Simplifying products of fractional powers of powers

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Abstract

Most computer algebra systems incorrectly simplify

$$\frac{w-w}{\frac{\sqrt{z^2}}{z^3}-\frac{1}{z\sqrt{z^2}}}$$

to 0 rather than to $0/0$. The reasons for this are:

1. The default simplification doesn't succeed in simplifying the denominator to 0.
2. There is a rule that 0 is the result of 0 divided by anything that doesn't simplify to either 0 or $0/0$.

Many of these systems have more powerful optional transformation and general purpose simplification functions. However that is unlikely to help this example even if one of those functions can simplify the denominator to 0, because the input to those functions is the result of *default* simplification, which has already incorrectly simplified the overall ratio to 0. Try it on your computer algebra systems!

Many of these extra transformation and general simplification functions *do* transform fractional powers of powers and products of such sub-expressions, but not always to an expression that is equivalent everywhere the input is defined. For example many systems unnecessarily rationalize the denominator of $1/\left(z\left(z^2\right)^{1/3}\right)$, giving $\left(z^2\right)^{2/3}/z^3$. The value of the original expression is an informative complex infinity at $z=0$, whereas the replacement expression is a useless nonequivalent $0/0$ at $z=0$. Moreover, the replacement expression requires a mental calculation to realize that the pole dominates the zero at $z=0$ by a multiplicity of $5/3$.

This article describes how to compute three good principal branch forms for products of the form $w^\alpha\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}$ where w is any real or complex expression and the exponents are rational numbers or various extensions thereof.

It might seem that surely good simplification of such a restrictive expression class must already be either published or built into at least one widely used computer-algebra system, but apparently this issue has been overlooked.

DRAFT

1 Introduction

The Appendix lists a sequence of *Mathematica*® rewrite rules that simplify sub-expressions of the form $w^\alpha\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}$, where w is any canonically simplified real or complex expression and

α, β_k and γ_k are rational numbers. All but one of these rules is automatically applied *before* default simplification so that, for example, the input

$$\frac{w-w}{\dfrac{\sqrt{z^2}}{z^3}-\dfrac{1}{z\sqrt{z^2}}}$$

correctly simplifies to `indeterminate` rather than to 0. Some of these rules are also automatically applied *during* default or optional simplification, but one rule that would interfere with others is applied only *after* all other simplification.

Table 1 shows the result of these rewrite rules on the expression $w^{\alpha}\left(w^2\right)^{\gamma}$ for $\alpha=-3,-2,\ldots,3$ together with $\gamma=-7/3,-4/3,\ldots,8/3$. Tables 2 through 9 show the results for various computer algebra default and optional simplification functions. The expression coloring in all of the tables in this article reflect the following partially conflicting goals, listing in decreasing order of importance:

1. The result must be equivalent to the input wherever the input is defined.[1]

2. When possible, use the identity

$$\left(w^{\beta}\right)^{\gamma}\equiv w^{m\beta}\left(w^{\beta}\right)^{\gamma-m}. \tag{1}$$

   with an appropriate integer m to avoid having w occur to both positive and negative net exponents.[2]

3. When possible, fully absorb the power of w into the power of w^{β} to have fewer factors.

4. Otherwise minimize the *magnitude* of γ in $\left(w^{\beta}\right)^{\gamma}$ to minimize the contribution of the troublesome nested power in favor of the better behaved unnested power of w.

5. When there is a choice between $\gamma=-1/2$ and $\gamma=1/2$, choose the latter so as to rationalize the denominator rather than the numerator.

A larger numbered goal is not fulfilled if the only way to fulfill it is to violate a smaller numbered goal. For example, fulfillment of goals 4 and/or 5 would often violate goals 1, 2 and/or 3.

The expression colors in all of the tables match the above goal text color of the smallest-numbered goal that is violated by each entry in the tables. A black entry complies with all of the goals. Many colored entries also violate less important larger-numbered goals than is indicated by the expression color.

The reasons for this ranking of goals are:

1. A red entry is most unsatisfactory because it is a result that is not equivalent to the input everywhere the input is defined.

2. A magenta entry is next most serious because it is a squandered opportunity to improve the result by removing a removable singularity, thereby making the result have the limiting value at $w=0$ rather than be undefined there. Not fulfilling this goal is analogous to not simplifying w/w^2 to $1/w$, or w^2/w to w, or w/w to 1.

[1]In this article, all finite and infinite complex numbers are regarded as defined, including the circle at complex infinity or any proper subset of it. Only $0/0$ is considered undefined.

[2]The net exponent of $\left(w^{\beta}\right)^{\gamma}$ is $\beta\gamma$.

3. A green entry is more complicated than need be.

4. A blue entry is less suitable for canonicality, because when there is more than one factor of the form $\left(w^{\beta}\right)^{\gamma}$, there might be more than one way to distribute only some of w^{α} into the nested powers, whereas there are at most two ways to minimize $|\gamma|$ by absorbing all of the excess values of γ into α.

5. A cyan entry is contrary to the custom of canonically rationalizing denominators that are square roots.[3]

Table 2 shows the results for *Mathematica* 8.0.0 default simplification.[4] Table 3 shows the results for the *Mathematica* FullSimplify[...] function. Table 4 shows the results for Maple[tm] 13 and wxMaxima 0.8.2 default simplification.[5] Table 5 shows the results for the Maple simplify(...) function. Table 6 shows the corresponding result for default *Derive*® 6 simplification after the declaration w :∈ `Complex`. Table 7 shows the corresponding result for default simplification in current Texas Instruments products that contain computer algebra.[6] Tables 8 and 9 show corresponding results for the wxMaxima fullratsimp(...) and rat(...) functions.

Mathematica, wxMaxima and Maple also respectively have PowerExpand[...], radcan(...) and simplify(..., symbolic) functions that always transform $(w^{\beta})^{\gamma}$ to $w^{\beta\gamma}$. However, because of such transformations these functions can return results that are not equivalent to the input everywhere it is defined. These systems, *Derive* and TI computer algebra also have safe ways to enable such desired transformations when justified by declaring, for example, that certain variables are real or positive.

Wherever an *input* is defined in Tables 1 through 9, it is equivalent to the *input* two rows below it, but one column left of it. However, the *results* are not all canonically identical in every row and two rows down shifted one column left, except for Table 6 (*Derive*) and Table 1 (the algorithm described in subsection 2.3).

Tables 1 through 8 use only powers of w and of w^{β}. Table 9 also uses powers of $\left(w^{\beta}\right)^{\gamma}$. However, the standard definition of $u^{m/n}$ for reduced integers m and n *is* $\left(u^{1/n}\right)^{m}$, which is consistent with the alternate definition $e^{\ln(u)m/n}$.[7] Consequently, the wxMaxima rat $(\ldots)$ function merely makes the standard interpretation of the input more explicit at the expense of clutter. Nonetheless, it might be helpful as a precursor to syntactically substituting a new expression for $(w^2)^{1/3}$.

To indicate how these systems and their optional simplification functions comply with goal 5, Table 10 lists their simplified values of $z/\sqrt{z^2}$.

Goals 1 through 5 are also applicable to fractional powers of negative powers, For example, the

[3] A now irrelevant reason for this tradition is probably because it is much easier, for example, to approximate $\sqrt{2}/2$ than $2/\sqrt{2}$ when done by table look up followed by manual division.

[4] Default simplification is called *evaluation* in *Mathematica* and in many other systems.

[5] All of the wxMaxima results follow a prior assignment `domain : complex` to force the principal rather than the default real branch; and a prior declaration `declare(w, complex)` to prevent unwanted transformations that might happen for real w.

