Some definite integrals containing the Tree T function

Robert M. Corless, Junrui Hu and D. J. Jeffrey
Department of Applied Mathematics, Western University, London, Canada
{jhu93,djeffrey}@uwo.ca

Abstract

We take up a numerical challenge regarding definite integrals, recently considered by Walter Gautschi. We show that symbolic computation can contribute to the evaluation of many special cases of the integrals.

1 Introduction

The paper [3] examines, in effect, numerical schemes for the evaluation of the integrals

$$I_0(\alpha,\beta) = \int_1^\infty T_0\left(xe^{-x}\right)^\alpha x^{-\beta}\,dx \tag{1}$$

and

$$I_1(\alpha,\beta) = \int_0^1 T_1\left(xe^{-x}\right)^\alpha x^{-\beta}\,dx\ . \tag{2}$$

Here, T_k is the Tree T function, satisfying $T(z)\exp(-T(z)) = z$. Also, α and β are restricted to values ensuring convergence[1]. The Tree T function is a cognate of the Lambert W function through $T_k(z) = -W_{-k}(-z)$; see [1] for more discussion. The notation in this paper makes a new convention for the signs of the branches: we realized with this work that the indices of the branches of the Tree T function should also be negated, as in the equation above. This means that while for Lambert W the only real-valued branches have indices $k = 0$ and $k = -1$, the corresponding indices for the Tree T function are $k = 0$ and $k = 1$. See figure 1.

One should note that in (1), $T_0\left(xe^{-x}\right) \neq x$. The equation $ye^{-y} = xe^{-x}$ has two real solutions for positive x, namely $y = T_k(xe^{-x})$. It might seem natural to define one branch of T to be the trivial solution $y = x$ and to find a method to denote (and compute) the other, nontrivial, solution; but if we are to re-use the predefined Tree T function or equivalently the Lambert W function, then the branch definitions for those functions impose a nondifferentiable corner at $x = 1$ for each branch. See figure 2. As we shall see, the nontrivial branch can be described parametrically by $y = v\exp(v)/(\exp(v)-1)$ and $x = v/(\exp(v)-1)$ where the branch difference $v = y - x$ runs from $-\infty$ to ∞.

[1] We make two changes in notation from Gautschi [3]. Gautschi used the Lambert W function while we use the Tree T function; we omit a subscript specifying the range of integration.

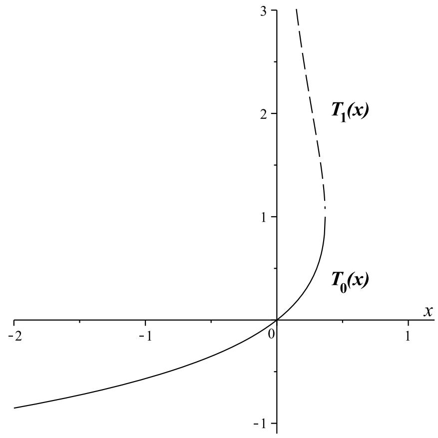

Figure 1: The Tree T function $T_k(z) = -W_{-k}(-z)$. The principal branch $T_0(z)$ satisfies $T_0(z) \le 1$, while the only other branch with real values has $T_1(z) \ge 1$. The branch point is at $z = 1/e$, where $T(z) = 1$. Note that the branch indices change sign from those of Lambert W.

1.1 Properties

We have

$$T_0(z) = \sum_{n\ge1} \frac{n^{n-1}}{n!} z^n \tag{3}$$

for $-1/e \le z < 1/e$. The Tree T function is the generating function for the number of rooted trees with n nodes. The series for $T_1(z)$ near $z = 0$ is also of interest:

$$T_1(z) = \ln\frac{1}{z} + \ln\ln\frac{1}{z} + \frac{\ln\ln\frac{1}{z}}{\ln\frac{1}{z}} + \frac{\ln\ln\frac{1}{z} - \frac{1}{2}\,\ln^2\ln\frac{1}{z}}{\ln^2\frac{1}{z}} + \cdots\,, \tag{4}$$

and the higher-order terms can be expressed in terms of Stirling numbers. This series converges for small enough $z > 0$, as proved first by Comtet in a different context.

