Contributions to Constructive Polynomial Ideal Theory XVIII: The Proof of Equality, Ideal Membership and Minimal Length Bases using Gjunter Bases *

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Dedicated to Professor Wilhelm Hauser on the occasion of his 100th birthday who, through his book [6] gave much encouragement for this series of articles.

Abstract

In this paper an algorithm is given for obtaining a characteristic basis (a *Gjunter basis*) for any H-ideal. With such a basis, we can decide the validity of $\mathfrak{a} = \mathfrak{b}$ and $F \in \mathfrak{a}$ for H-ideals, and $(\mathfrak{a}) = (\mathfrak{b})$ and $f \in (\mathfrak{a})$ for P-ideals. In many cases, computing the chain of syzygies is simplified by using Gjunter bases. We also show how to compute a minimal length basis for any P-ideal $(\mathfrak{a})$.

1 Preface: Goals, Relationships with Systems of Linear Equations

While systematically compiling the results thus far for this series of articles (which first appeared in volume **17** (1973) of this journal), the author noticed in [10, 3.2], that for effectively proving the equality of H-ideals, and hence the accompanying ideal membership $F \in \mathfrak{a}$, the method used by the author had not yet been published. This will now be rectified, which will also satisfy a claim by Krull in [8, p. 51]. The basic idea (of changing over to a characteristic basis), which goes back to Gjunter [5], is also useful for constructing minimal length bases of P-ideals from their associated equivalent H-ideals.

Let us first illustrate the basic idea mentioned above for the special case of homogeneous systems of linear equations. Here we call two systems of linear equations *equivalent* if they have the same solution (see [4, 5.1, Definition 4, p. 47] for example). However, with different parameter settings, this definition is less suitable for actually proving equivalence; determining the compete solution is needed here. Without this, the equivalence of systems of linear equations can be characterized by converting one system to the other and conversely (see [3, I.6, p. 29]) using module operations. In order to actually prove equivalence in this manner, it is advantageous to specify one characteristic system of equations among all equivalent systems as a representative of the class of equivalent systems of equations. We have in mind here the *trapezoidal form* of a linear system of equations, which however, just by changing to *diagonal form*, is unique (up to constant factors, thus without loss of generality, after scaling). Therefore, the representative system of equations has the form

$$x_i + b_{ir}x_r + \ldots + b_{in}x_n = 0 \qquad (i = 0, 1, \ldots, r-1). \tag{1}$$

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Then by replacing $x_{r+i} = t_i$ ($i = 0, \ldots, d$ with $d = n - r$), a complete solution is already given by (1) in which the last $d + 1 = n - r + 1$ variables $x_r, \ldots, x_n$ are set as parameters. But in order to achieve this, a permutation of the variables $x_0, \ldots, x_n$ may be necessary, which has never been explained in the literature as clearly as in [7, p. 340, left column, final theorem of VI.2].

For the nonlinear case, since a corresponding reordering of the power products may not always be achieved by variable transformations, knowing which normal forms in the linear case are achievable without permuting the variables, which corresponds to not setting the parameters in the last $d + 1$ variables, is of interest when changing over to the nonlinear case. For example, only after scaling, do we obtain the uniquely determined system of equations of the form

$$\left.\begin{array}{l} x_0 - x_2 - 3x_6 - x_7 + 7x_{12} - 8x_{13} + x_{14} = 0 \\ x_1 + x_2 + 4x_6 - 3x_7 - 2x_{12} + x_{13} - x_{14} = 0 \\ x_3 - x_6 + 2x_7 + x_{12} - 2x_{13} + 3x_{14} = 0 \\ x_4 = 0 \\ x_5 - 3x_6 + x_7 + 2x_{13} - x_{14} = 0 \\ x_8 - x_{12} + 2x_{14} = 0 \\ x_9 + 7x_{12} - 8x_{13} + 9x_{14} = 0 \\ x_{10} = 0 \\ x_{11} - x_{12} - x_{14} = 0 \end{array}\right\} \quad (2)$$

or of the form

$$\left.\begin{array}{l} x_0 - x_2 - 3x_6 - x_7 + 7x_{12} - 8x_{13} + x_{14} = 0 \\ \vdots \\ x_{11} - x_{12} - x_{14} = 0 \\ x_{15} = 0 \\ x_{16} = 0 \end{array}\right\}. \quad (3)$$