[6] TI CAS refers to the computer algebra built into some Texas Instruments calculators, for which there are also Windows and Macintosh versions. For these products w must be entered as $w\_$ to indicate that it is a complex variable, thereby using the principal rather than the real branch.

[7] Using $(u^m)^{1/n}$ is *not* equivalent to the principal branch of $u^{m/n}$ for all u.

transformation

$$\frac{1}{w\left(\frac{1}{w^2}\right)^{3/2}} \rightarrow \frac{w}{\sqrt{\frac{1}{w^2}}} \tag{2}$$

meets all of the goals, because:

1. The result is equivalent to the input wherever the input is defined.

2. At $w = 0$ the result improves from $0/0$ to $0/\sqrt{1/0^2} \rightarrow 0/\mathring{\infty} \rightarrow 0$, which is the limit of the input as $w \rightarrow 0$ from any direction, where $\mathring{\infty}$ denotes complex infinity.

Rationalizing the denominator of result (2) transforms the expression to

$$w^3\sqrt{\frac{1}{w^2}}, \tag{3}$$

which isn't as cumbersomely *tall* as result (2). Howeveer, result (3) sacrifices the more important improvement from $0/0$ to 0 at $w = 0$.

It is tempting to *oversimplify* this result by commuting the reciprocation with the fractional power to further transform result (2) to

$$\frac{w}{\left(\frac{1}{\sqrt{w^2}}\right)} \stackrel{?}{\rightarrow} w\sqrt{w^2}. \tag{4}$$

However, it is important to resist the temptation to transform $\left(w^{-\lambda}\right)^{\mu}$ to $\left(w^{\lambda}\right)^{-\mu}$ unless μ is integer, or $-1 \leq \lambda < 1$, or some declaration precludes w along the branch cuts of $w^{-\mu}$ where the two expressions are not equivalent.[8] The branch cuts of

$$\sqrt{\frac{1}{w^2}}$$

are the positive and negative imaginary semi-axes of w. So at $w = i$, for example, correct result (2) has value 1 whereas incorrect result (4) has value -1.[9]

Tables 11 through 15 show the result of applying the rewrite rules in the Appendix and most of the preceding system's default and optional simplification to the expression $w^{\alpha}\left(w^{-2}\right)^{\gamma}$ for $\alpha = -3, -2, \ldots, 3$ together with $\gamma = -5/2, \ldots, 5/2$. TI-CAS results require using w_ to indicate a complex variable, and the *Derive* results require the prior declaration $w :\in$ `Complex`.

[8]Rich and Jeffrey [5] suggest implementing a multi-valued interpretation of fractional powers for radicands that are non-numeric or *rational* numbers, but using a principal-valued interpretation for *floating-point* radicands. Although this makes substitution not commute with simplification for floating-point numbers, the scheme otherwise has some attractive properties. However, all of the major computer algebra systems currently strive to use a single-valued interpretation of fractional powers, with the default being either the principal or real branch. Consequently, it is a bug if these systems quietly employ transformations that don't preserve single-valued equivalence for that branch wherever the input is defined.

[9]Notice that even $\sqrt{1/z}$ is not equivalent to $1/\sqrt{z}$ along its branch cut, such as at $z = -1$. In contrast $\sqrt{1/z^{8/9}} \equiv 1/\sqrt{z^{8/9}} \equiv z^{-4/9}$ because $-1 < -8/9 \leq 1$. However, this simplification should have already been done by bottom-up default simplification.

Results for wmMaxima are omitted for this family of examples because its default simplification incorrectly commutes these fractional powers with reciprocation despite the assignment `domain : complex` and the declaration `declare(z, complex)`. The optional simplification functions unavoidably receive default-simplified inputs, so all of the wmMaxima entries would be colored **boldface** red because of being incorrect along entire semi-infinite rays rather than only at one point $w = 0$.

Wherever an *input* is defined in Tables 11 through 15, it is equivalent to the *input* two rows below it, but one column *right* of it. However, the *results* are not all canonically identical in every row and two rows down shifted one column right, except for Tables 15 (*Derive*) and 11 (the algorithm described in subsection 2.3).

Collectively, the many differences between and within Tables 1 through 15 indicate a general lack of thought about how best to simplify expressions of the form $w^{\alpha}\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}$.

The many colored entries and the lack of exhibited canonicality in most of the tables indicate that there is significant room for improvement in all of these computer algebra systems.[10]

If a class of expressions includes 0, then any reasonable canonical form for that class should return 0 for any expression in the class that is equivalent to 0. Therefore having default simplification transform expressions into a canonical form is one way to at least avoid incorrect results such as transforming

$$\frac{w-w}{\dfrac{\sqrt{z^2}}{z^3}-\dfrac{1}{z\sqrt{z^2}}}$$

to 0.

As discussed in Brown [1], Moses [3] and Stoutemyer [6], canonical forms are too costly and rigid for the entire class of expressions addressed by general-purpose computer algebra systems. However, canonical forms are acceptable and good for default simplification of certain simple classes of expressions such as the class of sub-expressions discussed in this article. Accordingly, this article describes three good canonical forms for this class and algorithms that can achieve them.

2 Three alternative forms

For consistency, computer algebra systems should use transformations that preserve equivalence everywhere in the infinite complex plane, including branch cuts and irremovable singularities. Otherwise users can obtain a different result if they substitute numbers into the transformed versus untransformed result. Most computer algebra systems use the principal branch for numerical expressions and this article discusses only that choice.[11]

This section discusses three alternative forms for sub-expressions of the form

$$W := w^{\alpha}\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}. \tag{5}$$

They each have different sets of advantages and disadvantages. Therefore it is worthwhile to make one of them be the default simplification and offer the others as options obtainable by transformation functions or a control variable.

[10] *Mea culpa*: I am a coauthor of *Derive* and TI-CAS.

[11] Some computer algebra systems optionally or by default use the *real* branch wherein for reduced integers m and n, $(-1)^{m/n} \to 1$ for m even, and $(-1)^{m/n} \to -1$ for m and n odd.

2.1 Form 1: One unnested power times a unit-magnitude factor

A universal principal-branch formula for transforming a nested power to an unnested power is

$$\left(w^{\beta}\right)^{\gamma} \to (-1)^{\tau}\, w^{\beta\gamma}, \tag{6}$$

where

$$\tau := \mathrm{mod}\left(\frac{\gamma\left(\arg\left(w^{\beta}\right) - \beta \arg(w)\right)}{\pi}, 2\right). \tag{7}$$

If w is nonnegative or $-1 < \beta \leq 1$ or γ is integer, then $(-1)^{\tau} \equiv 1$. However, such opportunities should have already been exploited with bottom-up simplification, in which case those factors will have already been simplified to 1. Thus without loss of generality we assume that w isn't known to be nonnegative, and that $\beta \leq -1$ or $\beta > 1$, and that γ is non-integer.

