Representation, simplification and display of fractional powers of rational numbers in computer algebra

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Abstract

Simplification of fractional powers of positive rational numbers and of sums, products and powers of such numbers is taught in beginning algebra. Such numbers can often be expressed in many ways, as this article discusses in some detail. Since they are such a restricted subset of algebraic numbers, it might seem that good simplification of them must already be implemented in all widely used computer algebra systems. However, the algorithm taught in beginning algebra uses integer factorization, which can consume unacceptable time for the large numbers that often arise within computer algebra. Therefore some systems apparently use various *ad hoc* techniques that can return an incorrect result because of not simplifying to 0 the difference between two equivalent such expressions. Even systems that avoid this flaw often do not return the same result for all equivalent such input forms, or return an unnecessarily bulky result that does not have any other compensating useful property. This article identifies some of these deficiencies, then describes the advantages and disadvantages of various alternative forms and how to overcome the deficiencies without costly integer factorization.

1 Why discuss such an elementary topic here?

First:

Definition. An *absurd number* is one that can be expressed as a rational number times a product of zero or more fractional powers of positive rational numbers.

Remark. We need a *brief* name for this subset of algebraic numbers, and the inspiration for this one is that *ab* means "from" in Latin, and "absurd numbers" continues the tradition started with whimsical names such as surds, imaginary numbers, radicals, irrational numbers, and surreal numbers.

This article discusses the advantages and disadvantages of alternative ways computer algebra systems can represent, simplify, and display absurd numbers. Although this is a topic taught in beginning algebra, some major computer algebra systems do an imperfect or surprisingly poor job; and we have suggestions for remedies.

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1.1 Simplification of equivalent forms of an absurd number

Table 1 shows the results produced by four systems for sixteen different input representations of the same absurd number.

Table 1: Results for simplifying 16 input representations of the same absurd number

#	Input	*Derive*	*Mathematica*	Maple		Maxima		
		default	default	default	simplify()	default	rootscontract()	radcan()
prime bases								
1	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
2	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
3	$\frac{14\cdot 2^{1/3}\,3^{1/3}\,5^{1/3}}{15\cdot 7^{1/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
4	$\frac{2\cdot 2^{1/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
coprime square free								
5	$\frac{2^{4/3}\,7^{2/3}}{15^{2/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\mathbf{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$	$\mathbf{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$	$\frac{2^{4/3}\,7^{2/3}}{15^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
6	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2\,7^{2/3}30^{1/3}}{15}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
7	$\frac{14\cdot 30^{1/3}}{15\cdot 7^{1/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2\,7^{2/3}30^{1/3}}{15}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
8	$\frac{2\cdot 2^{1/3}\,7^{2/3}}{15^{2/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\mathbf{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$	$\mathbf{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$	$\frac{2^{4/3}\,7^{2/3}}{15^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
9	$2\cdot 2^{1/3}\left(\frac{7}{15}\right)^{2/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\mathbf{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$	$\mathbf{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$	$\frac{2^{4/3}\,7^{2/3}}{15^{2/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
10	$\frac{14}{15}\left(\frac{30}{7}\right)^{1/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\frac{2\,7^{2/3}30^{1/3}}{15}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
non perfect powers								
11	$\left(\frac{28}{15}\right)^{2/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{1}{15}28^{2/3}15^{1/3}$	$\frac{1}{15}28^{2/3}15^{1/3}$	$\mathbf{\frac{28^{2/3}}{15^{2/3}}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
12	$\frac{28^{2/3}}{15^{2/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{1}{15}\,28^{2/3}\,15^{1/3}$	$\frac{1}{15}\,28^{2/3}\,15^{1/3}$	$\mathbf{\frac{28^{2/3}}{15^{2/3}}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
max reciprocal powers								
13	$\left(\frac{784}{225}\right)^{1/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\mathbf{\frac{1}{225}\,784^{1/3}\,225^{2/3}}$	$\mathbf{\frac{2}{15}\,98^{1/3}\,15^{1/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
14	$\frac{784^{1/3}}{225^{1/3}}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\mathbf{\frac{1}{225}\,784^{1/3}\,225^{2/3}}$	$\mathbf{\frac{2}{15}\,98^{1/3}\,15^{1/3}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\mathbf{\frac{2\,98^{1/3}}{225^{1/3}}}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
one integer power								
15	$\frac{2}{15}\,1470^{1/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{2}{15}\,1470^{1/3}$	$\frac{2}{15}\,1470^{1/3}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$
16	$\frac{1}{15}\,11760^{1/3}$	$\frac{2\cdot 1470^{1/3}}{15}$	$2\left(\frac{7}{15}\right)^{2/3}2^{1/3}$	$\frac{1}{15}\,11760^{1/3}$	$\frac{2}{15}\,1470^{1/3}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2\,1470^{1/3}}{15}$	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$

