The Theory of Bonds
Linkages with Paradoxical Mobility

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August 24, 2015

Dual Quaternions

$\mathbb{DH}$ is a non-commutative associative algebra over $\mathbb{R}$, of vector space dimension 8. Here is the multiplication table.

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	ϵ	$\epsilon\mathbf{i}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	ϵ	$\epsilon\mathbf{i}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$	$\epsilon\mathbf{i}$	$-\epsilon$	$\epsilon\mathbf{k}$	$-\epsilon\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$	$\epsilon\mathbf{j}$	$-\epsilon\mathbf{k}$	$-\epsilon$	$\epsilon\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1	$\epsilon\mathbf{k}$	$\epsilon\mathbf{j}$	$-\epsilon\mathbf{i}$	$-\epsilon$
ϵ	ϵ	$\epsilon\mathbf{i}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{k}$	0	0	0	0
$\epsilon\mathbf{i}$	$\epsilon\mathbf{i}$	$-\epsilon$	$\epsilon\mathbf{k}$	$-\epsilon\mathbf{j}$	0	0	0	0
$\epsilon\mathbf{j}$	$\epsilon\mathbf{j}$	$-\epsilon\mathbf{k}$	$-\epsilon$	$\epsilon\mathbf{i}$	0	0	0	0
$\epsilon\mathbf{k}$	$\epsilon\mathbf{k}$	$\epsilon\mathbf{j}$	$-\epsilon\mathbf{i}$	$-\epsilon$	0	0	0	0

Conjugation is an anti-automorphism defined by the table

q	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	ϵ	$\epsilon\mathbf{i}$	$\epsilon\mathbf{j}$	$\epsilon\mathbf{k}$
$\bar{q}$	1	$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$	ϵ	$-\epsilon\mathbf{i}$	$-\epsilon\mathbf{j}$	$-\epsilon\mathbf{k}$

Dual Quaternions and Euclidean Displacements

The function $N : \mathbb{DH} \to \mathbb{D}(= \mathbb{R} \oplus \epsilon\mathbb{R}) : q \mapsto q\bar{q}$ is multiplicative: $N(q_1 q_2) = N(q_1)N(q_2)$. We define

$$\mathbb{S}^* := \{q \mid N(q) \in \mathbb{R}^*\}.$$

The set $\mathbb{S}^*$ is a multiplicative group, and $\mathbb{R}^* \subset \mathbb{S}^*$ is a normal subgroup.

Theorem (Study 1891). The quotient group $\mathbb{S}^*/\mathbb{R}^*$ is isomorphic to the group of Euclidean displacements.

For $t \in \mathbb{R}$, $[t - \mathbf{i}]$ corresponds to a rotation around the first axis by an angle $2\mathrm{arccot}(t)$,
and $[1 - \epsilon\mathbf{i}t]$ corresponds to a translation by a distance $2t$ in the direction of the first axis.

Closed Linkages

Let $n \geq 4$ be an integer. An nR-loop is a linkage consisting of n links that are cyclically connected by n rotational joints.
Denavit-Hartenberg parameters:

- twist angles $\alpha_1, \ldots, \alpha_n$
- normal distances $d_1, \ldots, d_n$
- offsets $o_1, \ldots, o_n$

For $r = 1, \ldots, n$, the Euclidean displacement corresponding to

$$g_r := \left(1 - \frac{o_r}{2}\epsilon \mathbf{i}\right)\left(\frac{\cot(\alpha_r)}{2} - \mathbf{k}\right)\left(1 - \frac{d_r}{2}\epsilon \mathbf{k}\right)$$

transforms one rotation axis to the next.

