Invariant Theory, Photogrammetry, and Hexapods

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Problem 1: Recover an Object from Images

- *object:* unknown, typically in $\mathbb{R}^3$
- *object set:* set of possible objects
- *image:* known, typically in $\mathbb{R}^2$
- *image set:* set of possible images

Equivalent Objects/Images

Two images in the image set (objects in the object set) are equivalent if there is a transformation from one to the other.

Equivalent images give the equal information about the object.

We can hope to determine the object up to equivalence.

Possible transformation groups: Euclidean similarities or projective transformations

Invariant Theory

The quotient set of equivalence classes of objects/images modulo a group of transformations is called object (image) moduli space. By classical invariant theory, we can construct quotient varieties.

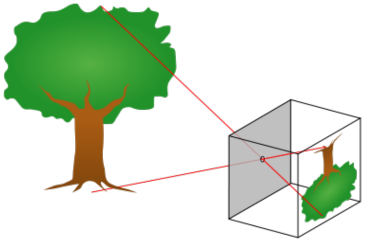
Example 1: If the image set is $(\mathbb{P}^1)^4$, and the transformation group G is the projective group, then the class is determined by the cross ratio. The quotient variety is $\mathbb{P}^1$.
The point $(0:1)$ in the quotient variety corresponds to two equivalence classes, one with $p_1 = p_2$ and the second with $p_3 = p_4$. (similarily the points $(1:0)$ and $(1:1)$).

Invariant Theory

Example 2: If the image set is $(\mathbb{P}^1)^6$, and the transformation group G is the projective group, then the quotient variety M_6 has dimension 3 and can be embedded into $\mathbb{P}^4$ as a cubic hypersurface. It has 10 singular points and 15 planes.

The planes correspond to 6-tuples with coincident points.
The singular points correspond to 6-tuples $(p_1, \ldots, p_6)$ such that either $p_1 = p_2 = p_3$ or $p_4 = p_5 = p_6$, possible after some permutation (10 cases).

Cameras



Abstractly, we may describe a camera as a function

$$F : (\text{object space} \times \mathbb{R}^3) \to \text{image moduli space}$$

If objects O_1 and O_2 are equivalent by a transformation T, then

$$F(O_1, p) = F(O_2, T(p)).$$

The Profile

For a fixed object O, the profile is defined as $\{F(O,p) \mid p \in \mathbb{R}^3\}$. It has the parametrization $p \mapsto F(O,p) =: F_O(p)$.

Exercise 1: Prove that equivalent objects have the same profile.

Problem 1 can be decomposed into two subproblem:

1. Given a (finite) set of points in the profile, compute the profile.
2. Given the profile, compute the object (up to equivalence).

Planar Surveying

		transformations:
Object set:	$(\mathbb{R}^2)^4$	Euclidean similarities
Image set:	$(S^1)^4$	rotations

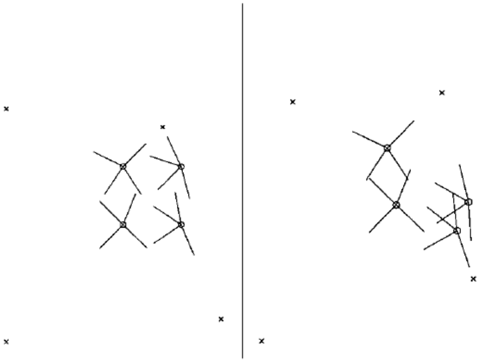
The image moduli space can be defined as $\{(z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^3(\mathbb{C}) \mid |z_1| = |z_2| = |z_3| = |z_4|\}$, the set of "equiabsolute" points.

Planar Surveying

Theorem 1. For general 4-tuples (the 4 points should not be collinear), the profile is a the set of equiabsolute points of the quadric surface passing through the 5 points $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$, $(0:0:0:1)$, $(1:1:1:1)$.

Corollary. Given 4 images, subproblem 1 can be solved by interpolating a quadric equation through nine points in $\mathbb{P}^3$.

Planar Surveying



By [Aström, Oskarsson 2000], subproblem 2 has in general 2 solutions.

6 Points in the Plane

		transformations:
Object set:	$(\mathbb{P}^2)^6$	projective
Image set:	$(\mathbb{P}^1)^6$	projective

If among the 6 points in $\mathbb{P}^2$ there are the two cyclic points $(0 : 1 : \pm\mathrm{i})$, then we get planar surveying.

6 Points in the Plane

The image is a set of 6 points in $\mathbb{P}^1$, or its equivalence class up to projective transformations. Hence the image modulo space is the Segre cubic M_6.