Regarding columns labeled "default":

Definition. *Default simplification* is the result of pressing Enter in Maple, Ctrl Enter in *Derive*, or Shift Enter in *Mathematica* or wxMaxima − with the factory-default mode settings and no transformational or simplification functions anywhere in the input expression.

For Maple 15, simplify(…, size) gave some different results than simplify(…) − not always smaller in terms of any easily discerned measure. For Maxima 5.24, the rootscontract(…) function is subject to a rootsconmode control variable, but its setting does not affect these examples.

The boldface results in Table 1 appear to be a consequence of happenstance more than intent, because they do not satisfy any easily discerned goal. For example:

- In $\boldsymbol{\frac{2\cdot 98^{1/3}}{225^{1/3}}}$, both radicands are composite with the same exponent and 225 is a perfect square, so why not either combine the two fractional powers or simplify the denominator to $15^{2/3}$?
- In $\boldsymbol{\frac{2}{15}\,2^{1/3}\,7^{2/3}\,15^{1/3}}$, prime 2 and composite 15 occurs to the same 1/3 power. Thus this pair could equally well be $6^{1/3}\,5^{1/3}$ or $2^{1/3}\,10^{1/3}$. Also, since 15 is already composite and has the same exponent as 2, why not combine the two factors into $30^{1/3}$?
- Similar remarks apply to $\boldsymbol{\frac{2}{15}\,98^{1/3}\,15^{1/3}}$.
- In $\boldsymbol{\frac{1}{225}\,784^{1/3}\,225^{2/3}}$, $784 = 28^2$ and $225 = 15^2$, so why not express this result more comprehensibly as $\frac{1}{15^2}28^{2/3}15^{4/3} \rightarrow \frac{28^{2/3}}{15^{2/3}} \rightarrow \left(\frac{28}{15}\right)^{2/3}$?

Definition. A *canonical form* for a class of expressions is one for which all equivalent expressions in the class are represented uniquely.

As discussed in [1, 3, 4], canonical forms are unnecessarily costly and rigid for the entire class of expressions addressed by general-purpose computer algebra systems. However, canonical forms are acceptable and good for the internal form systems use to represent some restricted classes of subexpressions such as absurd numbers.

Notice that the default *Derive* result is the same for all sixteen alternative inputs of the same absurd number, as is the default *Mathematica* result and the Maxima radcan() result.[1] This suggests that these three columns are a consequence of transforming absurd numbers to a canonical form.

In contrast, none of the other columns in Table 1 display the same result in all sixteen rows, which implies that they are not simplified to a canonical form. Such non-canonical internal representations might be defensible if caused by the goal of returning the closest result to the input that satisfies one of several alternative easily comprehended goals. However, we will explain how all of the *inputs* already exhibit one such alternative set of goals. Therefore maximum compliance with this goal would return the inputs unchanged. Moreover, the dramatic transformations of most inputs throughout Table 1 indicate that closeness to the input was not a goal for any of these systems.