The Closure Equation

The configuration set of a closed nR-loop given by $g_1, \dots, g_n$ is given by

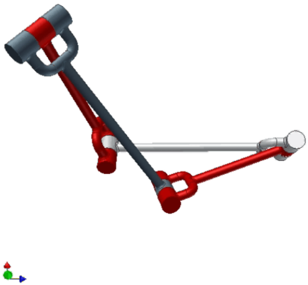
$$K := \{(t_1, \dots, t_n) \mid (t_1 - \mathbf{i})g_1(t_2 - \imath)g_2 \cdots (t_n - \mathbf{i})g_n \in \mathbb{R}^*\}.$$

The *mobility* of the linkage is $\dim(K)$. If $\dim(K) > 0$, then we say that the linkage is mobile.
The condition is a system of 7 scalar equations. But we know already that vector on the left hand side has norm in $\mathbb{R}$, hence only 6 equations are needed.
For generic parameters, $\dim(K) = \max(0, n - 6)$.

Paradoxical Mobility: Trivial Cases

- $\alpha_1 = d_1 = 0$ (two axes coincide) $\Rightarrow \dim(K) \geq 1$
- $\alpha_1 = \cdots = \alpha_n = 0$ (all axes parallel) $\Rightarrow \dim(K) \geq n - 3$
- $d_1 = \cdots = d_n = o_1 = \cdots = o_n = 0 \Rightarrow \dim(K) \geq n - 3$
- by adding an *idle* axis to an nR-linkage, we obtain an $(n+1)$R-linkage with bigger or equal mobility.

Paradoxical Mobility: the Bennett Linkage



$$o_1 = o_2 = o_3 = o_4 = 0, \alpha_1 = \alpha_3, \alpha_2 = \alpha_4,$$

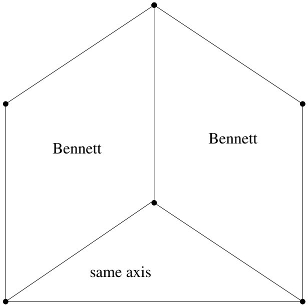
$$d_1 = d_3, d_2 = d_4, d_1 \sin(\alpha_2) = d_2 \sin(\alpha_1)$$

This movable 4R-linkage was discovered by G. Bennett (1914).
E. Delassus (1922) proved that every movable 4R linkage is either a Bennett linkage or trivial (in the sense of last slide).

The Link Graph

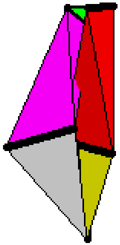
For every link, we draw a vertex. For any two links connected by a joint, we draw an edge.
For instance, the link graph of an nR-loop is an n-cycle.

Paradoxical Mobility: the Goldberg 5R Linkage

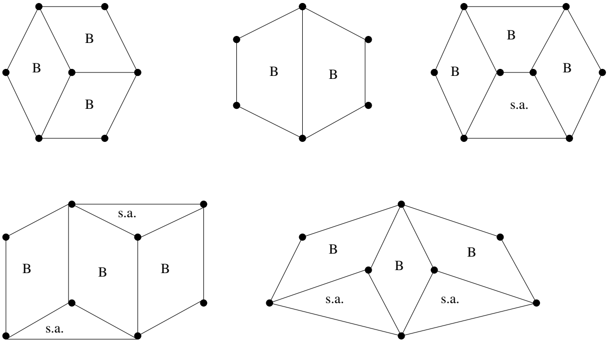


Goldberg (1943) constructed a 5R linkage by composing two Bennett linkages.
A. Karger (1998) proved that every movable 5R linkage is either a Goldberg 5R linkage or trivial.

Animation of a Goldberg 5R Linkage



Composing Bennett Linkages



Similarily, Goldberg, Waldron and others constructed paradoxically mobile 6R loops by composing Bennett linkages or trivial 4R linkages.

6R Loops

The set of all 6R loops can be mathematically described by $\mathbb{R}^{18}$ (using the Denavit-Hartenberg parameters).
For a generic values, the closure equations have 16 complex solutions; in particular the linkage is not mobile. Those values which give mobile linkages form a closed subset of $\mathcal{M} \subset \mathbb{R}^{18}$.