1.2 Change of variables relating the integrals

Note that in equation (1) $x \ge 1$, allowing the simplification $T_1(x\exp(-x)) = x$. Thus the integral can be written

$$I_0(\alpha,\beta) = \int_1^\infty T_0^\alpha\left(xe^{-x}\right) T_1^{-\beta}\left(xe^{-x}\right)\,dx\,. \tag{5}$$

Similarly, for equation (2) we have $x \le 1$, implying $T_0(x\exp(-x)) = x$. Thus the integral can be written

$$I_1(\alpha,\beta) = \int_0^1 T_1^\alpha\left(xe^{-x}\right) T_0^{-\beta}\left(xe^{-x}\right)\,dx\,. \tag{6}$$

This shows the relationship between I_0 and I_1.

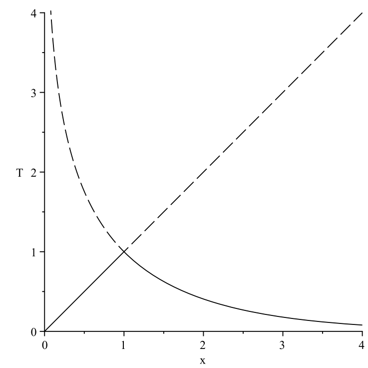

Figure 2: The equation $y\exp(-y) = x\exp(-x)$ has two solutions for y, which we may write as $y = T_k(x\exp(-x))$ for $k = 0$ (solid line) and $k = 1$ (dashed line). Notice that each curve has a corner at $x = 1$; at that point the trivial solution $y = x$ crosses the nontrivial solution, and the descriptions of each solution in terms of the Tree T function changes. A parametric description of the nontrivial solution is given by $y = v\exp(v)/(\exp(v) - 1)$ and $x = v/(\exp(v) - 1)$ where v runs from $-\infty$ to ∞. If $v \geq 0$, we recover the branch with $x \leq 1$, whereas if $v \leq 0$ then $x \geq 1$.

2 Branch Differences

In some applications, the difference between the branches is interesting. In [4] the series at the branch point was studied in terms of this difference. Here we use it to simplify the integrals.

In equation (5), put

$$v = T_1\left(xe^{-x}\right) - T_0\left(xe^{-x}\right) . \tag{7}$$

Since $x \geq 1$, we have $T_1(x\exp(-x)) = x$, implying $v = x - T_0(x\exp(-x))$ or $T_0 = x - v$. Suppose $v \neq 0$. Then, since $T_0\exp(-T_0) = (x - v)\exp(-x + v) = x\exp(-x)$, we cancel $\exp(-x)$ on both sides and get $(x - v)\exp(v) = x$, which is a linear equation in x that is easily solved. Therefore

$$T_0\left(xe^{-x}\right) = \frac{v}{e^v - 1} \qquad \text{and} \qquad T_1\left(xe^{-x}\right) = \frac{ve^v}{e^v - 1} \tag{8}$$

where we used again $x = T_1(x\exp(-x))$, and

$$x = \frac{ve^v}{e^v - 1} = \frac{v}{1 - e^{-v}} \; ; \tag{9}$$

$$dx = \frac{1 - e^{-v} - ve^{-v}}{\left(1 - e^{-v}\right)^2}\, dv \; . \tag{10}$$

The limit $x = 1$ corresponds to $v = 0$ and $x = \infty$ corresponds to $v = \infty$.

Using equations (9) and (10), the integral (5) becomes

$$\begin{aligned} I_0(\alpha, \beta) &= \int_0^\infty \left(\frac{v}{e^v - 1}\right)^\alpha \left(\frac{ve^v}{e^v - 1}\right)^{-\beta} \frac{(1 - e^{-v} - ve^{-v})}{\left(e^v - 1\right)^2}\, dv \\ &= \int_0^\infty e^{-\alpha v} \left(\frac{v}{1 - e^{-v}}\right)^{\alpha - \beta} \frac{(1 - e^{-v} - ve^{-v})}{\left(1 - e^{-v}\right)^2}\, dv \; . \end{aligned} \tag{11}$$

By inspection, the terms have at most removable singularities at the origin. A sufficient condition for convergence is $\alpha > 0$. If $\alpha = 0$, then the integral converges if and only if $\beta > 1$.