1.2 Differences of equivalent forms of an absurd number

It is difficult to fully simplify an expression that contains different internal representations of the same absurd number, because syntactic comparison is then insufficient to assess equivalence. This

[1] Full disclosure: We were two of the authors of *Derive*.

can lead to a disastrously incorrect result, because if a numerator and denominator are both equivalent to 0 but default simplification transforms only the numerator to 0, then most default simplification will incorrectly return 0 rather than the result of 0/0.[2] For example, Table 2 displays the results of default simplification of all differences of input forms from Table 1 for Maple and Maxima. The entry "0,$\frac{0}{0}$" indicates that Maxima simplified the expression to 0, but Maple did not.

Table 2: Default simplification of $\frac{0}{\mathrm{form}_j - \mathrm{form}_k}$ for Maple & Maxima. The correct result is $\frac{0}{0}$.

primal	$\frac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	1	$\mathbf{\frac{0}{0}}$															
	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	2	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$														
	$\frac{14\cdot 2^{1/3}\,3^{1/3}\,5^{1/3}}{15\cdot 7^{1/3}}$	3	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$													
	$\frac{2\cdot 2^{1/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	4	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$												
coprime square free distinct exponents	$\frac{2^{4/3}\,7^{2/3}}{15^{2/3}}$	5	0	0	0	0	$\mathbf{\frac{0}{0}}$											
	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	6	0	0	0	0	0	$\mathbf{\frac{0}{0}}$										
	$\frac{14\cdot 30^{1/3}}{15\cdot 7^{1/3}}$	7	0	0	0	0	0	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$									
	$\frac{2\cdot 2^{1/3}\,7^{2/3}}{15^{2/3}}$	8	0	0	0	0	$\mathbf{\frac{0}{0}}$	0	0	$\mathbf{\frac{0}{0}}$								
	$2\cdot 2^{1/3}\left(\frac{7}{15}\right)^{2/3}$	9	0	0	0	0	$\mathbf{\frac{0}{0}}$	0	0	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$							
	$\frac{14}{15}\left(\frac{30}{7}\right)^{1/3}$	10	0	0	0	0	0	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$	0	0	$\mathbf{\frac{0}{0}}$						
imperfect powers	$\left(\frac{28}{15}\right)^{2/3}$	11	0	0	0	0	0	0	0	0	0	0	$\mathbf{\frac{0}{0}}$					
	$\frac{28^{2/3}}{15^{2/3}}$	12	0	0	0	0	0	0	0	0	0	0	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$				
reciprocal exponents	$\left(\frac{784}{225}\right)^{1/3}$	12	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{\frac{0}{0}}$			
	$\frac{784^{1/3}}{225^{1/3}}$	14	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$		
1 integer base	$\frac{2}{15}\,1470^{1/3}$	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbf{\frac{0}{0}}$	
	$\frac{1}{15}\,11760^{1/3}$	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0,$\mathbf{\frac{0}{0}}$	$\mathbf{\frac{0}{0}}$
	form_j ↑	↑ form#:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
↑ category ⟶			primal				coprime square free distinct exponents						imperfect powers		reciprocal exponents		1 integer base	

The five categories of forms, such as the primal form, are described in Section 2. Notice that the

[2]Maple and Maxima both throw an error for 0/0. Less disruptively, *Mathematica* returns the symbol `Indeterminate` and *Derive* returns the symbol "?".

only successes had both forms from the same category, as evidenced the $\frac{0}{0}$ entries being confined to blocks along the diagonal.

Maple's default simplification recognizes only 17% of the 0 denominators and Maxima's default simplification recognizes only 18%. If Maple and Maxima default simplification simplified absurd numbers to a canonical form, then all of the entries would be $\frac{0}{0}$, as they are for *Mathematica* and *Derive*.

The Maple simplify(...) function *does* simplify all of these denominators to 0 despite the fact that it doesn't transform all sixteen forms to the same form. Therefore a canonical form isn't absolutely necessary for zero recognition. However zero recognition is much more difficult to implement and often slower to execute without a canonical form. Clearly the extra effort invested in simplify(...) was not invested in the Maple default simplification. Unfortunately, that can cause simplify(...) to return incorrect results despite its admirable sophisticated zero-recognition for absurd numbers:

$$\mathrm{simplify}\left(\frac{0}{\mathrm{form}_j - \mathrm{form}_k}\right) \tag{1}$$

incorrectly returns 0 rather than the result of 0/0 for 83% of the differences because default simplification has already incorrectly simplified the entire argument to 0 before simplify(...) has a chance to simplify the denominator to 0.