The transformation given by formulas (6) and (7) can be derived from the identities

$$\begin{aligned} |p| &\equiv (-1)^{-\arg(p)/\pi} p, && (8)\\ |q^{\alpha}|^{\beta} &\equiv |q|^{\alpha\beta}. && (9) \end{aligned}$$

Using transformation (6) on every $\left(w^{\beta_k}\right)^{\gamma_k}$ in W defined by (5) gives

$$\overline{W} = (-1)^{\sigma}\, w^{\alpha+\beta_1\gamma_1+\cdots+\beta_n\gamma_n}, \tag{10}$$

where σ is a canonically simplified sum of terms of the form (7). Since $(-1)^{\sigma}$ has period 2 in σ, for canonicality we should reduce all of the coefficients in the expanded terms of σ modulo 2 to a standard interval such as (-1, 1]. For example,

$$\begin{aligned} z^{-3/2}\left(z^{-4/3}\right)^{-4/5} &\to (-1)^{\mathrm{mod}\left((-4/5)\left(\arg\left(z^{-4/3}\right)-(-4/3)\arg z\right)/\pi,\,2\right)}\ z^{-3/2+(-4/3)(-4/5)} \\ &\to (-1)^{\left((-4/5)\arg\left(z^{-4/3}\right)-(1/15)\arg z\right)/\pi}\ z^{-13/30}. \end{aligned} \tag{11}$$

If $\arg(0)$ simplifies to 0 as it does in *Mathematica*, then this canonical form has the benefit of candidly removing removable singularities entirely: For example in the input of transformation (11), the net negative total degree of $|z|$ is -3/2 and the net positive total degree of $|z|$ is $(-4/3)(-4/5) = 16/15$, whereas the result has only a net negative total degree of -13/30. Thus an *input* that is 0/0 at $z = 0$ has been improved to a *result* that is a well-defined complex infinity at $z = 0$.

As two other examples of this benefit of $\arg(0) \to 0$,

$$\begin{aligned} \frac{z^3}{\left(z^2\right)^{1/2}} &\to (-1)^{\mathrm{mod}\left((-1/2)\left(\arg\left(z^2\right)-2\arg z\right)/\pi,2\right)}\ z^{3-2(1/2)} \\ &\to (-1)^{\left(-\arg\left(z^2\right)/2+\arg z\right)/\pi}\ z^2, \end{aligned} \tag{12}$$

which improves the input from 0/0 to 0 at $z = 0$, and

$$\begin{aligned} \frac{\left(z^2\right)^{1/2}}{z} &\to (-1)^{(1/2)\left(\arg\left(z^2\right)-2\arg z\right)/\pi}\ z^{1-2/2} \\ &\to (-1)^{\left(\arg\left(z^2\right)/2-\arg z\right)/\pi}, \end{aligned} \tag{13}$$

which improves the input from $0/0$ to 1 at $z=0$. Therefore $\arg(0) \to 0$ has replaced troublesome partial functions with total functions that have the most appropriate value where the input has a removable singularity.[12]

Unfortunately most computer algebra systems leave $\arg(0)$ as is, or treat it as an unknown value in the interval $(-\pi, \pi]$, or – worse yet – throw an error. For such systems, this canonical form can have the serious disadvantage of producing a result that is not equivalent to an input that is defined at $w=0$, or throwing an error that prevents any result at all without awkward programming that is beyond amateur expertise and very troublesome to experts. Two such examples are

$$\left(z^2\right)^{1/2} \to (-1)^{\left(\arg\left(z^2\right)/2-\arg z\right)/\pi}\, z$$

for which the input is a well-defined 0 at $z=0$, and

$$\left(z^2\right)^{-1/2} \to \frac{(-1)^{\left(-\arg\left(z^2\right)/2+\arg z\right)/\pi}}{z}$$

for which the input is a well-defined complex infinity at $z=0$.

Sometimes $\arg(\ldots)$ and therefore the $\arg(0) \not\to 0$ issue can be avoided:

1. When w is real, $(-1)^\sigma$ can often be simplified to either 1 or to a piecewise expression of the form
$$\begin{cases} 1, & \text{if } w :: 0, \\ -1 & \text{otherwise,} \end{cases}$$
where "::" is one of the comparison operators "$>$", "$\geq$", $=$, "$\leq$", "$<$", or "$\neq$".

2. For fractional powers that are half integers or quarter integers, $(-1)^\sigma$ can be expressed as a piecewise expression depending on the real and/or imaginary parts of w. For example,
$$\frac{\left(w^2\right)^{1/2}}{w} \to \begin{cases} 1 & \text{if } \Re(w) > 0 \vee \Re(w) \geq 0 \wedge \Im(w) \geq 0, \\ -1 & \text{otherwise;} \end{cases} \tag{14}$$
$$\frac{\left(w^4\right)^{1/4}}{w} \to \begin{cases} 1 & -\Re(w) < \Im(w) \leq \Re(w), \\ -i & \text{if } -\Im(w) < \Re(w) \leq \Im(w), \\ -1 & \Re(w) < \Im(w) \leq -\Re(w), \\ i & \text{otherwise.} \end{cases}$$
Maple has a built-in function named csgn that is defined by the right side of (14), and simplify(...) uses this function for half powers of squares, as illustrated in Table 10. For example, simplify $\left(\left(w^2\right)^{7/2}\right) \to w^7 \operatorname{csgn}(w)$, which is more candid than the other results in that table, albeit in a notation that is probably familiar only to experienced Maple users.

[12] Some users object to such domain enlargements. However, many of them would inconsistently complain if z/z didn't automatically simplify to 1 as it already does in most computer algebra systems, after which the substitution $1 \mid z=0$ unavoidably simplifies to 1. The remedy is to implement optional *domain-enlargement provisos* that are automatically attached to intermediate and final results, making these three example results contain the attached proviso "$\mid z \neq 0$". However, implementation of that is beyond the scope of this article.

For systems that don't automatically simplify $\arg(0)$ to 0 and can't be made to do so, form (10) should be avoided when $w = 0$ is possible and the input is defined there.

Moreover, when $\arg(\ldots)$ can't be avoided, most implementers might want to avoid form (10) even when $\arg(0) \to 0$, because when a result contains $\arg(\ldots)$, then $(-1)^{\sigma}$ is likely to be rather complicated:

1. It will probably contain complicated square roots and arctangents if the real and imaginary parts of w are given as exact numbers.

2. It will probably also contain piecewise sign tests if given real and imaginary parts are non-numeric, such as for $w = x + iy$ with non-numeric real x and y.

3. It will probably contain radicals nested at least one deep if $\arg(w)$ is given as a simple enough rational multiple of π.

4. Otherwise it will contain perhaps bulky sub-expressions $\arg(w)$ and $\arg\left(w^{\beta}\right)$ — or, worse yet, expressions involving square roots, arctangents, piecewise sign tests, and sub-expressions of the form $\Re(w)$ and $\Im(w)$.

As espoused by Corless and Jeffrey [2], expression τ can alternatively be defined in terms of the unwinding *function* κ as:

$$\tau := 2\gamma\kappa\left(\beta \ln w\right). \tag{15}$$

This is more concise than definition (7), but a function that computes unwinding numbers isn't currently available in most computer algebra systems. Also, unless the system automatically transforms $\ln 0$ to $-\infty$, as is done in *Mathematica* but not most systems, then definition (15) has the same disadvantages as using $\arg(\ldots)$.

2.2 Form 2: Reduction of outer fractional exponents to (-1/2, 1/2]

Proposition 1. *For $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R} - \mathbb{Z}$, and arbitrary expression $w \in \mathbb{C}$,*

$$\left(w^{\beta}\right)^{\gamma} \equiv w^{\mathrm{Ip}(\gamma)\,\beta}\left(w^{\beta}\right)^{\mathrm{Fp}(\gamma)} \tag{16}$$

where $\mathrm{Ip}(\ldots)$ *denotes the integer part and* $\mathrm{Fp}(\ldots)$ *denotes the fractional part.*

Proof. We have

$$\left(w^{\beta}\right)^{\gamma} \equiv \left(w^{\beta}\right)^{\mathrm{Ip}(\gamma)}\left(w^{\beta}\right)^{\mathrm{Fp}(\gamma)} \tag{17}$$

because:

1. With $\gamma \in \mathbb{R} - \mathbb{Z}$, $\operatorname{sign}\mathrm{Ip}(\gamma) = \operatorname{sign}\mathrm{Fp}(\gamma)$,

2. For any expression $u \in \mathbb{C}$ and $r_1, r_2 \in R \mid \operatorname{sign} r_1 = \operatorname{sign} r_2$,

$$u^{r_1+r_2} \equiv u^{r_1}u^{r_2}, \tag{18}$$

   even at $u = 0$ with r_1 and r_2 both negative, making both sides of (18) be complex infinity.