Thus in general, the transformed coefficient matrix (i.e. the $n + 1$ columns corresponding to the $n + 1$ variables in the sequence $x_0, x_1, \ldots, x_n$) has the form

$$\begin{pmatrix} D_1 & B_{11} & 0 & B_{12} & 0 & B_{13} & 0 & B_{14} & \ldots & 0 & B_{1m} \\ 0 & 0 & D_2 & B_{22} & 0 & B_{23} & 0 & B_{24} & \ldots & 0 & B_{2m} \\ 0 & 0 & 0 & 0 & D_3 & B_{33} & 0 & B_{34} & \ldots & 0 & B_{3m} \\ 0 & 0 & 0 & 0 & 0 & 0 & D_4 & B_{44} & \ldots & 0 & B_{4m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & D_m & B_{mm} \end{pmatrix}, \quad (4)$$

where the blocks $D_i \neq 0$ are diagonal matrices and for fixed $k \in \{1, \ldots, m-1\}$, the block matrices B_{ik} are non-zero for at least one $i \in \{1, \ldots, k\}$. For type (2), this also holds for B_{im}, while for type (3), all $B_{im} = 0$, and hence may be omitted.

We designate type (4) as the *pseudo-diagonal form*, and therefore after scaling, we have obtained a characteristic system for equivalent linear systems of equations; that is, two systems of linear equations are *equivalent* if they have the same scaled pseudo-diagonal form.

We can characterize pseudo-diagonal forms more briefly using the following two properties:

(PD1)* The first variables on the left hand sides of equations in the system of equations are all distinct and arranged in natural order.

(PD2)* No second, third, ... variable appears in a later position than the first variable.

We will now carry over these facts to K-modules of forms of the same degree, and in particular, to the module $\mathfrak{M}(t; \mathfrak{a})$ of the H-ideal $\mathfrak{a}$.

2 Pseudo-Diagonal Bases for Modules of Forms

We now consider forms of degree t in $x_0, x_1, \ldots, x_n$, thus linear combinations of the $\binom{t+n}{n}$ power products of degree t in $x_0, x_1, \ldots, x_n$. Then we can choose the basis forms of this module analogous to (4), whereby the $\binom{t+n}{n}$ columns of the transformed coefficient matrix now correspond to the $\binom{t+n}{n}$ power products in lexicographic order. Analogous to (PD1)* and (PD2)*, we can now interpret (4) using

Definition 1. A set $(H_1, \ldots, H_k)$ of forms $H_1, \ldots, H_k$ of the same degree t in $K[x_0, x_1, \ldots, x_n]$ is called a *pseudo-diagonal basis* (*PD-basis*) of the K-module generated by $H_1, \ldots, H_k$ if

(PD1) The leading power products of the forms $H_1, \ldots, H_k$ are all distinct and are arranged in lexicographic order.

(PD2) No second, third, ... power product (in lexicographic order) appears in a later position than the leading power product.

For $t = 1$, we again obtain (PD1)* and (PD2)* from this. As in the linear case, we have for arbitrary t

Theorem 1. *The PD-basis of a K-module of forms is uniquely determined up to a common factor in K.*

Thus in particular, we have found a standard basis for the module $\mathfrak{M}(t; \mathfrak{a})$ of forms of degree t of an H-ideal $\mathfrak{a} \subset K[x_0, x_1, \ldots, x_n]$, and just as in the special case, the basis forms of $\mathfrak{a}$ all have the same degree. Now our goal is to carry over these observations to the case of basis forms of $\mathfrak{a}$ with different degrees.