For similar reasons, in Maxima

$$\mathrm{radcan}\left(\frac{0}{\mathrm{form}_j - \mathrm{form}_k}\right) \tag{2}$$

incorrectly returns 0 rather than the result of 0/0 for 82% of the differences despite the fact that radcan (...) produces a canonical form.

Thus as much as practical, it is important for *default* simplification to simplify the difference between equivalent forms to 0. By far the easiest way to implement this is to default simplify equivalent inputs to a canonical internal form.

Some systems use a canonical form based on factoring radicands, but don't attempt complete factorization when it becomes too costly. For example, Table 3 compares results for the expression

$$\sqrt{12345701^2 \cdot 12345709} - 12345701\sqrt{12345709}\,, \tag{3}$$

which is equivalent to 0:

Table 3: Simplification of $\sqrt{12345701^2 \cdot 12345709} - 12345701\sqrt{12345709}$:

Derive	Maple		*Mathematica*			Maxima		
default	default	simplify	default	Simplify	FullSimplify	default	rootscontract	radcan
0	non-0	0	non-0	non-0	0	non-0	non-0	non-0

Therefore, simplify(...), FullSimplify[...], rootscontract(...) and radcan(...) all *incorrectly* give 0 for the argument

$$\frac{0}{\sqrt{12345701^2 \cdot 12345709} - 12345701\sqrt{12345709}}$$

because their system's default simplification did not simplify to 0 a subexpression that is equivalent to 0. The damage was done *before* entering these four *extra simplification* functions.

Even with some non-canonical internal forms, gcds can be used to determine when two surds are rational multiples of each other, then combine them. For expression (3) with the left radicand expanded, the gcd of the two radicands is 12345709, so the expression is equivalent to

$$\sqrt{15241557021803401} \cdot \sqrt{12345709} - 12345701\sqrt{12345709}, \tag{4}$$

then $\gcd(15241557021803401, 12345701) \rightarrow 12345701$ and

$$15241557021803401/12345701 \quad \rightarrow \quad 12345701 \tag{5}$$

giving

$$\sqrt{12345701^2} \cdot \sqrt{12345709} - 12345701\sqrt{12345709} \quad \rightarrow \quad 0 \tag{6}$$