Questions on 6R Loops

- What is the dimension of $\mathcal{M}$? (14; linkages with equal axes)
- What is the dimension of the largest nontrivial component? (11; Hooke's linkage)
- What is the number of components and the dimension of each component?
- Give a construction for each component!
- Write down a system of equations defining $\mathcal{M}$!

Bricard's Line Symmetric Linkage

Theorem (Bricard, 189?). A linkage with parameters

$$g_1 = g_4, g_2 = g_5, g_3 = g_6$$

is mobile. [These linkages form a 9-dimensional component of $\mathcal{M}$.]

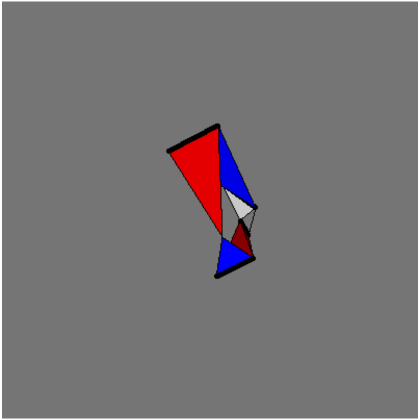
Proof. A dual quaternion in $\mathbb{S}$ represents an involution iff its first and its fifth coordinate vanish. Hence the set

$$I = \{(t_1, t_2, t_3) \mid ((t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2(t_3 - \mathbf{i})g_3)^2 \in \mathbb{R}\}$$

has dimension 1. But then

$$K = \{(t_1, \dots, t_6) \mid (t_1, t_2, t_3) \in I, t_1 = t_4, t_2 = t_5, t_3 = t_6\}.$$

Animation of Line Symmetric Linkage



Bricard's Orthogonal Linkage

Bricard also showed that a linkage with parameters

$$\alpha_1 = \cdots = \alpha_6 = \frac{\pi}{2}, o_1 = \cdots = o_6 = 0, d_1^2 - d_2^2 + d_3^2 - d_4^2 + d_5^2 - d_6^2 = 0$$

is movable. The set of these linkages has dimension 5.

Hegedüs, Li, Schröcker and S. showed that these linkages are contained in a 6-dimensional component of $\mathcal{M}$. This was done by *bond theory*.

Definition of Bonds

For a fixed nR-linkage given by $(g_1, \ldots, g_n)$, the set B of bonds is defined as the set of all complex points $(t_1, \ldots, t_n)$ in the Zariski closure of K that satisfy the *primary bond equation*

$$(t_1 - \mathbf{i})g_1(t_2 - \mathbf{i})g_2 \cdots (t_n - \mathbf{i})g_n = 0.$$

There are no bonds with only real coordinates. Bonds always occur in paits of complex conjugates.

The set of a bonds of a mobile linkage is always non-empty. More generally, we have $\dim(B) = \dim(K) - 1$.

First Property of Bonds

Proposition 1. If $(t_1, \dots, t_n)$ is a bond, then there is an index $r \in \{1, \dots, n\}$ such that $t_r = \pm \mathrm{i}$.

Proof. The norm of the left hand side of the primary bond equation is $N(g_1 \dots g_n)(t_1^2 + 1) \cdots (t_n^2 + 1)$.

Second Property of Bonds

Proposition 2. If $(t_1, \ldots, t_n)$ is a bond, then there are two indices $r < s \in \{1, \ldots, n\}$ such that $t_r = \pm\mathrm{i}$ and $t_s = \pm\mathrm{i}$.

Proof. For any $t \in \mathbb{C}$ such that $t^2 + 1 \neq 0$, the element $t - \mathbf{i}$ is invertible:

$$(t - \mathbf{i})^{-1} = \frac{t + \mathbf{i}}{t^2 + 1}.$$

Also, $g_1, \ldots, g_n$ are invertible because they correspond to group elements. Assume, indirectly, that r is the only index such that $t_r = \pm\mathrm{i}$. Then we bring all invertible factors on the left hand side of the primary bond equation to the other side, by multiplying with the inverse from the left or from the right. We obtain $t_r - \mathbf{i} = 0$, a contradiction.