We also have $\left(w^{\beta}\right)^{\mathrm{Ip}(\gamma)} \equiv w^{\mathrm{Ip}(\gamma)\,\beta}$ because $\mathrm{Ip}(\gamma) \in \mathbb{Z}$.

□

Remark: If r_1 is positive and r_2 is negative, then $0^{r_1+r_2} \not\equiv 0^{r_1}0^{r_2}$ because

$$0^{r_1+r_2} = \begin{cases} 0 & \text{if } r_1 + r_2 > 0, \\ 1 & \text{if } r_1 + r_2 = 0, \\ \text{complex infinity} & \text{otherwise}, \end{cases}$$

whereas $0^{r_1}0^{r_2}$ is always $0/0$.

Therefore Phase 1 of the rewrite rules in the Appendix is to transform $w^{\alpha}\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}$ toward canonicality by transforming every positive fraction γ_k to the interval $(0,1)$ and every negative fraction γ_k to the interval $(-1,0)$. The various $w^{\mathrm{Ip}(\gamma_k)\,\beta_k}$ are combined with the original w^{α}, giving a transformed expression

$$\widehat{W} := w^{\hat{\alpha}}\left(w^{\beta_1}\right)^{\hat{\gamma}_1}\cdots\left(w^{\beta_n}\right)^{\hat{\gamma}_n}$$

where $\hat{\alpha}$ might be 0.

In general, any or all exponents can be fractions and/or negative. For example,

$$\begin{aligned} w^{-3/10}\left(w^{-3/2}\right)^{-7/3} &\rightarrow w^{-3/10}\left(w^{-3/2}\right)^{-2}\left(w^{-3/2}\right)^{-7/3+2} \\ &\rightarrow w^{-3/10}w^{6/2}\left(w^{-3/2}\right)^{-1/3} \\ &\rightarrow w^{27/10}\left(w^{-3/2}\right)^{-1/3}. \end{aligned}$$

This transformation of each nested power is context independent and therefore fast and easy to implement.

Also, the result of this phase has the advantage that if $\left(w^{\beta}\right)^{\hat{\gamma}}$ is subsequently raised to any power λ, then we can simplify it to the simplified value of $\left(w^{\beta}\right)^{\gamma\lambda}$ because $-1 < \hat{\gamma} < 1$. For example,

$$\left(\left(w^2\right)^{3/4}\right)^{7/6} \rightarrow \left(w^2\right)^{7/8}.$$

Expression $\widehat{W}$ given by definition (2.2) is equivalent to expression W everywhere that W is defined, because at $w = 0$:

1. Expressions W and $\widehat{W}$ are both 0 if $\alpha \geq 0$ and all of the $\beta_k\gamma_k$ are positive.

2. Otherwise expression W and $\widehat{W}$ are both complex infinity if $\alpha \leq 0$ and all of the $\beta_k\gamma_k$ are negative.

3. Otherwise if all $\beta_k\gamma_k > 0$ and $\hat{\alpha} \geq 0$, then W is $0/0$ but $\widehat{W}$ has improved to 0.

4. Otherwise if all $\beta_k\gamma_k < 0$ and $\hat{\alpha} \leq 0$, then W is $0/0$ but $\widehat{W}$ has improved to complex infinity.

5. Otherwise both W and $\widehat{W}$ are $0/0$. However, the magnitude of the multiplicity of the removable singularity is less for $\widehat{W}$ if for any γ_k, $|\gamma_k| \geq 1$.

Expression $\widehat{W}$ is canonical in cases 1 through 4, but not necessarily for case 5: For example,

1. The different equivalent expressions $z\left(z^2\right)^{-1/2}$ and $z^{-1}\left(z^2\right)^{1/2}$ both have outer exponents in (-1, 1), and the multiplicities of the uncanceled portion of their removable singularity at $z=0$ are both 1. Of these two alternatives, the latter is slightly preferable because it has a traditionally rationalized denominator rather than a rationalized numerator.

2. The different equivalent expressions $z^{-1}\left(z^2\right)^{2/3}$ and $z\left(z^2\right)^{-1/3}$ both have outer exponents in (-1, 1). Of these two alternatives, the latter is preferable for most purposes because the $|-2/3| < |4/3|$, making multiplicity of the uncanceled portion of the "removable" singularity have a smaller magnitude. Thus a rationalized numerator is sometimes preferable to a rationalized denominator.

3. The different expressions $z^{-3}\left(z^2\right)^{1/2}$ and $z^{-1}\left(z^2\right)^{-1/2}$ both have outer exponents in (-1, 1), and they are equivalent wherever the first alternative is defined. However, the latter is preferable because the former is indeterminate at $z=0$ where the latter is defined and equal to the complex infinity limit of the former.

Thus after phase 1 we can sometimes add 1 to a negative $\hat{\gamma}_k$ or subtract 1 from a positive $\hat{\gamma}_k$, then adjust α accordingly to reduce the magnitude of the overall removable singularity – perhaps entirely.

Let

$$\begin{aligned}\Delta_k &:= \beta_k\hat{\gamma}_k,\\ \Delta &:= \alpha+\Delta_1+\cdots+\Delta_n.\end{aligned}$$

Transforming any of the $\left(w^{\beta_k}\right)^{\gamma_k}$ to $w^{m_k\beta_k}\left(w^{\beta_k}\right)^{\gamma_k-m_k}$ for any integer m_k leaves Δ unchanged.

Our primary goal is, whenever possible, to make all of the Δ_k have the same sign and for α to have either the same sign or be 0.

A secondary goal is to prefer $-1/2 < \hat{\gamma}_k \le 1/2$.

Therefore, Phase 2 is:

1. If $\Delta > 0$, then for each $\Delta_k < 0$, add $\operatorname{sign}(\beta_k)$ to $\hat{\gamma}_k$ and subtract $|\beta_k|$ from α.

2. Otherwise if $\Delta < 0$, then for each $\Delta_k > 0$, subtract $\operatorname{sign}(\beta_k)$ from $\hat{\gamma}_k$ and add $|\beta_k|$ to α.

3. All of the $\hat{\gamma}_k$ are still in the interval (-1, 1), but now all Δ_k have the same sign. If α has the same sign as Δ, then return this result.

4. For each $\hat{\gamma} \le -1/2$, add 1 to $\hat{\gamma}_k$ and subtract β_k from α.

5. For each $\hat{\gamma}_k > 1/2$, subtract 1 from $\hat{\gamma}_k$ and add β_k to α.

The resulting form is a canonical form for expressions of the form $w^\alpha\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}$.

For strict adherence to this canonical form, some effort might be required either to prevent default simplification from combining two factors of the form $w^\alpha\left(w^\alpha\right)^\beta$ or to prevent reduction of the exponent to the standard interval when default simplification would undo the transformation, thus causing an infinite recursion. For example, default simplification might transform $x^2\left(x^2\right)^{1/2}$ to $\left(x^2\right)^{5/2}$. The form is canonical even if this exception to the standard exponent interval is allowed in results, and this exception is probably acceptable to most users.