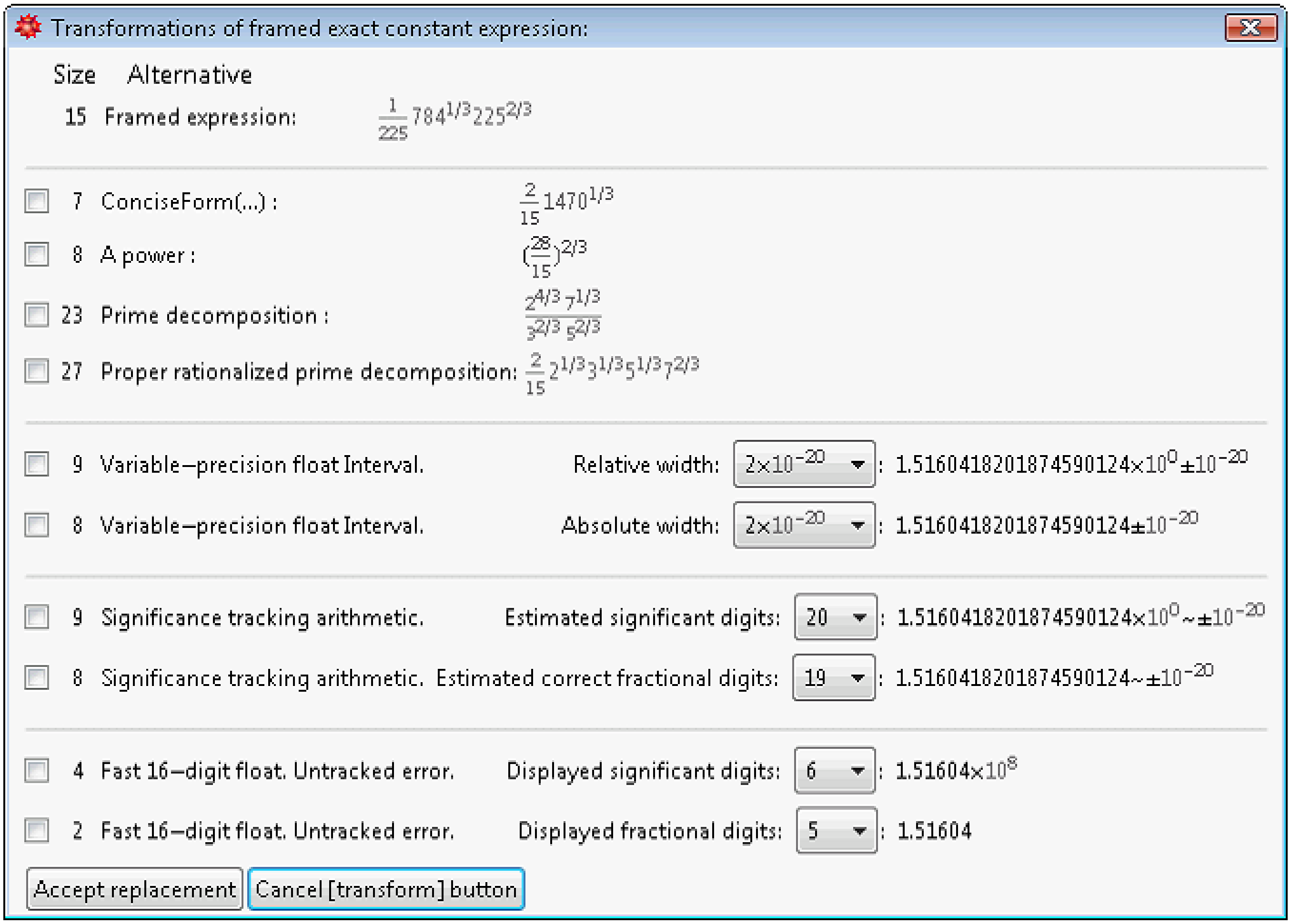
As indicated by the great variety of inputs and results in Table 1, there are a bewildering number of ways absurd numbers can be internally stored and displayed. However, the sixteen canonical inputs can be organized into a spectrum of categories discussed in Section 2, which compares their algorithms, advantages, and disadvantages, with conclusions in Section 3.

This article is complementary to [5], which instead addresses products of fractional powers of *rational powers of non-numeric expressions*. The difficulties there are different, entailing the need to be correct even when a subsequent substitution makes a denominator 0 or a radicand non-positive.

2 A spectrum of representation categories

For efficiency and ease of implementation, most computer algebra systems use an internal representation during simplification that does not completely correspond to displayed results. For example, often subtraction is represented using multiplication by a negative numeric coefficient.

We can do this for absurd numbers too: We can choose an internal representation that is easy to implement and/or fast to simplify; but for each example display the most concise of some alternative forms. As proposed in [6, 7], we could also have a transformation wizard that opens an Alternative Transformation dialog box for a highlighted subexpression. For example if the highlighted subexpression is $\frac{1}{225}784^{1/3}\,225^{2/3}$, then the dialog box might be



This is a mock up created with the *Mathematica* CreateDialog[. . .] function. The size column would be some easily-computed measure that correlates positively reasonably well with the area used to display the alternative.