The Bond Diagram

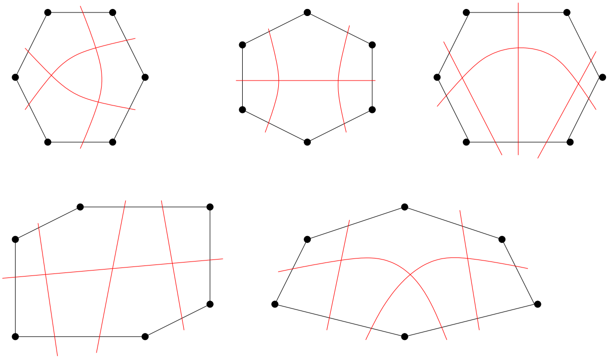
Most bonds have exactly two coordinates with value $\pm\mathrm{i}$. If the indices are $r < s$, then the *secondary bond equations* hold:

$$(t_r - \mathbf{i})g_r \cdots (t_s - \mathbf{i}) = (t_s - \mathbf{i})g_s \cdots (t_r - \mathbf{i}) = 0.$$

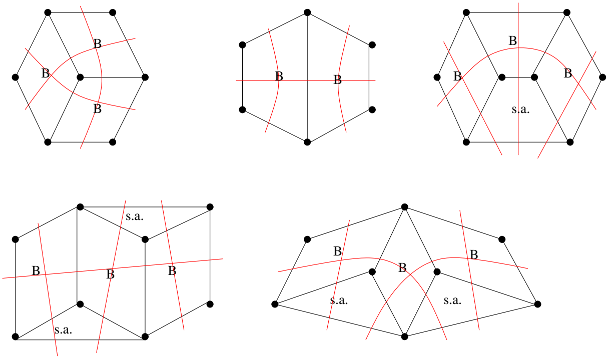
We say that the bond connects the r-th and s-th joint.

Starting from the link graph, we draw each bond (actually: each conjugated pair of bonds) as a line between the edges corresponding to the joints that are connected by the bond.

Examples of Bond Diagrams



Examples of Bond Diagrams



Parameter Conditions from the Bond Diagram

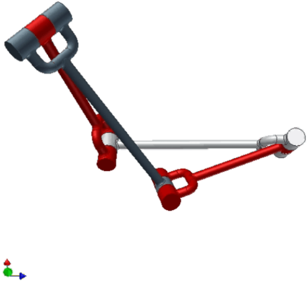
The secondary bond equations imply conditions on the Denavit-Hartenberg parameters. To state them we assume $r = 1$, without loss of generality; and that there is a connection between joints 1 and s.

$s = 2$ (very short connection). Then $\alpha_1 = d_1 = 0$, i.e. the axes coincide (trivial case).

$s = 3$ (short connection). Then $o_2 = 0$ and
$d_1 \sin(\alpha_2) = d_2 \sin(\alpha_1)$.
Compare with slide "Paradoxical Mobility: the Bennett Linkage"

$s = 3$ (long connection): the equations are known, but very complicated.

Paradoxical Mobility: the Bennett Linkage



$$o_1 = o_2 = o_3 = o_4 = 0, \alpha_1 = \alpha_3, \alpha_2 = \alpha_4,$$

$$d_1 = d_3, d_2 = d_4, d_1 \sin(\alpha_2) = d_2 \sin(\alpha_1)$$

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Degree of Motions

The projectivization of $\mathbb{DH}$ is a 7-dimensional projective space. The Study quadric is defined by dual part of the norm. The group of Euclidean displacements can be considered as an open subset of the Study quadric.