However, if an expression also contains sub-expressions of the form either $(-1)^{\sigma}$ with σ being a canonically simplified sum of terms of the form (7), or a piecewise equivalent, then the differing canonical forms should be unified. For example,

$$\begin{aligned}
(-1)^{\left(\arg\left(z^2\right)/2-\arg z\right)/\pi}+\frac{\left(z^2\right)^{1/2}}{z} &\rightarrow \frac{2\left(z^2\right)^{1/2}}{z},\\
(-1)^{\left(\arg\left(z^2\right)/2-\arg z\right)/\pi}+\frac{\left(z^2\right)^{1/2}}{z} &\rightarrow 2(-1)^{\left(\arg\left(z^2\right)/2-\arg z\right)/\pi},\\
\frac{\left(w^2\right)^{1/2}}{w}-\begin{cases}1 & \text{if } \Re(w)>0 \vee \Re(w)\geq 0 \wedge \Im(w)\geq 0,\\ -1 & \text{otherwise;}\end{cases} &\rightarrow 0.
\end{aligned}$$

It is easier to transform from form 2 to form 1 than vice versa, but the rewrite rules in the Appendix don't address this issue.

2.3 Form 3: Also fully absorb w^{α} into a fractional power when possible.

Although Form 2 is canonical, it can result in an expression such as $z^4\left(z^2\right)^{1/2}$, for which many users would regard $\left(z^2\right)^{5/2}$ as a simpler result because it has one less factor.

We can often absorb at least some of z^{α} into one of the $\left(z^{\beta_k}\right)^{\gamma_k}$ by the transformation

$$z^{\alpha}\left(z^{\beta_k}\right)^{\gamma_k} \longrightarrow z^{\beta_k \operatorname{Fp}(\alpha/\beta_k)}\left(z^{\beta_{\mathbf{k}}}\right)^{\gamma_k+\operatorname{Ip}(\alpha/\beta_k)},$$

which doesn't change the domain of definition. However, this transformation seems inadvisable unless $\operatorname{Fp}(\alpha/\beta_k)=0$, because it increases the contribution of a troublesome nested power without reducing the number of factors.

Routine use of this transformation also seems inadvisable *during* intermediate computations even if $\operatorname{Fp}(\alpha/\beta_k)=0$, because when there is more than one nested power, then more than one might be eligible, making it awkward to maintain canonicality.

However, this transformation *does* seem advisable as a last step just before displaying a result.

When there is more than one nested power of w, then there might be more than one way to absorb α completely into those nested powers. For example,

$$\begin{aligned}
w^6\left(w^2\right)^{1/2}\left(w^3\right)^{1/2}\left(w^4\right)^{1/2} &\equiv \left(w^2\right)^{\mathbf{7/2}}\left(w^3\right)^{1/2}\left(w^4\right)^{1/2}\\
&\equiv \left(w^2\right)^{1/2}\left(w^3\right)^{\mathbf{5/2}}\left(w^4\right)^{1/2}\\
&\equiv \left(w^2\right)^{\mathbf{3/2}}\left(w^3\right)^{1/2}\left(w^4\right)^{\mathbf{3/2}}.
\end{aligned} \tag{19}$$

In general, the possible resulting expressions are the given by

$$\left(w^{\beta_1}\right)^{\gamma_1+m_1}\left(w^{\beta_2}\right)^{\gamma_2+m_2}\cdots\left(w^{\beta_n}\right)^{\gamma_n+m_n},$$

where the tuple of integers $\langle m_1, m_2, \ldots, m_n\rangle$ is a solution to the linear Diophantine equation

$$m_1\beta_1+m_2\beta_2+\cdots+m_n\beta_n=\alpha.$$

Solutions exist if and only if α is an integer multiple of $\gcd(\beta_1,\beta_2,\ldots\beta_n)$, in which case there might be a countably infinite number of tuples. However, to avoid introducing removable singularities or increasing the magnitude of their multiplicity, we are only interested in solutions for which

$\text{sign}(m_j\beta_j) \equiv \text{sign}(\alpha)$ for $j = 1, 2, \ldots, n$. Papp and Vizvari [4] describe an algorithm for such sign-constrained problems, and the *Mathematica* `Reduce`[...] function can solved such equations. For example, suppose our canonical form 2 result is

$$z^{14}\left(z^{6/7}\right)^{1/2}\left(z^{10/7}\right)^{1/3}.$$

In *Mathematica*, we can determine the family of integers $m_1 \geq 0$ to add to $1/2$ and $m_2 \geq 0$ to add to $1/3$ that together absorb z^{14} as follows:

$$\mathsf{In}[1]: \;=\; \texttt{Reduce}\left[\frac{6}{7}m_1 + \frac{10}{7}m_2 == 14\ \{m_1, m_2\}\ \&\&\ \frac{6}{7}m_1 \geq 0\ \&\&\ \frac{10}{7}m_2 \geq 0,\ \texttt{Integers}\right]$$
$$//\texttt{TraditionalForm}$$

$$\mathsf{Out}[1]//\text{TraditionalForm} = (m_1 = 3 \wedge m_2 = 8) \vee (m_1 = 8 \wedge m_2 = 5) \vee (m_1 = 13 \wedge m_2 = 2)$$

With more than one solution, we should choose one in a canonical way. One way to do so is to order the β_j in some canonical way, such as the way they order in $\left(w^{\beta_1}\right)^{\gamma_1} \cdots \left(w^{\beta_n}\right)^{\gamma_n}$, then to choose the solution for which m_1 is smallest, with ties broken according to which m_2 is smallest, etc.

The rewrite rules in the Appendix only partially implement this canonical form, because such simplifications should be part of the built-in optional simplification or, better yet, the default simplification: In reality, not many users are liable to learn about and routinely use an add-on package for this narrowly-scoped issue. Therefore the rewrite rules in the Appendix are only the minimal amount necessary to generate the results in Tables 1, 10 and – and hopefully inspire decision makers to build them into their systems.

However, these rules also return canonical form 3 for some but not all examples containing more than one fractional power of a power.

3 Extensions

Although not implemented in the rules of the Appendix, more generally the exponents for forms 1 through 4 can be Gaussian fractions or even symbolic, in which case we can still apply these transformations to the numeric real parts of the exponents. For example,

$$w^{3\xi+\rho}\left(w^{\xi}\right)^{3/2+\omega\pi i} \to \left(w^{\xi}\right)^{3} w^{\rho}\left(w^{\xi}\right)^{1+1/2+\omega\pi i} \to \left(w^{\xi}\right)^{4} w^{\rho}\left(w^{\xi}\right)^{1/2+\omega\pi i} \to w^{4\xi+\rho}\left(w^{\xi}\right)^{1/2+\omega\pi i}.$$

As another example, if a user has declared the variable n to be an integer,

$$w^{-n}\left(w^{2}\right)^{n+1/2} \to w^{n}\left(w^{2}\right)^{1/2}.$$

4 Summary

This article:

1. shows that many widely-used computer algebra systems have significant room for improvement at simplifying sub-expressions of the form $w^{\alpha}\left(w^{\beta_1}\right)^{\gamma_1} \cdots \left(w^{\beta_n}\right)^{\gamma_n}$;

2. defines three different simplified canonical forms with good properties;
3. explains how to compute these forms;
4. includes a partial implementation of one of these forms via *Mathematica* rewrite rules.

Acknowledgment

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References

[1] Brown, W.S: On computing with factored rational expressions. *Proceedings of EUROSAM '74, ACM SIGSAM Bulletin* 8 (3), pp. 26-34, 1974.

