Gallimaufries

Josef Schicho, RICAM-Linz

joint work with Herwig Hauser, Vienna

Rees Algebras and Blowups

Let k be a field of characteristic zero.
Let W be a nonsingular variety over k.
A Rees algebra on W is an algebra of $\mathcal{O}_W$-ideals

$$A = \oplus_{i=0}^{\infty} A_i, \quad A_i \subseteq \mathcal{O}_W, \quad A_i A_j \subseteq A_{i+j},$$

such that $A_0 = \mathcal{O}_W$.

Rees Algebras and Blowups

Let $Z \subseteq W$ be a nonsingular subvariety of W, $I = \mathrm{Ideal}(Z)$.
The blowup algebra of Z is $\alpha(Z) = \oplus_{j=0}^{\infty} I^j$.
The blowup of W at Z is $\mathbf{Proj}(\alpha(Z))$.

Assume $A \subseteq \alpha(Z)$. Then the transformed Rees algebra is

$$\tilde{A} = \oplus_{i=0}^{\infty} \tilde{A}_i, \quad \tilde{A}_i = \oplus_{j=i}^{\infty} A_i I^{j-i} \subseteq \oplus_{j=0}^{\infty} I^j$$

The Resolution Problem for Rees Algebras

The singular set of A is the set of all points x such that $A \subseteq \alpha(x)$.
If $Z \subseteq \mathrm{Sing}(A)$, then $A \subseteq \alpha(Z)$ and we can pass to the transform $\tilde{A}$.
The resolution problem is to find a sequence of nonsingular varieties in the singular set such that after finitely many steps the singular set is empty.

Hironaka's Theorem

The resolution problem has a solution for finitely generated Rees algebras.

The proof was more and more simplified by Hironaka, Aroca/Vincente/H., Villamayor, Bierstone/Milman, Cutkosky, Encinas/Hauser, Wlodarczyk, Kollar.

Closure of Rees Algebras

A Rees algebra is closed under derivations if $\delta(A_{i+1}) \subseteq A_i$, for all $i \geq 0$ and all derivations $\delta : \mathcal{O}_W \to \mathcal{O}_W$. The smallest closed Rees algebra which contains A as a subalgebra is the closure of A, written $\bar{A}$. If A is finitely generated, then $\bar{A}$ is also finitely generated.

Villamayor's Lemma

Lemma 1. $\mathrm{Sing}(A) = \mathrm{Sing}(\bar{A})$.

Lemma 2. The transform of a closed Rees algebra is not necessarily closed, but $\tilde{\bar{A}} \subseteq \bar{\tilde{A}}$.

This reduced the resolution problem to closed Rees algebras (passing to the closure after each transform).

Zooms

A nonsingular subvariety $V \subseteq W$ such that $\alpha(V) \subseteq A$ is called a *zoom*.

Trivial Example: W is a zoom.

A zoom contains the singular locus.

The proper transform of a zoom V under blowup of a nonsingular subvariety of V is again a zoom.

Gallimaufries

A gallimaufry is a pair (A, d), where A is a closed finitely generated Rees algebra and $d \leq \dim W, d \geq 0$ is an integer – the dimension – such that for every point p, there exists locally – in some neighborhood of p – a zoom of dimension d.

If (A, d) is a gallimaufry ($d < \dim W$), then $(A, d + 1)$ is also a gallimaufry.

Restrictions to Zooms

Let V be a zoom of A. Then we define the restricted zoom $A|_V$ as $i^*(A)$, where $i : V \hookrightarrow W$ is the inclusion map.

Theorem. For fixed V, restriction is a bijective operator from closed $\mathcal{O}_W$-algebras containing $\alpha(V)$ to closed $\mathcal{O}_V$-algebras.

The extension of B can be constructed either as $r^*(B) + \alpha(V)$,
where $r : W \to V$ is a left inverse of i,
or as the largest closed algebra contained in $i_*(B)$.

The Order Function

For a Rees algebra A, we define the order function $\mathrm{ord}_A : W \to \mathbb{R}_{\geq 0}$ by $p \mapsto \inf\{\mathrm{ord}_p(f)/i \mid f \in A_i\}$.

The order function is rational for finitely generated Rees algebras.

It is ≥ 1 exactly for $p \in \mathrm{Sing}(A)$.

The Order Function

In general $\operatorname{ord}_A \neq \operatorname{ord}_{\bar{A}}$, but they coincide in the singular set.

For gallimaufries, the order function is defined as the order function of the restriction of a zoom of dimension d. We restrict it to the singular set, then it does not depend on the choice of the zoom.

[It is easy to show that if $d < \dim W$, then the $\operatorname{ord}_A(p) = 1$ for any $p \in \mathrm{Sing}$.]

Tight Algebras

A finitely generated algebra is tight iff $\mathrm{ord}(p) = 1$ for any $p \in \mathrm{Sing}$.

The transform of a tight algebra is tight.

There is a well-known strategy to reduce the resolution problem to the tight case, but on some cost: nonsingularity of blowup centers or zooms is not enough, one has to require in addition that the centers fulfill a normal crossing condition with respect to the exceptional hypersurfaces from all previous blowups.

In this talk, we ignore this complication and concentrate on the tight case.

The Step where Char 0 Enters

Theorem. Assume $\mathrm{ord}(p) = 1$. Then there is locally in some neighborhood of p a zoom of codimension 1.

Proof: There must be an integer i and an element in A_i of order i. Because A is closed, there is also on element $f \in A_1$ of order 1. The zero set of f is the zoom.

Corollary. If (A, d) is tight, then $(A, d-1)$ is a gallimaufry.

Proof of Hironaka's Result

(modulo the complication with the normal crossing condition)

Lemma 1. If there is a resolution for tight gallimaufries of dimension d, then there is a resolution for all gallimaufries of dimension d.

Lemma 2. If there is a resolution for gallimaufries of dimension d, then there is a resolution for tight gallimaufries of dimension $d + 1$.

On the Complication

The methods for fighting the complication arising from the normal crossing conditions are entirely combinatorial. No algebra is used, only incidence relations between various sets (exceptional divisors, singular sets of involved gallimaufries, sets of constant order).

The combinatorics is complicated and can be described by the game "Salmagundy". It is played between two players, Dido and Mephisto, who change a certain graph by successive moves. Dido has won when a final configuration is reached; Mephisto cannot win. The complicated part is to show that Dido has a winning strategy.

Best Place to Learn Details

Summer 2012, Obergurgl, Austria:

Clay Summer School on Resolution of Singularities

Organizers: H. Hauser, S. Mori, S.