Assume we have a linkage with mobility 1. Let A_1, A_2 be two links. When we fix A_1 and the linkage moves, then the link A_2 experiences a motion, which can be described as a curve in the Study quadric in $\mathbb{P}^7$. This curve has a degree, and we call it the degree of the motion of A_2 relative to A_1.

For instance, if A_1 and A_2 are connected by a joint, then the motion is a rotation with fixed axis. This is a line in the Study quadric, hence the motion degree is 1.

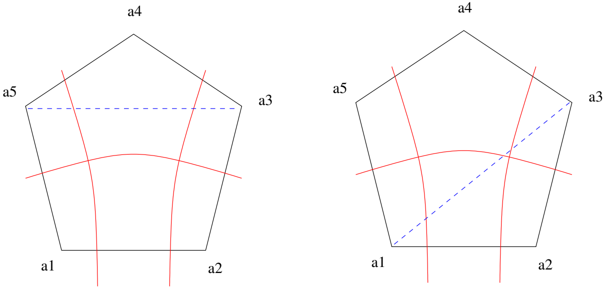
The Main Theorem of Bond Theory

Theorem. Let A_1, A_2 be two links. Let a_1, a_2 be the corresponding vertices of the bond diagram. Then the following two numbers are equal:
(a) the degree of the relative motion of A_2 with respect to A_1, multiplied with the number of configurations when the positions of both links are fixed;
(b) the number of bond connections that cross a line between vertices a_1 and a_2.

In particular, a joint is not connected to any other joint by a bond if and only if it is idle (i.e. it can be frozen without changing the mobility).

Example for Main Theorem

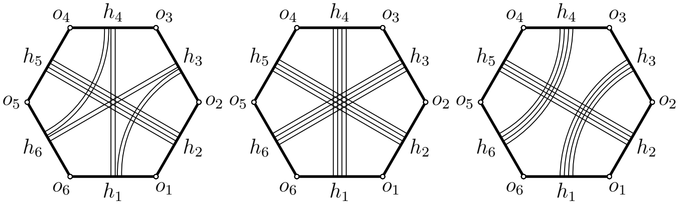
Here is the bond diagram of the Goldberg 5R linkage.



The motion of A_5 relative to A_3 has degree 2. The motion of A_1 relative to A_3 has degree 3.

Bounds on the Bond Diagram of a 6R Loop

- the total number of bond connections is at most 12.
- the number of connections to a fixed joint is at most 4.
- the largest possible degree of a motion is 12.



(b) (d) (a)

Maximal Degree 6R Loops

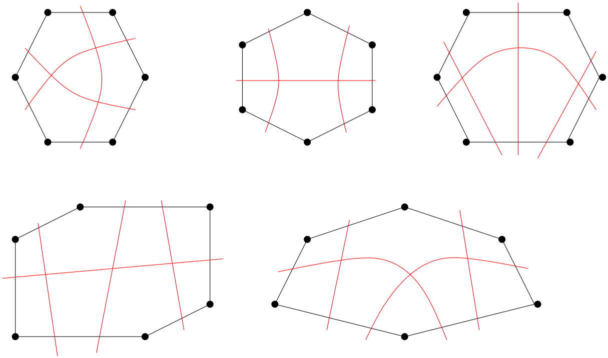
The 6R loops with a motion of degree 12 are known. There are 4 families:

- Hooke's linkage: composed by two spherical 4R linkages. Bond diagram (a). Dimension of family: 11
- Dietmaier's linkage (found by a computer search). Bond diagram (b). Dimension of family: 9
- Generalization of Bricard's orthogonal linkage (found by bond theory). Bond diagram (d). Dimension of family: 6
- Generalization of double line symmetric (found by bond theory). Bond diagram (d). Dimension of family: 7