[2] Corless, R.M. and Jeffrey, D.J., Editor's corner: The unwinding number, *Communications in Computer Algebra* 30 (2), pp. 28-35, 1996.

[3] Moses, J: Algebraic simplification a guide for the perplexed. *Proceedings of the second ACM symposium on symbolic and algebraic manipulation*, pp. 282-304.

[4] Papp, D. and Vizvari, B: Effective solution of linear Diophantine equation systems with an application to chemistry, *Journal of Mathematical Chemistry* 39 (1), pp. 15-31, 2006.

[5] Rich, A.D. and Jeffrey, D.J., Function evaluation on branch cuts, *Communications in Computer Algebra* 30 (2), pp. 25-27, 1996.

[6] Stoutemyer, D.R., Ten commandments for good default expression simplification, *Journal of Symbolic Computation*, 46 (7), pp. 859-887, 2011.

Appendix: *Mathematica* rewrite rules for $w^{\alpha}\left(w^{\beta_1}\right)^{\gamma_1}\cdots\left(w^{\beta_n}\right)^{\gamma_n}$

```
   (* EXTRA SIMPLIFICATION DONE BEFORE ORDINARY EVALUATION: *)
PreProductOfPowersOfPowers [(w_^b_)^(g_Rational /; g <= -1 || g >= 1)] :=
   w^(IntegerPart[g]*b) * (w^b)^FractionalPart[g];

PreProductOfPowersOfPowers [(w_^b_)^(g_Rational /; g<=-1 || g>=1) * w_^a_. * u_]:=
  PreProductOfPowersOfPowers[w^(a+IntegerPart[g]*b) * (w^b)^FractionalPart[g] * u];

PreProductOfPowersOfPowers [(w_^b_)^g_ * w_^a_. * u_. /; Sign[a] != Sign[b*g] &&
    (Sign [a+b*Sign[g]] == Sign [b*(g-Sign[g])] ||
     Min [Abs[a], Abs[b*g]] > Min [Abs [a+b*Sign[g]], Abs [b*(g-Sign[g])]] ||
     g == -1/2 && Min [Abs[a], Abs[b/2]] == Min [Abs[a-b], Abs[b/2]])] :=
   w^(a+b*Sign[g]) * (w^b)^(g-Sign[g]) * u;

PreProductOfPowersOfPowers [f_[args__]] :=
   Apply [f, Map [PreProductOfPowersOfPowers, {args}]];
```

```
PreProductOfPowersOfPowers [anythingElse_] := anythingElse;
$Pre = PreProductOfPowersOfPowers;

   (* EXTRA SIMPLIFICATION DURING DURING ORDINARY EVALUATION: *)
Unprotect [Times];

(w_^b_)^g_ * w_^a_. * u_. /; Sign[a] != Sign[b*g] &&
    (Sign [a+b*Sign[g]] == Sign [b*(g-Sign[g])] ||
     Min [Abs[a], Abs[b*g]] > Min [Abs [a+b*Sign[g]], Abs [b*(g-Sign[g])]] ||
     Min [Abs[a], Abs[b*g]] == Min [Abs [a+b*Sign[g]], Abs [b*(g-Sign[g])]] &&
       Abs[g] > Abs [g-Sign[g]] ||
     g == -1/2 && Min [Abs[a], Abs[b/2]] == Min [Abs[a-b], Abs[b/2]]) :=
   w^(a+b*Sign[g]) * (w^b)^(g-Sign[g]) * u;

(w_^b1_)^g1_ * (w_^b2_)^g2_ * u_. /; Sign[b1*g1] != Sign[b2*g2] &&
     Abs[b2] > Abs[b1] && Sign [b2*(g2-Sign[g2])] == Sign [b1*g1 + b2*Sign[g2]] :=
   (w^b2)^(g2-Sign[g2]) * ((w^(b2*Sign[g2]) * (w^b1)^g1) * u);
Protect [Times];

   (* EXTRA SIMPLIFICATION DONE AFTER ORDINARY EVALUATION: *)
PostProductOfPowersOfPowers [w_^a_ * (w_^b_)^g_ * u_. /; IntegerQ [a/b]] :=
    PostProductOfPowersOfPowers [(w^b)^(g+a/b) * u];

PostProductOfPowersOfPowers [f_[args__]] :=
   Apply [f, Map [PostProductOfPowersOfPowers, {args}]];
PostProductOfPowersOfPowers [anythingElse_] := anythingElse;
$Post = PostProductOfPowersOfPowers;
```

Tables

15.

Table 1: Results of Appendix rewrite rules for 1st row $\times$ 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^7(w^2)^{1/3}}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$
$\frac{1}{w^2}$	$\frac{1}{(w^2)^{10/3}}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$
$\frac{1}{w}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$
1	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
w	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$
w^2	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$	$(w^2)^{11/3}$
w^3	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$	$w^7\,(w^2)^{2/3}$

Table 2: *Mathematica* 8 default simplify for 1st row $\times$ 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^3(w^2)^{7/3}}$	$\frac{1}{w^3(w^2)^{4/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{5/3}}{w^3}$	$\frac{(w^2)^{8/3}}{w^3}$
$\frac{1}{w^2}$	$\frac{1}{w^2(w^2)^{7/3}}$	$\frac{1}{w^2(w^2)^{4/3}}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^2}$	$\frac{(w^2)^{5/3}}{w^2}$	$\frac{(w^2)^{8/3}}{w^2}$
$\frac{1}{w}$	$\frac{1}{w(w^2)^{7/3}}$	$\frac{1}{w(w^2)^{4/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w}$	$\frac{(w^2)^{5/3}}{w}$	$\frac{(w^2)^{8/3}}{w}$
1	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
w	$\frac{w}{(w^2)^{7/3}}$	$\frac{w}{(w^2)^{4/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w\,(w^2)^{5/3}$	$w\,(w^2)^{8/3}$
w^2	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$	$(w^2)^{11/3}$
w^3	$\frac{w^3}{(w^2)^{7/3}}$	$\frac{w^3}{(w^2)^{4/3}}$	$\frac{w^3}{(w^2)^{1/3}}$	$w^3\,(w^2)^{2/3}$	$w^3\,(w^2)^{5/3}$	$w^3\,(w^2)^{8/3}$

Table 3: *Mathematica* 8 FullSimplify[...] for 1st row $\times$ 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{(w^2)^{2/3}}{w^9}$	$\frac{(w^2)^{2/3}}{w^7}$	$\frac{(w^2)^{2/3}}{w^5}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$
$\frac{1}{w^2}$	$\frac{(w^2)^{2/3}}{w^8}$	$\frac{(w^2)^{2/3}}{w^6}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$
$\frac{1}{w}$	$\frac{(w^2)^{2/3}}{w^7}$	$\frac{(w^2)^{2/3}}{w^5}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$
1	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
w	$\frac{w}{(w^2)^{7/3}}$	$\frac{w}{(w^2)^{4/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w\,(w^2)^{5/3}$	$w\,(w^2)^{8/3}$
w^2	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$	$(w^2)^{11/3}$
w^3	$\frac{w^3}{(w^2)^{7/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^3\,(w^2)^{5/3}$	$w^3\,(w^2)^{8/3}$