3 The Gjunter Basis of an H-Ideal

We will order the basis forms of a minimal basis of the H-ideal $\mathfrak{a} \subset K[x_0, x_1, \ldots, x_n]$ by increasing degrees and denote them using double indices, i.e. $h(F_{t,i}) := \deg(F_{t,i}) = t$. If m_0 is the minimal degree and M the maximal degree of the basis forms, then we can express the minimal basis of $\mathfrak{a}$ in the form

$$\mathfrak{a} = \left(F_{m_0,1}, \ldots, F_{t,1}, \ldots, F_{t,s_t}, F_{t+1,1}, \ldots, F_{t+1,s_{t+1}}, \ldots, F_M, s_M\right), \tag{5}$$

where we can omit any $F_{g,i}$ for which the degree g does not appear. For $t = m_0 + 1, \ldots, M$, we have in each case

$$\mathfrak{M}(t; \mathfrak{a}) \quad = \quad (x_0, x_1, \ldots, x_n)\, \mathfrak{M}(t-1; \mathfrak{a}) \, \cup \, (F_{t,1}, \ldots, F_{t,s_t}). \tag{6}$$

Our goal is to obtain a basis $(G_{m_0,1}, \ldots, G_{M,s_M})$ for $\mathfrak{a}$ from (5), in which the $G_{m_0,1}, \ldots, G_{M,s_M}$ are defined so that while forming the module (6) with additional basis forms, no leading power product already present in $(x_0, x_1, \ldots, x_n)\, \mathfrak{M}(t-1; \mathfrak{a})$ appears and otherwise, power products of as high a number as possible (according to the lexicographic order) are used. Thus we define four transformation rules (GJ1), (GJ2), (GJ3), (GJ4):

(GJ1) Replace $F_{m_0,1}, \ldots, F_{m_0,s_{m_0}}$ by $G_{m_0,1}, \ldots, G_{m_0,s_{m_0}}$, where $\left(G_{m_0,1}, \ldots, G_{m_0,s_{m_0}}\right)$ is a PD-basis of the K-module generated by $F_{m_0,1}, \ldots, F_{m_0,s_{m_0}}$ of forms of minimal degree m_0.

(GJ2) Replace $F_{t,1}, \ldots, F_{t,s_t}$ by $F^{(2)}_{t,1}, \ldots, F^{(2)}_{t,s_t}$ for all t such that $m_0 + 1 \leq t \leq M$, where $\left(F^{(2)}_{t,1}, \ldots, F^{(2)}_{t,s_t}\right)$ is a PD-basis of the K-module generated by $F_{t,1}, \ldots, F_{t,s_t}$ of forms of degree t.

(GJ3) Replace $F^{(2)}_{t,1}, \ldots, F^{(2)}_{t,s_t}$ by $F^{(3)}_{t,1}, \ldots, F^{(3)}_{t,s_t}$ for all t such that $m_0 + 1 \leq t \leq M$, in such a way that no leading power product of the PD-basis of $(x_0, x_1, \ldots, x_n)\, \mathfrak{M}(t-1; \mathfrak{a})$ appears.

(GJ4) Replace $F^{(3)}_{t,1}, \ldots, F^{(3)}_{t,s_t}$ by $G_{t,1}, \ldots, G_{t,s_t}$ for all t such that $m_0 + 1 \leq t \leq M$, where $(G_{t,1}, \ldots, G_{t,s_t})$ is a PD-basis of the K-module generated by $F^{(3)}_{t,1}, \ldots, F^{(3)}_{t,s_t}$ of forms of degree t.

Remarks.

(a) In general, the transformation (GJ4) is by no means redundant because the PD property originally created by (GJ2) can vanish under the transformation (GJ3).

(b) Let $(H_1, \ldots, H_k)$ be a PD-basis of $(x_0, x_1, \ldots, x_n)\,\mathfrak{M}(t-1;\mathfrak{a})$. Then it is not true in general that $(H_1, \ldots, H_k, G_{t,1}, \ldots, G_{t,s_t})$ is a PD-basis of $\mathfrak{M}(t-1;\mathfrak{a})$; thus (GJ3) and (GJ4) cannot be replaced by a corresponding claim. In the following example, we consider $G_{1,1} = x_1 + x_3$ with $x_0^2 G_{1,1} = x_0^2 x_1 + x_0^2 x_3 \in (x_0, x_1, x_2, x_3)\,\mathfrak{M}(2;\mathfrak{a})$ and $G_{3,1} = x_0^2 x_3$. This simple example shows the obvious reason for this restrictive remark: basis forms of lower degree influence those of higher degree, but not conversely!

