2 * x^2 + -7 * x * y + x^2 * y + x^3 * y + -2 * y^2 + -2 * x * y^2 + 2 * x^2 * y^2$$

(Formula (Test): B1.2.structure) is equivalent to

(Formula (Test): B1.2.implication)

$$\forall_{x,y} ((\text{Poly}[1] = 0) \wedge (\text{Poly}[2] = 0) \Rightarrow (\text{Poly}[3] = 0) \vee (\text{Poly}[4] = 0)).$$

(Formula (Test): B1.2.implication) is equivalent to

(Formula (Test): B1.2.not-exists)

$$\nexists_{x,y} (((\text{Poly}[1] = 0) \wedge (\text{Poly}[2] = 0)) \wedge ((\text{Poly}[3] \neq 0) \wedge (\text{Poly}[4] \neq 0))).$$

By introducing the slack variable(s)

$\{\xi 1, \xi 2\}$

(Formula (Test): B1.2.not-exists) is transformed into the equivalent formula

(Formula (Test): B1.2.not-exists-slack)

$$\nexists_{x,y,\xi 1,\xi 2} (((\text{Poly}[1] = 0) \wedge (\text{Poly}[2] = 0)) \wedge \{-1 + \xi 1\, \text{Poly}[3] = 0, -1 + \xi 2\, \text{Poly}[4] = 0\}).$$

Hence, we see that the proof problem is transformed into the question on whether or not a system of polynomial equations has a solution or not. This question can be answered by checking whether or not the (reduced) Groebner basis of

$$\{\text{Poly}[1], \text{Poly}[2], -1 + \xi 1\, \text{Poly}[3], -1 + \xi 2\, \text{Poly}[4]\}$$

is exactly {1}.

Hence, we compute the Groebner basis for the following polynomial list:

$$\{-1 + 3\, x\, \xi 1 + x^2\, y\, \xi 1, -1 + -2\, x^2\, \xi 2 + -7\, x\, y\, \xi 2 + x^2\, y\, \xi 2 + x^3\, y\, \xi 2 + -2\, y^2\, \xi 2 + -2\, x\, y^2\, \xi 2 + 2\, x^2\, y^2\, \xi 2, -3\, x + x^2\, y, x + y + x\, y\}$$

The Groebner basis:

$\{1\}$

Hence, (Formula (Test): B1.2) is proved.

Since all of the individual subtheorems are proved, the original formula is proved.

A.2 A Proof by the PCS Method

Prove:

(Proposition (limit of sum)) $\forall_{f,a,g,b} (\text{limit}[f, a] \wedge \text{limit}[g, b] \Rightarrow \text{limit}[f + g, a + b]),$

under the assumptions:

(Definition (limit:)) $\underset{f,a}{\forall}\left(\mathrm{limit}[f, a] \Leftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{N}{\exists}\ \underset{\substack{n \\ n\geq N}}{\forall} (|f[n]-a|<\epsilon)\right)$,

(Definition (+:)) $\underset{f,g,x}{\forall}((f+g)[x]=f[x]+g[x])$,

(Lemma (|+|)) $\underset{x,y,a,b,\delta,\epsilon}{\forall}(|(x+y)-(a+b)|<\delta+\epsilon \Leftarrow (|x-a|<\delta \wedge |y-b|<\epsilon))$,

(Lemma (max)) $\underset{m,M1,M2}{\forall}(m\geq \max[M1, M2] \Rightarrow m\geq M1 \wedge m\geq M2)$.

We assume

(1) $\mathrm{limit}[f_0, a_0] \wedge \mathrm{limit}[g_0, b_0]$,

and show

(2) $\mathrm{limit}[f_0+g_0, a_0+b_0]$.

Formula (1.1), by (Definition (limit:)), implies:

(3) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{N}{\exists}\ \underset{\substack{n \\ n\geq N}}{\forall}(|f_0[n]-a_0|<\epsilon)$.

By (3), we can take an appropriate Skolem function such that

(4) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{\substack{n \\ n\geq N_0[\epsilon]}}{\forall}(|f_0[n]-a_0|<\epsilon)$,

Formula (1.2), by (Definition (limit:)), implies:

(5) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{N}{\exists}\ \underset{\substack{n \\ n\geq N}}{\forall}(|g_0[n]-b_0|<\epsilon)$.

By (5), we can take an appropriate Skolem function such that

(6) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{\substack{n \\ n\geq N_1[\epsilon]}}{\forall}(|g_0[n]-b_0|<\epsilon)$,

Formula (2), using (Definition (limit:)), is implied by:

(7) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{N}{\exists}\ \underset{\substack{n \\ n\geq N}}{\forall}(|(f_0+g_0)[n]-(a_0+b_0)|<\epsilon)$.

We assume

(8) $\epsilon_0 > 0$,

and show

(9) $\underset{N}{\exists}\ \underset{\substack{n \\ n\geq N}}{\forall}(|(f_0+g_0)[n]-(a_0+b_0)|<\epsilon_0)$.

We have to find N_2^* such that

(10) $\underset{n}{\forall}\ (n \geq N_2^* \Rightarrow |(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon_0)$.

Formula (10), using (Definition (+:)), is implied by:

(11) $\underset{n}{\forall}\ (n \geq N_2^* \Rightarrow |(f_0[n] + g_0[n]) - (a_0 + b_0)| < \epsilon_0)$.

Formula (11), using (Lemma (|+|)), is implied by:

(12) $\underset{\substack{\delta, \epsilon \\ \delta + \epsilon = \epsilon_0}}{\exists}\ \underset{n}{\forall}\ (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta \wedge |g_0[n] - b_0| < \epsilon)$.

We have to find δ_0^*, ϵ_1^* and N_2^* such that

(13) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall}\ (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta_0^* \wedge |g_0[n] - b_0| < \epsilon_1^*)$.

Formula (13), using (6), is implied by:

$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall}\ (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta_0^* \wedge (\epsilon_1^* > 0 \wedge n \geq N_1[\epsilon_1^*]))$,

which, using (4), is implied by:

$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall}\ (n \geq N_2^* \Rightarrow (\delta_0^* > 0 \wedge n \geq N_0[\delta_0^*]) \wedge (\epsilon_1^* > 0 \wedge n \geq N_1[\epsilon_1^*]))$,

which, using (Lemma (max)), is implied by:

(14) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall}\ (n \geq N_2^* \Rightarrow \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge n \geq \max[N_0[\delta_0^*], N_1[\epsilon_1^*]])$.

Formula (14) is implied by

(15) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \delta_0^* > 0 \bigwedge \epsilon_1^* > 0 \bigwedge \underset{n}{\forall}\ (n \geq N_2^* \Rightarrow n \geq \max[N_0[\delta_0^*], N_1[\epsilon_1^*]])$.

Partially solving it, formula (15) is implied by

(16) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \wedge \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge (N_2^* = \max[N_0[\delta_0^*], N_1[\epsilon_1^*]])$.

Now,

$(\delta_0^* + \epsilon_1^* = \epsilon_0) \wedge \delta_0^* > 0 \wedge \epsilon_1^* > 0$

can be solved for δ_0^* and ϵ_1^* by a call to Collins cad–method yielding the solution

$0 < \delta_0^* < \epsilon_0$,

$\epsilon_1^* \leftarrow \epsilon_0 + -1 * \delta_0^*$.

Let us take

$N_2^* \leftarrow \max[N_0[\delta_0^*], N_1[\epsilon_0 + -1 * \delta_0^*]]$.

Formula (16) is solved. Hence, we are done.