Turning our attention to the second integral, we have $x = T_0(x\exp(-x))$ because now $x \le 1$ and this time, we put $v = T_1(x\exp(-x)) - x$, which is again greater than or equal to zero, and find that the equations (8) still hold[2], but now $x = T_0(x\exp(-x))$ and so the substitution is different:

$$x = \frac{v}{e^v - 1}\;; \tag{12}$$

$$dx = \frac{e^v - 1 - ve^v}{(e^v-1)^2}\,dv\,. \tag{13}$$

Now when $x = 0$ we have $v = \infty$ and when $x = 1$ we have $v = 0$. Using this substitution the integral (6) becomes

$$\begin{aligned} I_1(\alpha,\beta) &= \int_\infty^0 \left(\frac{ve^v}{e^v-1}\right)^{\alpha}\left(\frac{v}{e^v-1}\right)^{-\beta}\frac{(e^v-1-ve^v)}{(e^v-1)^2}\,dv \\ &= -\int_0^\infty e^{(\beta-1)v}\left(\frac{v}{1-e^{-v}}\right)^{\alpha-\beta}\frac{(1-v-e^{-v})}{(1-e^{-v})^2}\,dv\,. \end{aligned} \tag{14}$$

Again this integral contains only removable singularities at $v = 0$ and a sufficient condition for convergence is that $\beta < 1$. If $\beta = 1$, then the integral converges if and only if $\alpha < -1$.

These definite integrals no longer contain the Tree T function and will be amenable to direct methods or contour integration methods. Note that although several definite integrals for I_0 and I_1 were given in [3], the expressions given here appear to be new.

Because the integrals for I_0 and I_1 are so similar, we can easily establish that

$$I_0(\alpha,\beta) + I_1(1-\beta,1-\alpha) = \int_0^\infty e^{-\alpha v}\left(\frac{v}{1-e^{-v}}\right)^{\alpha-\beta+1}dv\,. \tag{15}$$

In the case $\alpha = \beta$ this can be shown to be $\Psi_1(\alpha)$, the trigamma function. One method to see this is to write $1/(1-\exp(-v))$ as a geometric series and then integrate term by term to get $1/(\alpha+k)^2$; summing over k gives the trigamma function. Similarly, one can see with one integration by parts that $I_0(\alpha,\alpha) = \alpha\Psi_1(\alpha)$, and from thence that $I_1(\alpha,\alpha) = \alpha\Psi_1(1-\alpha)$.

3 Curiosities

As just discussed, the paper [3] notes that the integral is a trigamma function if $\alpha = \beta$. Neither Maple nor Wolfram Alpha can identify the trigamma function from the integrals presented here. However, both systems can evaluate the integrals for a variety of integer values of α and β. For instance, Maple easily evaluates

$$I_0(1,2) = \int_0^\infty \frac{\mathrm{e}^{-v}\,(1-(1+v)\,\mathrm{e}^{-v})}{v\,(1-\mathrm{e}^{-v})}dv = 1-\gamma\,, \tag{16}$$

[2] That is, $T_1 = v\exp(v)/(\exp(v)-1)$ and $T_0 = v/(\exp(v)-1)$ as before, but remember now that $x \ge 1$ and $v = T_1(x\exp(-x)) - x$ instead of $v = x - T_0(x\exp(-x))$; this change is the same as reversing the sign of the parametrization of the nontrivial branch.

$$I_0(2,3) = \int_0^\infty \frac{\mathrm{e}^{-2\,v}\,(1-(1+v)\,\mathrm{e}^{-v})}{v\,(1-\mathrm{e}^{-v})} dv = \tfrac{3}{2} - \gamma - \ln 2\,, \tag{17}$$

and

$$I_0(3,2) = \int_0^\infty \frac{\mathrm{e}^{-3\,v} v\,(1-(1+v)\,\mathrm{e}^{-v})}{(1-\mathrm{e}^{-v})^3} dv = \tfrac{5}{2} - 6\zeta(3) + \tfrac{1}{2}\pi^2\,. \tag{18}$$

Similarly, I_1 is easily evaluated in Maple if α and β are integer values (or are equal).

$$I_1(2,0) = -\int_0^\infty \frac{v^2\mathrm{e}^{-v}\,(1-v-\mathrm{e}^{-v})}{(1-\mathrm{e}^{-v})^4} dv = \tfrac{1}{3} + 2\zeta(3) + \tfrac{1}{3}\pi^2\,. \tag{19}$$

An interesting variation is that Maple can sometimes evaluate the integrals if α and β differ by an integer but are not themselves integers:

$$I_0\left(\tfrac{3}{2},\tfrac{1}{2}\right) = \int_0^\infty e^{-3v/2}\frac{(1-e^{-v}-ve^{-v})}{(1-e^{-v})^3}\,dv = -\tfrac{1}{2} + \tfrac{3}{4}\pi^2 - \tfrac{21}{4}\zeta(3)\,. \tag{20}$$

This does not always work, however. If $\alpha = 1/4$ and $\beta = -3/4$, then the difference is an integer but neither Maple nor Mathematica is able to evaluate the integral. For the simpler integral $I = I_0(\alpha,\beta) + I_1(1-\beta,1-\alpha)$ this also happens.