(c) The transformation (GJ2) is not essential, but does simplify transformations (GJ3) and (GJ4) significantly, and even makes (GJ4) completely redundant in many cases.

(d) If (5) is not a minimal basis, (GJ1), (GJ2), (GJ3) and (GJ4) still lead to a minimal basis.

Definition 2. The basis representation

$$\mathfrak{a} = \left(G_{m_0,1}, \ldots, G_{t,1}, \ldots, G_{t,s_t}, G_{t+1,1}, \ldots, G_{t+1,s_{t+1}}, \ldots, G_{M,s_M}\right) \tag{7}$$

of an H-ideal $\mathfrak{a} \subset K[x_0, x_1, \ldots, x_n]$ obtained from (5) using (GJ1), (GJ2), (GJ3) and (GJ4) (where $h(G_{t,i}) = t$, and any $G_{g,i}$ for which the degree g does not appear is omitted) is called a *Gjunter basis* (*GJ-basis*) of the H-ideal $\mathfrak{a}$.

Then as the main result analogous to Theorem 1, we have

Theorem 2. *The elements $G_{h,i}$ of the Gjunter basis* (7) *of an H-ideal $\mathfrak{a} \subset K[x_0, x_1, \ldots, x_n]$ are uniquely determined up to a common factor in K.*

Before drawing some conclusions from this theorem, we consider an

Example. Let

$$\left.\begin{array}{c} \mathfrak{a}^* = (F_{1,1}, F_{2,1}, F_{2,2}, F_{3,1}), \quad \text{where} \\ F_{1,1} = x_1 + x_3, \qquad F_{2,1} = x_0 x_2 + x_1^2, \qquad F_{2,2} = x_1 x_2, \qquad F_{3,1} = x_0^2 x_1 + x_1^3. \end{array}\right\} \tag{8}$$

Here, $m_0 = 1$ and $M = 3$. (GJ1) does not apply because $\mathfrak{M}(1;\mathfrak{a}) = (x_1 + x_3)$. Hence $G_{1,1} = x_1 + x_3$. Similarly, (GJ2) does not apply because the power products of $F_{2,1}$ and $F_{2,2}$ are all distinct. (GJ3) produces $F_{2,1}^{(3)} = F_{2,1} - (x_1 - x_3)G_{1,1} = x_0 x_2 + x_3^2$ and $F_{2,2}^{(3)} = x_2 G_{1,1} - F_{2,2} = x_2 x_3$ for $t = 2$ and

$$F_{3,1}^{(3)} \;=\; (x_0^2 + x_1^2 - x_1 x_3 + x_3^2)G_{1,1} - x_3 F_{2,1}^{(3)} + x_0 F_{2,2}^{(3)} - F_{3,1} \;=\; x_0^2 x_3$$

for $t = 3$. (GJ4) does not apply; therefore, the GJ-basis for (8) is

$$\left.\begin{array}{c} \mathfrak{a}^* = (G_{1,1}, G_{2,1}, G_{2,2}, G_{3,1}), \quad \text{where} \\ G_{1,1} = x_1 + x_3, \qquad G_{2,1} = x_0 x_2 + x_3^2, \qquad G_{2,2} = x_2 x_3, \qquad G_{3,1} = x_0^2 x_3. \end{array}\right\} \tag{9}$$

4 Applications of GJ-Bases

The advantage of going from (8) to (9) in this example is evident through the sharp decrease in new forms. This will substantially simplify the formation of the module $\mathfrak{M}(t;\mathfrak{a})$ and the computation of the second syzygy module (see [13, 5.6]). We illustrate this in our example. If we start from the basis representation (8), then we have the three new forms

$$\begin{array}{rcccc} V_1 & = & x_2F_{1,1} - F_{2,2} & = & x_2x_3, \\ V_2 & = & x_0^2F_{1,1} - F_{3,1} & = & x_0^2x_3 - x_1^3, \\ V_3 & = & (x_1^2 - x_1x_3 + x_3^2)F_{1,1} - x_1F_{2,1} + x_0F_{2,2} & = & x_3^2. \end{array}$$