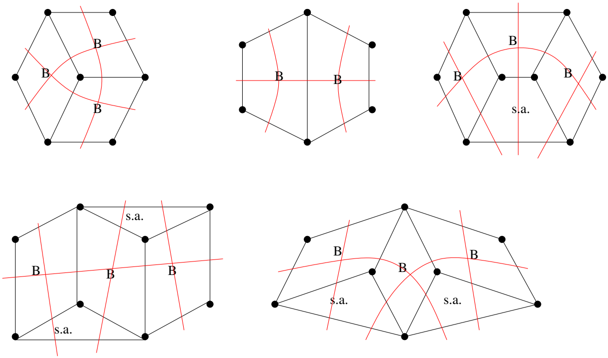
Bonds for Multiply Closed Linkages

The statement “a bond connects two joints” is no more appropriate.
It is a replaced by the following: a bond can be drawn as a line in the bond diagram that separates the set of vertices (=links) into two subsets.
Compare with slide “Examples of Bond Diagrams”

Examples of Bond Diagrams



Examples of Bond Diagrams



Bonds for Prismatic Joints

The main theorem still holds. There are some differences in the parameter conditions.

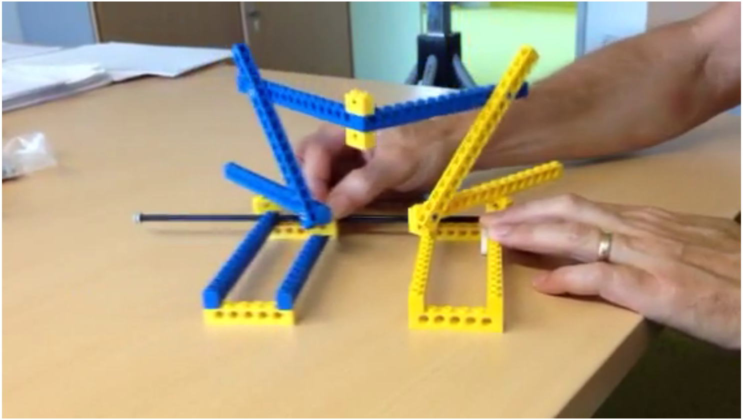
Very short connections between an R-joint and a P-joint are not possible, but very short connections between two P-joints do exist.

If an R-joint and a P-joint are connected by a short connection and the joint in the middle is an R-joint, then the two R-joints have parallel axes.

Classification of 5-loops with P- and R-joints

Theorem (Ahmadinezhad, Li, S.). Assume we have a nontrivial 5-loop with revolute and prismatic joints, at least one prismatic. Then two cases are possible.
(a) All rotational axes are parallel (there may be 2, 3, or 4).
(b) We have one prismatic joint and four revolute joints with pairwise parallel axes.

Movie of PRRRR Linkage



Helical Joints

The motions restrictions posed by helical joints are not algebraic, so bond theory does not apply directly. But we have a

Transfer Theorem (Ahmadinezhad, Li, S.). For any linkage with helical joints, we do not decrease the mobility when we replace all H-joints either by R-joints or by P-joints.

The proof is based on a Theorem of Ax on the transcendence degree of function fields. The theorem of Ax is a function-theoretical analogue of a conjecture about transcendental numbers.

Classification of 5-loops with P- , R- and H-joints

Theorem (Ahmadinezhad, Li, S.). Assume we have a nontrivial 5-loop with revolute, prismatic and helical joints, at least one helical. Then two cases are possible.
(a) All rotational axes and all helical axes are parallel (there may be 2, 3, or 4).
(b) We have one prismatic joint and four helical with pairwise parallel axes; moreover, the pitches of joints with parallel axes are equal.

Conclusion

- The investigation of closed linkages with paradoxical mobility leads to big systems of polynomial equations which are often difficult to solve.
- Bond theory is an approach that is easy to use and leads to much simpler systems of equations.
- Bonds are *not* geometrically intuitive. But for the purpose of explaining paradoxical mobility, constructing examples, or classification of paradoxical linkage, bond theory is unmatched.