At the time of writing, we do not know if any computer algebra system can evaluate these integrals for values of α and β that have non-integer differences, or even merely for arbitrary α and β whose difference is an integer.

4 Series

We pointed out earlier that when $\alpha = \beta$ the integrals for I_0 and I_1 could be identified as containing Ψ_1, the trigamma function, by using a series expansion. One is tempted to try the same thing for $\alpha \neq \beta$. We are successful in writing $I_0(\alpha,\beta) + I_1(1-\beta,1-\alpha)$ as a series, as follows.

The integrand in equation (15) can be expanded in a convergent series if $v > 0$.

$$\begin{aligned} e^{-\alpha v}\left(\frac{v}{1-\exp(-v)}\right)^{\alpha-\beta+1} &= v^{\alpha-\beta+1}e^{-\alpha v}\left(1-e^{-v}\right)^{\beta-1-\alpha} \\ &= v^{\alpha-\beta+1}e^{-\alpha v}\sum_{\ell\geq 0}\binom{\beta-1-\alpha}{\ell}(-1)^\ell e^{-(\alpha+\ell)v}\,. \end{aligned} \tag{21}$$

Now

$$\int_0^\infty v^{\alpha-\beta+1}e^{-(\alpha+\ell)v}\,dv = \frac{\Gamma(\alpha-\beta+2)}{(\alpha+\ell)^{\alpha-\beta+2}} \tag{22}$$

so the series for the sum $I_0(\alpha,\beta) + I_1(1-\beta,1-\alpha)$ is

$$\begin{aligned} I &= \sum_{\ell\geq 0}(-1)^\ell\binom{\beta-1-\alpha}{\ell}\frac{\Gamma(\alpha-\beta+2)}{(\alpha+\ell)^{\alpha-\beta+2}} \\ &= \Gamma(\alpha-\beta+2)\sum_{\ell\geq 0}\binom{\alpha-\beta+\ell}{\ell}\frac{1}{(\alpha+\ell)^{\alpha-\beta+2}}\,. \end{aligned} \tag{23}$$

Notice that we need $\alpha - \beta > -2$ for the integral (22) to converge and for the series to be valid. This is a new restriction, additional to the $\alpha > 0$ needed for convergence of the original integrals $I_0(\alpha, \beta)$ and $I_1(1-\beta, 1-\alpha)$.

When $\alpha = \beta$ this reduces, as claimed, to

$$\Psi_1(\alpha) = \sum_{\ell \geq 0} \frac{1}{(\alpha + \ell)^2} \,. \tag{24}$$

If we introduce a new variable m with the definition $\beta = \alpha - m$, this sum becomes

$$S(m) = \sum_{\ell \geq 0} \binom{m+\ell}{\ell} \frac{1}{(\alpha+\ell)^{m+2}} \,. \tag{25}$$

For explicit integers m, Maple can evaluate this sum in terms of known special functions such as the polygamma functions $\Psi_j(\alpha)$ for $j \leq m+1$. For example,

$$\begin{aligned} S(4) =& \tfrac{1}{24}\, \Psi_1(\alpha) + \left(\tfrac{1}{12}\, \alpha - \tfrac{5}{24}\right) \Psi_2(\alpha) + \left(-\tfrac{5}{24}\, \alpha + \tfrac{1}{24} \alpha^2 + \tfrac{35}{144}\right) \Psi_3(\alpha) \\ &+ \tfrac{1}{288}\, (2\,\alpha - 5) \left(\alpha^2 - 5\,\alpha + 5\right) \Psi_4(\alpha) \\ &+ \tfrac{1}{2880}\, (\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)\, \Psi_5(\alpha) \,. \end{aligned} \tag{26}$$

Indeed, for explicit integers $m \geq 0$, this sum is proportional to the value of a hypergeometric function:

$$S(m) = \frac{1}{\alpha^{m+2}} \cdot {}_{m+3}F_{m+2} \left(\left. \begin{matrix} \alpha, \alpha, \ldots, \alpha, m+1 \\ \alpha+1, \alpha+1, \ldots, \alpha+1 \end{matrix} \right| 1 \right) \tag{27}$$