Table 4: Maple 13 & wxMaxima 0.8.2 default simplify for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^3(w^2)^{7/3}}$	$\frac{1}{w^3(w^2)^{4/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{5/3}}{w^3}$	$\frac{(w^2)^{8/3}}{w^3}$
$\frac{1}{w^2}$	$\frac{1}{w^2(w^2)^{7/3}}$	$\frac{1}{w^2(w^2)^{4/3}}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^2}$	$\frac{(w^2)^{5/3}}{w^2}$	$\frac{(w^2)^{8/3}}{w^2}$
$\frac{1}{w}$	$\frac{1}{w(w^2)^{7/3}}$	$\frac{1}{w(w^2)^{4/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w}$	$\frac{(w^2)^{5/3}}{w}$	$\frac{(w^2)^{8/3}}{w}$
1	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
w	$\frac{w}{(w^2)^{7/3}}$	$\frac{w}{(w^2)^{4/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w\,(w^2)^{5/3}$	$w\,(w^2)^{8/3}$
w^2	$\frac{w^2}{(w^2)^{7/3}}$	$\frac{w^2}{(w^2)^{4/3}}$	$\frac{w^2}{(w^2)^{1/3}}$	$w^2\,(w^2)^{2/3}$	$w^2\,(w^2)^{5/3}$	$w^2\,(w^2)^{8/3}$
w^3	$\frac{w^3}{(w^2)^{7/3}}$	$\frac{w^3}{(w^2)^{4/3}}$	$\frac{w^3}{(w^2)^{1/3}}$	$w^3\,(w^2)^{2/3}$	$w^3\,(w^2)^{5/3}$	$w^3\,(w^2)^{8/3}$

Table 5: Maple 13 simplify(...) for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^7(w^2)^{1/3}}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$
$\frac{1}{w^2}$	$\frac{1}{w^6(w^2)^{1/3}}$	$\frac{1}{w^4(w^2)^{1/3}}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^2}$	$(w^2)^{2/3}$	$w^2\,(w^2)^{2/3}$
$\frac{1}{w}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$
1	$\frac{1}{w^4(w^2)^{1/3}}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$w^2\,(w^2)^{2/3}$	$w^4\,(w^2)^{2/3}$
w	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$
w^2	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{1}{(w^2)^{1/3}}$	$\frac{w^2}{(w^2)^{1/3}}$	$w^2\,(w^2)^{2/3}$	$w^4\,(w^2)^{2/3}$	$w^6\,(w^2)^{2/3}$
w^3	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$\frac{w^3}{(w^2)^{1/3}}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$	$w^7\,(w^2)^{2/3}$

Table 6: *Derive* 6 default simplify for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{(w^2)^{2/3}}{w^9}$	$\frac{(w^2)^{2/3}}{w^7}$	$\frac{(w^2)^{2/3}}{w^5}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$
$\frac{1}{w^2}$	$\frac{1}{(w^2)^{10/3}}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$
$\frac{1}{w}$	$\frac{(w^2)^{2/3}}{w^7}$	$\frac{(w^2)^{2/3}}{w^5}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$
1	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
w	$\frac{(w^2)^{2/3}}{w^5}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$
w^2	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$	$(w^2)^{11/3}$
w^3	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$	$w^7\,(w^2)^{2/3}$

Table 7: TI-CAS default simplify for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^7(w^2)^{1/3}}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$
$\frac{1}{w^2}$	$\frac{1}{(w^2)^{10/3}}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$
$\frac{1}{w}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$
1	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
w	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$
w^2	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$	$(w^2)^{11/3}$
w^3	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$\frac{w^3}{(w^2)^{1/3}}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$	$w^7\,(w^2)^{2/3}$

Table 8: wxMaxima 0.8.2 fullratsimp(...) for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^7(w^2)^{1/3}}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^3}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$
$\frac{1}{w^2}$	$\frac{1}{w^6(w^2)^{1/3}}$	$\frac{1}{w^4(w^2)^{1/3}}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w^2}$	$(w^2)^{2/3}$	$w^2\,(w^2)^{2/3}$
$\frac{1}{w}$	$\frac{1}{w^5(w^2)^{1/3}}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{(w^2)^{2/3}}{w}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$
1	$\frac{1}{w^4(w^2)^{1/3}}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$w^2\,(w^2)^{2/3}$	$w^4\,(w^2)^{2/3}$
w	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$w\,(w^2)^{2/3}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$
w^2	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{1}{(w^2)^{1/3}}$	$\frac{w^2}{(w^2)^{1/3}}$	$w^2\,(w^2)^{2/3}$	$w^4\,(w^2)^{2/3}$	$w^6\,(w^2)^{2/3}$
w^3	$\frac{1}{w(w^2)^{1/3}}$	$\frac{w}{(w^2)^{1/3}}$	$\frac{w^3}{(w^2)^{1/3}}$	$w^3\,(w^2)^{2/3}$	$w^5\,(w^2)^{2/3}$	$w^7\,(w^2)^{2/3}$

Table 9: wxMaxima 0.8.2 rat(...) for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{(w^2)^{7/3}}$	$\frac{1}{(w^2)^{4/3}}$	$\frac{1}{(w^2)^{1/3}}$	$(w^2)^{2/3}$	$(w^2)^{5/3}$	$(w^2)^{8/3}$
$\frac{1}{w^3}$	$\frac{1}{w^3\left((w^2)^{1/3}\right)^7}$	$\frac{1}{w^3\left((w^2)^{1/3}\right)^4}$	$\frac{1}{w^3(w^2)^{1/3}}$	$\frac{\left((w^2)^{1/3}\right)^2}{w^3}$	$\frac{\left((w^2)^{1/3}\right)^5}{w^3}$	$\frac{\left((w^2)^{1/3}\right)^8}{w^3}$
$\frac{1}{w^2}$	$\frac{1}{w^2\left((w^2)^{1/3}\right)^7}$	$\frac{1}{w^2\left((w^2)^{1/3}\right)^4}$	$\frac{1}{w^2(w^2)^{1/3}}$	$\frac{\left((w^2)^{1/3}\right)^2}{w^2}$	$\frac{\left((w^2)^{1/3}\right)^5}{w^2}$	$\frac{\left((w^2)^{1/3}\right)^8}{w^2}$
$\frac{1}{w}$	$\frac{1}{w\left((w^2)^{1/3}\right)^7}$	$\frac{1}{w\left((w^2)^{1/3}\right)^4}$	$\frac{1}{w(w^2)^{1/3}}$	$\frac{\left((w^2)^{1/3}\right)^2}{w}$	$\frac{\left((w^2)^{1/3}\right)^5}{w}$	$\frac{\left((w^2)^{1/3}\right)^8}{w}$
1	$\frac{1}{\left((w^2)^{1/3}\right)^7}$	$\frac{1}{\left((w^2)^{1/3}\right)^4}$	$\frac{1}{(w^2)^{1/3}}$	$\left((w^2)^{1/3}\right)^2$	$\left((w^2)^{1/3}\right)^5$	$\left((w^2)^{1/3}\right)^8$
w	$\frac{w}{\left((w^2)^{1/3}\right)^7}$	$\frac{w}{\left((w^2)^{1/3}\right)^4}$	$\frac{w}{(w^2)^{1/3}}$	$w\left((w^2)^{1/3}\right)^2$	$w\left((w^2)^{1/3}\right)^5$	$w\left((w^2)^{1/3}\right)^8$
w^2	$\frac{w^2}{\left((w^2)^{1/3}\right)^7}$	$\frac{w^2}{\left((w^2)^{1/3}\right)^4}$	$\frac{w^2}{(w^2)^{1/3}}$	$w^2\left((w^2)^{1/3}\right)^2$	$w^2\left((w^2)^{1/3}\right)^5$	$w^2\left((w^2)^{1/3}\right)^8$
w^3	$\frac{w^3}{\left((w^2)^{1/3}\right)^7}$	$\frac{w^3}{\left((w^2)^{1/3}\right)^4}$	$\frac{w^3}{(w^2)^{1/3}}$	$w^3\left((w^2)^{1/3}\right)^2$	$w^3\left((w^2)^{1/3}\right)^5$	$w^3\left((w^2)^{1/3}\right)^8$