On the other hand, we have only one new form $V_1 = x_3G_{2,1} - x_0G_{2,2} = x_3^3$ when using (9). This example also shows that introducing the GJ-basis cannot prevent new forms up to the maximal degree M of the basis forms from appearing; this can also occur in rational prime ideals (the *Abhyankar ideal* $\mathfrak{p}_{70}$ with generic zero $(t_0^6,\ t_0^5t_1,\ t_0^3t_1^3 + t_0t_1^5,\ t_1^6)$ is an example, see [12]). Still, we do have as a result the

Theorem 3. *The number of new forms is reduced by introducing Gjunter bases.*

The introduction of GJ-bases can certainly increase the number of terms in the basis forms, causing Gauss' algorithm for computing syzygies to require more steps, but this is negligible when using computers.

Practical provable criteria for equality and ideal membership are important consequences[1] that result from Theorem 2, analogous to systems of linear equations:

Theorem 4. *Two H-ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *in* $K[x_0, x_1, \ldots, x_n]$ *are equal if and only if they have the same Gunter bases. Two P-ideals* $(\mathfrak{a})$ *and* $(\mathfrak{b})$ *in* $K[x_1, \ldots, x_n]$ *are equal if and only if the equivalent H-ideals are equal.*

Theorem 5. *The validity or invalidity of* $F \in \mathfrak{a}$ *can be proved in practice using* $F \in \mathfrak{a} \iff (\mathfrak{a}, F) = \mathfrak{a}$ *and Theorem 4; the same holds for* $f \in (\mathfrak{a})$ *for P-ideals* $(\mathfrak{a})$.

Finally, we examine the problem posed in [10, section 7] of constructing minimal length bases for an inhomogeneous P-ideal $(\mathfrak{a}) \subset K[x_1, \ldots, x_n]$. Therefore, let

$$(\mathfrak{a}) \;=\; (f_1, \ldots, f_t) \tag{10}$$

be a basis representation of $(\mathfrak{a})$, where $f_1, \ldots, f_t$ is a minimal length basis. If F_i is the form resulting from f_i under homogenization, then it is well-known (see [9, 5.12, (67)]) that as the final member $\mathfrak{a}_k$ of the divisor chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$, where $\mathfrak{a}_{i+1} = \mathfrak{a}_i : (x_0)$ and $\mathfrak{a}_1 := (F_1, \ldots, F_t)$, the equivalent H-ideal is determined by

$$\mathfrak{a} \;=\; \mathfrak{a}_k \;=\; \mathfrak{a}_1 : (x_0^k) \;=\; (F_1, \ldots, F_t, F_{t+1}, \ldots, F_s). \tag{11}$$

Now if a minimal basis is given by (11), then (10) can also be derived from (11) by considering syzygies with coordinates of the form $ax_0^g, bx_0^h, \ldots$. However, it can happen for $\deg F_{t+i} \le M$, that F_{t+i} is exchanged with one of the forms $F_1, \ldots, F_t$, resulting in a minimal length basis for $(\mathfrak{a})$ with smaller degree. Eliminating more than one of the forms $F_1, \ldots, F_t$ is not possible since $f_1, \ldots, f_t$ was assumed to be a minimal length basis.

Conversely, if we have computed the equivalent H-ideal $\mathfrak{a}$, then for a suitable choice of H-basis using (11), $\mathfrak{a}$ results from an H-ideal $\mathfrak{a}_1$, where $(\mathfrak{a}_1)_{x_0=0} = (f_1, \ldots, f_t)$ is represented by a minimal length basis $f_1, \ldots, f_t$. Thus we have

Theorem 6. *A minimal length basis for a given P-ideal* $(\mathfrak{a}) \subset K[x_1, \ldots, x_n]$ *can be computed with a suitable choice of basis for the H-ideal* $\mathfrak{a} \subset K[x_0, x_1, \ldots, x_n]$ *equivalent to* $(\mathfrak{a})$ *by considering terms* ax_0^g *in the second syzygy module of* $\mathfrak{a}$.

[1] In all of these consequences of Theorem 2, only the uniqueness of Gjunter bases was used; thus they also follow from other uniquely determined normal forms. One such was given by Buchberger [1], with its uniqueness proof given in [2, Theorem 3.6]; see also Trinks [14, p. 476]. I thank Professors Roquette (Heidelberg) and Leopoldt (Karlsruhe) for the approporiate references during my 1978 academic visit to the Banach Center in Warsaw. The numbering of power products used by the author in [9] can also be found in the papers of Buchberger and Trinks.