To show this, we recall that

$$ {}_pF_q \left(\left. \begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{matrix} \right| z \right) = \sum_{k=0}^{\infty} \frac{\alpha_1^{\overline{k}} \alpha_2^{\overline{k}} \ldots \alpha_p^{\overline{k}}}{\beta_1^{\overline{k}} \beta_2^{\overline{k}} \ldots \beta_q^{\overline{k}}} \cdot \frac{z^k}{k!}$$

Then we can write

$$\binom{m+\ell}{\ell} \frac{1}{(\alpha+\ell)^{m+2}} = \frac{(m+1)^{\overline{\ell}}}{\ell!} \left(\frac{\alpha^{\overline{\ell}}}{\alpha(\alpha+1)^{\overline{\ell}}} \right)^{m+2} ,$$

and the result (27) follows.

Finally, at least one of the integrals with $\alpha = 1/4$ and $\beta = -3/4$ that Maple was unable to integrate explicitly before can be found using these sums, namely

$$\begin{aligned} I\left(\frac{1}{4}, -\frac{3}{4}\right) &= \Gamma\left(\frac{1}{4} - \left(-\frac{3}{4}\right) + 2\right) S(1) \\ &= 2S(1) = 24\, i \left(P_3(-i) - P_3(i)\right) + 2\pi^2 + 16C + 42\zeta(3) \end{aligned} \tag{28}$$

where $P_a(z) = \sum_{n \geq 1} z^n / n^a$ is the polylogarithm function and

$$C = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2}$$

is Catalan's constant. This last example, of which we are sure there are many more, points the way to improve some of Maple's evaluation of definite integrals.

5 Comparison with Examples from [3]

We have seen here a reference solution for some examples in which α and β differ by an integer. For those integrals for which we cannot derive a symbolic solution, we could use numerical methods such as those in Maple's `evalf/Int`, but this is less interesting. The tables show that Gautschi's results are as accurate as he claimed.

6 Concluding Remarks

One aim of this paper is to provide reference expressions for the integrals (1) and (2) in terms of quantities such as γ, the Euler-Mascheroni Constant, and evaluations of functions such as the Riemann ζ function, which we consider to be known and partially understood.[3]

Since the discovery of these special forms was the result of examining the properties of the two real branches of the Tree T function, as well as the properties of their difference, the exploration of branch relations in other Lambert W integrals may lead to further development of solutions to special cases.

References

[1] Robert M. Corless, Gaston H. Gonnet, D. E. G. Hare, David J. Jeffrey, and Donald E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[2] Steven R. Finch. Mathematical Constants. *Encyclopedia of Mathematics and Its Applications 96*, pg 29, 2003.

[3] Walter Gautschi. The Lambert W-functions and some of their integrals: a case study of high-precision computation. *Numerical Algorithms*, 57(1):27–34, 2011.

[4] J. Karamata. Sur quelques problèmes posés par Ramanujan. *J. Indian Math. Soc.*, 24:343–365, 1960.

[3]Finch [2] points out that 'only' several hundred million digits of γ are known.

Table 1: The relative error of Gautschi's approximations $G(\alpha,\beta)$ when compared with exact values for $I_0(\alpha,\beta)$. Note that a dash indicates integrals for which we do not have symbolic expressions.

α	β	$\left\lvert\frac{I_0(\alpha,\beta)-G(\alpha,\beta)}{I_0(\alpha,\beta)}\right\rvert$
2	2	3.5804×10^{-32}
	0	2.0444×10^{-31}
	-2	1.3960×10^{-32}
1	1	7.4593×10^{-33}
	0	2.2001×10^{-32}
	-1	2.3135×10^{-32}
$\frac{1}{2}$	2	-
	0	-
	-2	-

Table 2: The relative error of Gautschi's approximations $G(\alpha,\beta)$ when compared with exact values for $I_1(\alpha,\beta)$. Note that a dash indicates integrals for which we do not have symbolic expressions.

α	β	$\left\lvert\frac{I_1(\alpha,\beta)-G(\alpha,\beta)}{I_1(\alpha,\beta)}\right\rvert$
2	$\frac{1}{2}$	-
	0	2.7055×10^{-33}
	-2	1.3116×10^{-32}
$\frac{1}{2}$	$\frac{1}{2}$	1.0897×10^{-32}
	0	-
	-1	-
$-\frac{1}{2}$	0	-
	-1	-
	-2	-