The integer or rational number bases of fractional powers can be treated similar to variables in a data structure, but perhaps with additional rules for when an exponent becomes integer. Therefore we can represent expressions having more than one such prime base in either distributed or recursive form. For example,

$$\frac{7}{5}2^{2/3}\,3^{4/5}\,5^{1/2}-\frac{8}{5}3^{4/5}\,5^{1/2}+3^{2/3}\,5^{1/2}+3^{1/4}+6$$

versus

$$\left(\left(\frac{7}{5}2^{2/3}-\frac{8}{5}\right)3^{4/5}+3^{2/3}\right)5^{1/2}+3^{1/4}+6.$$

For brevity the forms and algorithms discussed in this section assume only distributed form. However they can be adapted to recursive form, which has certain advantages such as often being more concise. For all forms we assume some canonical ordering of the factors in that form, such as in order of increasing base magnitude.

This section discusses the sixteen numbered input forms in Table 1, in that order. These examples are partitioned into five categories depending on the properties of the radicands and their exponents. To help show the relationships of the alternatives between and within each category,

Table 4 shows the same input forms in the same order, with descriptive phrases describing the properties of the radicands, their exponents and restrictions on any rational coefficient. This list of alternative forms is incomplete. However, it fits on one page; it collectively suggests additional forms; and we hope that it includes all of the most useful forms in general – not only for this example.

2.1 Primal forms

There are several canonical forms for absurd numbers based on prime factorization.

2.1.1 Pure primal form

Definition. *Pure primal* form is 0, 1, or a product of one or more distinct primes raised to nonzero reduced rational powers, with the factors in increasing order of their prime bases, or this form preceded by a minus sign.

Remark. For example, input #1 in Table 4 has this form.

Proposition 1. *Pure primal form is canonical for absurd numbers.*

Proof. Adapt almost any proof of the fundamental theorem of arithmetic from positive integer to reduced rational exponents, and employ the chosen canonical ordering of factors. □

The algorithm for *multiplication* of two absurd numbers represented using pure primal forms is obvious, easy to implement, and fast – as is raising a pure primal form to a rational power. Unfortunately for computer algebra implementers:

1. Conversion of composite radicands to primes requires integer factoring.
2. Factorization of large integers can be prohibitively time consuming.
3. Large integers occur rather often within computer algebra – even when the input and final result do not contain large integers.

Addition of pure primal forms is also quite slow because there is very little opportunity for making results more concise by merely combining *syntactically* similar terms: The only syntactically similar terms are ones that are identical or differ only in their signs, such as

$$2^{3/2}\,5^{-7/3} + \left(-2^{3/2}\,5^{-7/3}\right) \;\rightarrow\; (1-1)2^{3/2}\,5^{-7/3} \;\rightarrow\; 0\,.$$

Thus to simplify $2^{5/2}\,5^{-5/3} - 2^{1/2}\,5^{-2/3}$ to this canonical form we must recognize that the differences in their corresponding exponents are all integers, then *temporarily* use a form that makes them syntactically similar such as the proper-exponent primal form discussed in the next subsection. Then in general we must factor the resulting rational coefficient to convert the result to pure primal form:

$$2^{5/2}\,5^{-5/3} - 2^{1/2}\,5^{-2/3} \;\rightarrow\; \frac{4}{25}\left(2^{1/2}\,5^{1/3}\right) - \frac{1}{5}\left(2^{1/2}\,5^{1/3}\right) \;\rightarrow\; -\frac{1}{25}\left(2^{1/2}\,5^{1/3}\right) \;\rightarrow$$
$$-\,5^{-2}\left(2^{1/2}\,5^{1/3}\right) \;\rightarrow\; -2^{1/2}\,5^{-5/3}.$$

If the differences in the exponents are not all integer, then the two numbers are not commensurate and their sum or difference must be represented as a more general expression than a single absurd number.

To incur the cost of integer factorization not only initially but also after most additions of similar terms makes pure primal form a costly internal form, but it can be useful as an optional *result* form. Most computer algebra systems contain a rational-number factorization function that returns the pure primal form.

2.1.2 Proper-exponent primal form

Definition. *Proper-exponent primal* form is a rational number times a pure primal form in which all of the exponents are in the interval $(0, 1)$.