Theorem 2. For general 6-tuple in $\mathbb{P}^2$, the profile is a hyperplane section (cubic surface). 15 of its 27 lines are intersections with the 15 planes.

Corollary. Given 4 images, subproblem 1 can be solved by passing a hyperplane to 4 points in $\mathbb{P}^4$.

6 Points in the Plane

Subproblem 2 can be solved by parametrizing the cubic surface by a cubic parametrization. There are 72 such parametrization (up to reparametrizing by projective linear transformations). There are 2 among them that do not blow down any of the 15 “good” lines. Then the 6-tuple object consists of the 6 base points of the parametrization.

6 Points in Space

		transformations:
Object set:	$(\mathbb{P}^3)^6$	projective
Image set:	$(\mathbb{P}^2)^6$	projective

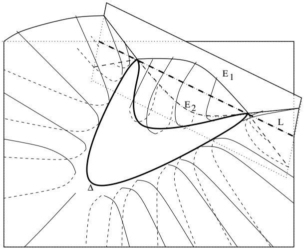
The image moduli space is a double cover of $\mathbb{P}^4$ branched over a quartic hypersurface I_4, known as the *Igusa quartic*. The ramification points correspond to 6-tuples of points lying on a plane conic.

6 Points in Space

For 6 general points in $\mathbb{P}^3$, there exists a unique cubic space curve C passing through them. If the camera is located on C, then the 6 points of the image lie on a conic. Any two such images are equivalent.

Apart from contracting C to a point, the profile parametrization map is injective.

6 Points in Space



Theorem 3. For a general 6-tuple of points in $\mathbb{P}^3$, the profile is the double cover of a hyperplane in $\mathbb{P}^4$ that is tangential to I_4. The tangential point is the image of the contracted curve C.

Corollary. Given 3 images, subproblem 1 amounts to finding a tangent hyperplane passing through 3 points; or dually, finding a point on the dual hypersurface lying on a known line. Since the dual hypersurface has degree 3 (again the Segre cubic), there are in general 3 solutions.

The Möbius Camera

Here the camera model is a projection followed by a Möbius transformation (group generated by even products of inversions on circles and Euclidean reflections). The image of a line is a line or a circle or a point. The contour of a sphere is a circle.

The Möbius Camera

		transformations:
Object set:	$(\mathbb{R}^3)^6$	Euclidean
Image set:	$(\mathbb{P}^1(\mathbb{C}))^6$	complex projective

The image module space $M_6(\mathbb{C})$ has dimension 6, but it is also a complex variety of dimension 3.

The Möbius Camera

The profiles are varieties of dimension 3, which cannot have a complex structure. So we define a restricted profile, where the camera is constrained to a fixed plane. In order to preserve object equivalence, we choose the plane at infinity, which means parallel projections.

The Möbius Camera

Theorem 4. Let $(p_1, \ldots, p_6) \in (\mathbb{R}^3)^6$ be a 6-tuple of point such that no 4 are collinear and not all 6 are coplanar. Then the restricted profile is a rational sextic complex algebraic curve in $M_6(\mathbb{C})$ defined by real equations. Its nonsingular points come in conjugate pairs which differ by reflection (or inversion). Moreover, the restricted profile parametrization

$$F_O : S^2 \to M_6(\mathbb{C})$$

assigning each unit vector u the Möbius class of the image by projections orthogonal to u is holomorphic, when we identify S_2 with the Riemann sphere $\mathbb{P}^1(\mathbb{C})$.

The Möbius camera

In order to solve subproblem 2, we get a complex parametrization of the restricted profile curve

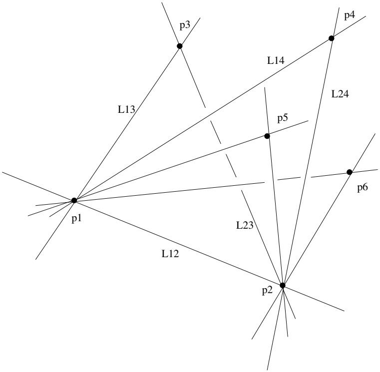
$$\phi : \mathbb{P}^1(\mathbb{C}) \to X \subset M_6(\mathbb{C})$$

mapping antipodal points to conjugated points. Algebraically, this is a birational map

$$\{(x : y : z) \in \mathbb{P}^2(\mathbb{C}) \mid x^2 + y^2 + z^2 = 0\} \to X$$

defined over the real numbers. Such a parametrization always exists and is unique up reparametrization by special orthogonal linear maps.