The author has been unable to prove that the Gjunter basis of $\mathfrak{a}$ represents just such a "suitable basis". Further computations using our example following Theorem 2 shows that such a conjecture seems likely.[2] For the second syzygy module of $\mathfrak{a}^*$ we obtain for each choice of basis:

$$(F_{11}, F_{21}, F_{22}, F_{31}),$$

$$\begin{pmatrix} x_0x_2 + x_1^2 & x_1x_2 & x_0^2x_1 & x_0x_2^2 + x_1x_2x_3 & x_0^3x_2 + x_0^2x_1x_3 & 0 \\ -x_1 - x_3 & 0 & x_1^2 + x_1x_3 & -x_2x_3 & -x_0^2x_3 + x_0x_1x_2 + x_1x_3^2 & x_1x_2 \\ 0 & -x_1 - x_3 & -x_0x_1 - x_0x_3 & -x_0x_2 - x_3^2 & -x_0^2x_2 - x_0x_3^2 & x_0^2 - x_0x_2 \\ 0 & 0 & -x_1 - x_3 & 0 & -x_0x_2 - x_3^2 & -x_2 \end{pmatrix}$$

and

$$(G_{11}, G_{21}, G_{22}, G_{31}),$$

$$\begin{pmatrix} x_0x_2 + x_3^2 & x_2x_3 & x_0^2x_3 & 0 & 0 & 0 \\ -x_1 - x_3 & 0 & 0 & x_2x_3 & x_0^2x_3 & 0 \\ 0 & -x_1 - x_3 & 0 & -x_0x_2 - x_3^2 & 0 & x_0^2 \\ 0 & 0 & -x_1 - x_3 & 0 & -x_0x_2 - x_3^2 & -x_2 \end{pmatrix}$$

If the Gjunter basis for $\mathfrak{a}^*$ is chosen, then a syzygy with coordinate x_0^{-2} appears, but not for the minimal basis $F_{11}, F_{21}, F_{22}, F_{31}$. If we set $f_{ik} := (F_{ik})_{x_0=1}$ and $g_{ik} := (G_{ik})_{x_0=1}$, then $f_{11}, f_{21}, f_{22}, f_{31}$ is a minimal basis for $(\mathfrak{a}) := (\mathfrak{a}^*)_{x_0=1}$, but not a minimal length basis, since the element g_{22} can be omitted from the equivalent basis $g_{11}, g_{21}, g_{22}, g_{31}$. Then $(\mathfrak{a}) = (g_{11}, g_{21}, g_{31}) = (x_1 + x_3, x_2 + x_3^2, x_3)$ and g_{11}, g_{21}, g_{31} is a minimal length basis, so a simple transformation leads to the P-ideal of the principal class $(x_1, x_2, x_3) = (\mathfrak{a})$. Thus here, the GJ-basis of $\mathfrak{a}^*$ leads to a minimal length basis, making the formation of the equivalent H-ideal using (11) unnecessary.

We share the results of such a computation anyway:

$$\begin{aligned} (G_{11}, G_{21}, G_{22}, G_{31}) : (x_0) &= (x_1 + x_3,\ x_0x_2 + x_3^2,\ x_0x_3,\ x_2^2,\ x_2x_3) \\ (F_{11}, F_{21}, F_{22}, F_{31}) : (x_0) &= (x_1 + x_3,\ x_0x_1 + x_1x_2,\ x_0x_2 + x_1^2,\ x_1x_2,\ x_2^2) \\ &= (x_1 + x_3,\ x_0x_1,\ x_0x_2^2 + x_1^2,\ x_1x_2,\ x_2^2) \\ \mathfrak{a}^* : (x_0^2) &= (x_1,\ x_3,\ x_0x_2,\ x_2^2) \quad \text{for both basis representations} \end{aligned}$$

and finally

$$\mathfrak{a} \;=\; \mathfrak{a}^* : (x_0^2) \;=\; (x_1, x_2, x_3).$$

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