Table 10: Simplification of $w/\sqrt{w^2}$

system	function	$\frac{w}{\sqrt{w^2}}$
	Appendix rewrite rules	$\frac{\sqrt{w^2}}{w}$
Mathematica	default	$\frac{w}{\sqrt{w^2}}$
Mathematica	FullSimplify	$\frac{w}{\sqrt{w^2}}$
Maple	default	$\frac{w}{\sqrt{w^2}}$
Maple	simplify	$\operatorname{csgn}(w)$
Derive	default	$\frac{\sqrt{w^2}}{w}$
TI-CAS	default	$\frac{w}{\sqrt{w^2}}$
wxMaxima	default	$\frac{w}{\sqrt{w^2}}$
wxMaxima	fullratsimp	$\frac{w}{\sqrt{w^2}}$
wxMaxima	rat	$\frac{w}{\sqrt{w^2}}$

Table 11: Appendix rewrite rules for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
$\frac{1}{w^3}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^5}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^7}$
$\frac{1}{w^2}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$	$\left(\frac{1}{w^2}\right)^{7/2}$
$\frac{1}{w}$	$\frac{w^3}{\sqrt{\frac{1}{w^2}}}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^5}$
1	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
w	$\frac{w^5}{\sqrt{\frac{1}{w^2}}}$	$\frac{w^3}{\sqrt{\frac{1}{w^2}}}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$
w^2	$\frac{1}{\left(\frac{1}{w^2}\right)^{7/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$
w^3	$\frac{w^7}{\sqrt{\frac{1}{w^2}}}$	$\frac{w^5}{\sqrt{\frac{1}{w^2}}}$	$\frac{w^3}{\sqrt{\frac{1}{w^2}}}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$

Table 12: *Mathematica* 8 default simplify for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
$\frac{1}{w^3}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}w^3}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}w^3}$	$\frac{1}{\sqrt{\frac{1}{w^2}}w^3}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w^3}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w^3}$
$\frac{1}{w^2}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$	$\left(\frac{1}{w^2}\right)^{7/2}$
$\frac{1}{w}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}w}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}w}$	$\frac{1}{\sqrt{\frac{1}{w^2}}w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w}$
1	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
w	$\frac{w}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w$	$\left(\frac{1}{w^2}\right)^{3/2}w$	$\left(\frac{1}{w^2}\right)^{5/2}w$
w^2	$\frac{w^2}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w^2}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w^2}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w^2$	$\left(\frac{1}{w^2}\right)^{3/2}w^2$	$\left(\frac{1}{w^2}\right)^{5/2}w^2$
w^3	$\frac{w^3}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w^3}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w^3}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w^3$	$\left(\frac{1}{w^2}\right)^{3/2}w^3$	$\left(\frac{1}{w^2}\right)^{5/2}w^3$

Table 13: *Mathematica* 8 FullSimplify[...] for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
$\frac{1}{w^3}$	$\sqrt{\frac{1}{w^2}}w^3$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w^3}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w^3}$
$\frac{1}{w^2}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$	$\left(\frac{1}{w^2}\right)^{7/2}$
$\frac{1}{w}$	$\sqrt{\frac{1}{w^2}}w^5$	$\sqrt{\frac{1}{w^2}}w^3$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w}$
1	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
w	$\frac{w}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}w$	$\left(\frac{1}{w^2}\right)^{3/2}w$	$\left(\frac{1}{w^2}\right)^{5/2}w$
w^2	$\sqrt{\frac{1}{w^2}}w^8$	$\sqrt{\frac{1}{w^2}}w^6$	$\sqrt{\frac{1}{w^2}}w^4$	$\sqrt{\frac{1}{w^2}}w^2$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$
w^3	$\sqrt{\frac{1}{w^2}}w^9$	$\sqrt{\frac{1}{w^2}}w^7$	$\sqrt{\frac{1}{w^2}}w^5$	$\sqrt{\frac{1}{w^2}}w^3$	$\sqrt{\frac{1}{w^2}}w$	$\left(\frac{1}{w^2}\right)^{5/2}w^3$

Table 14: TI-CAS default, Maple simplify(...) & default for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
$\frac{1}{w^3}$	$\frac{1}{w^3\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{w^3\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{w^3\sqrt{\frac{1}{w^2}}}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w^3}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w^3}$
$\frac{1}{w^2}$	$\frac{1}{w^2\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{w^2\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{w^2\sqrt{\frac{1}{w^2}}}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^2}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w^2}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w^2}$
$\frac{1}{w}$	$\frac{1}{w\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{w\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{w\sqrt{\frac{1}{w^2}}}$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\left(\frac{1}{w^2}\right)^{3/2}}{w}$	$\frac{\left(\frac{1}{w^2}\right)^{5/2}}{w}$
1	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
w	$\frac{w}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w}{\sqrt{\frac{1}{w^2}}}$	$w\sqrt{\frac{1}{w^2}}$	$w\left(\frac{1}{w^2}\right)^{3/2}$	$w\left(\frac{1}{w^2}\right)^{5/2}$
w^2	$\frac{w^2}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w^2}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w^2}{\sqrt{\frac{1}{w^2}}}$	$w^2\sqrt{\frac{1}{w^2}}$	$w^2\left(\frac{1}{w^2}\right)^{3/2}$	$w^2\left(\frac{1}{w^2}\right)^{5/2}$
w^3	$\frac{w^3}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{w^3}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{w^3}{\sqrt{\frac{1}{w^2}}}$	$w^3\sqrt{\frac{1}{w^2}}$	$w^3\left(\frac{1}{w^2}\right)^{3/2}$	$w^3\left(\frac{1}{w^2}\right)^{5/2}$

Table 15: *Derive* default simplify for 1st row × 1st column

$\downarrow\overrightarrow{\times}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
$\frac{1}{w^3}$	$w^3\sqrt{\frac{1}{w^2}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^5}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^7}$
$\frac{1}{w^2}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$	$\left(\frac{1}{w^2}\right)^{7/2}$
$\frac{1}{w}$	$w^5\sqrt{\frac{1}{w^2}}$	$w^3\sqrt{\frac{1}{w^2}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^5}$
1	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$	$\left(\frac{1}{w^2}\right)^{5/2}$
w	$w^7\sqrt{\frac{1}{w^2}}$	$w^5\sqrt{\frac{1}{w^2}}$	$w^3\sqrt{\frac{1}{w^2}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$	$\frac{\sqrt{\frac{1}{w^2}}}{w^3}$
w^2	$\frac{1}{\left(\frac{1}{w^2}\right)^{7/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{5/2}}$	$\frac{1}{\left(\frac{1}{w^2}\right)^{3/2}}$	$\frac{1}{\sqrt{\frac{1}{w^2}}}$	$\sqrt{\frac{1}{w^2}}$	$\left(\frac{1}{w^2}\right)^{3/2}$
w^3	$w^9\sqrt{\frac{1}{w^2}}$	$w^7\sqrt{\frac{1}{w^2}}$	$w^5\sqrt{\frac{1}{w^2}}$	$w^3\sqrt{\frac{1}{w^2}}$	$\sqrt{\frac{1}{w^2}}w$	$\frac{\sqrt{\frac{1}{w^2}}}{w}$