The Möbius camera



Let $E_{12}, \ldots, E_{56} \subset M_6(\mathbb{C})$ be the planes corresponding to 6-tuples with coinciding points. Then $\phi^{-1}(E_{ij}) \in S^2$ is the pair of two antipodal points in the direction of the line L_{ij} through p_1, p_2. This allows to reconstruct $p_1, \ldots, p_6$ up to similarity.

The Möbius camera

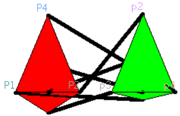
Corollary. If the profiles of two 6-tuples have infinitely many common points, then the restricted profiles are equal and the 6-tuples are equivalent up to Euclidean similarity.

Using intersection theory, one can show that the number of common points of two rational sextics in $M_6(\mathbb{C})$ is at most 14. For a general rational sextic C_1, there is a unique sextic C_2 intersecting C_1 in 14 points.

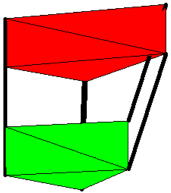
Hexapods

A hexapod consists of two rigid bodies (platform and base), connected by legs. The legs have fixed length and are anchored at platform/base spherical joints.

Movable Hexapods



Butterfly Linkage



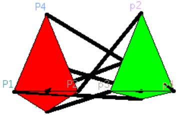
Bricard-Borel Linkage

General hexapods have 40 (not necessarily real) configurations. Special hexapods have infinitely many configurations. Several families (described as irreducible algebraic varieties in parameter space) are known. One would like to know how many maximal families exist, and to have a precise description of each maximal family.

In the following theorem, we assume: no 4 base points, no 4 platform points are collinear, not all base points are coplanar, not all platform points are collinear.

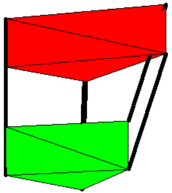
Theorem 5. If a hexapod is movable, then the restricted profile X_B of the base points and the restricted profile X_P of the platform points have common points. The number of common points is bigger than or equal to the degree of the motion as a curve in $\mathbb{P}^3$ embedded by the Euler parameters.

Mobile Hexapods and Möbius Photogrammetry



Example 1. If p_1, p_2, p_3 are collinear, or p_4, p_5, p_6 are collinear, then the profile contains one of the 10 singular points of M_6. This is the single common point of the profiles of base and platform of the Butterfly Linkage. The motion is a rotation, of degree 1 in the Euler parameters.

Mobile Hexapods and Möbius Photogrammetry



Example 2. In a Bricard-Borel Linkage, the projections of base and platform along the rotation axes are related by an inversion. Therefore the profiles have two common points which are conjugate to each other. The rotational part of the motion is again a rotation of degree 1. There is an algebraic explanation for the number of common images being 2, not 1: the map from the motion to its Euler parameters is 2:1.

Mobile Hexapods and Möbius Photogrammetry

Corollary. The degree of a motion of a mobile hexapod is at most 14.

Corollary. If a non-planar hexapod has a two-dimensional mobility, then base and platform are similar.

Mobile Hexapods and Möbius Photogrammetry

For a general 6-tuple of base points, we think that there exist mobile hexapods with a motion of maximal degree 14. Here is a construction which worked for every example tested so far.

1. Compute the restricted profile X of the base.
2. Compute the unique sextic rational curve X' intersecting X in 14 points.
3. Use Möbius photogrammetry to recover the platform points up to similarity, from X'.
4. Compute the similarity factor (solution is unique in all tested cases).
5. Compute leg lengths (three-dimensional set of solutions in all tested cases).

Known Families Mobile Hexapods

family	found by	degree	profile
planar a. s.	Duporcq 1898	20	5
butterfly		1	1
Schönflies	Bricard 1906, Borel 1906	1	2
	Husty/Zs.-Murray 1994		
translational	Nawratil 2012	0	2
flexible octahedra	Bricard 1897	6	6
planar affine	Karger 2002		
	Nawratil 2010		
number theory	Geis/Schreyer 2009	6	
leg surgery	Nawratil 2011		
point symmetric	Dietmaier, Nawratil 2013	12	12
this talk	Gallet/Nawratil/S 2015	14	14

Conclusion

- Profiles can be a useful stepping stone in photogrammetry problem.
- Möbius photogrammetry is a powerful technique for constructing mobile hexapods.
- The reason behind the success of this technique is the unexpected appearance of complex structures. They allow "dimension halfing".