Remark. For example, input #2 in Table 4 is proper-exponent primal form.

We were taught in beginning algebra to simplify a fractional power of a positive rational number r^α by converting it to this form. The algorithm *can* be expressed as

1. Factor r.
2. Represent the factored r as a product containing negative powers rather than as a ratio.
3. Distribute α over the factors.
4. Extract and multiply together the rational numbers corresponding to the *floor* of each resulting exponent.

Using the floor automatically rationalizes the denominator.

Proposition 2. *Proper-exponent primal form is canonical.*

Proof. Step 3 above gives the canonical pure primal form. The floor function is defined and single valued for all reals. Thus the extracted rational parts, their product, and any residual fractional powers are unique. Conversely, if we start with the proper-exponent primal form, factor the rational part, combine similar primes and order the bases, then the result is unique. □

The algorithms for multiplication of two proper-exponent primal forms and for raising one to a rational power are nearly as obvious, easy to implement, and fast as for pure primal form. Moreover addition of proper-exponent primal forms is much easier and more efficient than for pure primal forms: Sums of proper-exponent primal forms can be collected to make another such form if and only if the irrational factors are identical which is fast to check. Moreover, when the irrational factors are identical, there is no need to factor the resulting rational coefficient.

Unfortunately, integer factorization is still generally needed to transform a fractional power of a positive rational number to proper-exponent primal form.

Pure primal form often requires less display area than other primal forms because there is no rational factor formed from expanding a product of integer powers of the prime bases having exponents not in a designated interval. However, users often feel that extracting that rational factor makes the result "simpler". Students are also taught to rationalize denominators. Therefore, display of proper-exponent primal form complies with users' comfort zones.

"... *the customer is always right.*"
– Marshall Field

2.1.3 Tight balanced-exponent primal form

Any fixed near-unit-width exponent interval can be used instead of $(0,1)$ for primal and other categories of forms. The nearly balanced interval $(-1/2,1/2]$ has some appeal because the magnitude of the exponents never exceeds $1/2$.

Definition. *Tight balanced-exponent primal* form is a rational number times a pure primal form in which all of the exponents are in the interval $(-1/2,1/2]$.

Remark. For example, input #3 in Table 4 has this form. As another example with input $2^{2/3}$ we rationalize the *numerator* to return $2/2^{1/3}$.

2.1.4 Loose balanced-exponent primal form

Definition. *Loose balanced-exponent primal* form is a rational coefficient times a pure primal form in which all of the exponents are in the interval $(-1,1)$, none of the numerator radicands divide the denominator of the rational coefficient, and none of the denominator radicands divide the numerator of the rational coefficient.

Remark. For example, input #4 in Table 4 has this form.

This form can be derived from pure primal form by separately making the numerator and denominator exponents proper, and *not* rationalizing any denominators or numerators. For example, $\sqrt{2}$, $1/\sqrt{2}$, and $5\times 7^{2/3}/(2\times 3^{4/5})$ have this form, but $\sqrt{2}/6$ does not because $\gcd(2,6)\neq 1$. Loose balanced-exponent form is often more concise than proper-exponent or tight-balanced primal form because rationalizing denominators or numerators often increases bulk. For example, compare $1/\sqrt{1234567891}$ with $\sqrt{1234567891}/1234567891$ and compare $\sqrt{9876543211}$ with

$$9876543211/\sqrt{9876543211}.$$

Such rationalizations are also inconsistent with customary simplification of *non-numeric* radicands: Most people prefer $1/u^{1/3}$ or $u^{-1/3}$ to the unreduced $u^{2/3}/u$, and avoiding unnecessary form changes upon substitution of numbers is a virtue. We can of course use a product containing negative exponents rather than a ratio for the internal form.

However, addition is harder with loose balanced-exponent primal form than with proper-exponent primal form, because mere syntactic comparison does not reveal all commensurate absurd numbers. For example, $2^{1/2}$ and $2^{-1/2}$ are not syntactically similar, but $2^{1/2}+2^{-1/2}\rightarrow 2^{1/2}+2^{1/2}/2\rightarrow (3/2)2^{1/2}\rightarrow 3\times 2^{-1/2}$. Therefore we do not recommend this as an *internal* form, but it is a good display form.

2.2 coprime square-free distinct-exponent forms

Gcd calculations are sufficient to make all fractional power bases in a result mutually coprime. For example, $\gcd(30,42)\rightarrow 6$, so

$$30^{1/2}42^{1/3}\ \rightarrow\ (5\cdot 6)^{1/2}\,(6\cdot 7)^{1/3}\ \rightarrow\ 5^{1/2}\,6^{1/2}\,6^{1/3}\,7^{1/3}\ \rightarrow\ 5^{1/2}\,6^{5/6}\,7^{1/3}.$$

This *particular* result is canonical, but relative primality alone is not sufficient to guarantee canonicality. For example, the bases in the equivalent forms $5^{1/2}\,6^{1/2}\,7^{1/3}$ and $3^{1/2}\,7^{1/3}\,10^{1/2}$ are coprime. Combining factors having the same exponents to make all of the exponents distinct makes this example canonical: We combine $5^{1/3}$ and $6^{1/3}$ or combine $3^{1/3}$ and $10^{1/3}$ giving $7^{1/3}\,30^{1/2}$ either way.

However, 24 and 5 are coprime in $24^{1/2}\,5^{1/3}$ with distinct exponents, as are 2, 3 and 5 in $2^{3/2}\cdot 3^{1/2}\,5^{1/3}$, but both products are equivalent. So we need an additional criterion for canonicality:

Definition. An integer > 1 is *square free* if none of its prime factors occurs more than once.

Remark. For example, 6 is square free, but $24 = 2^3\,3$ is not.

Definition. *coprime square-free integer bases distinct-exponent* form is a product of coprime positive square-free integers raised to distinct rational exponents, or -1 times that, or 0.

Remark. For example, input #5 in Table 4 has this form.

This form can be computed from pure primal form as follows: For each distinct exponent, combine all of the factors having that exponent, raising the product of their primes to their shared exponent. There is clearly only one way to do this, the resulting bases are square-free because they are a product of distinct primes, and the resulting bases are coprime because each prime occurs in only one of the bases.

Conversely, to compute the pure primal form from this form, factor each base then distribute the distinct exponent of that square-free base over the resulting product of primes. Each distinct prime can occur in only one of the coprime factors, so the distinct exponent for each base will be the final exponent of all the primes in that base. This result is clearly unique when the bases are ordered canonically, so the coprime square-free integer bases distinct-exponent form is canonical.

When multiplying two such forms, if a base b in one form is not identical to a base in the other form, then it is important to compute the gcd of b with the bases in the other form to check for coprimeness and act appropriately if any of these gcds is not 1. It is also important to check for identical exponents as well as identical bases. Therefore multiplication is slower and not as easy to implement as for primal forms.

There are also proper, tight balanced, and loose balanced variants analogous to those based on prime radicands. For example,

- Input #6 is *coprime square-free integer bases distinct proper-exponent* form. Two such forms are commensurate for addition if and only if their irrational parts are identical, making this variant the best choice of internal form for this class.
- Input #7 is *coprime square-free integer bases distinct tight balanced-exponent* form.
- Input #8 is *coprime square-free integer bases distinct loose balanced-exponent* form.

As with primal forms, the proper variant is most efficient for adding absurd numbers because syntactic comparison is sufficient to decide similarity.

We can also combine factors whose exponents differ only in sign, for further sharing of common exponents, giving forms that are more concise and faster for subsequent floating-point approximation:

Definition. *coprime square-free rational bases distinct proper-exponent* form is a rational number times a product of coprime square-free positive rational numbers raised to distinct proper exponents, or -1 times that, or 0.

Remark. For example, input #9 is that form, obtained by combining a numerator and denominator factor of input #5.

Definition. *coprime square-free rational bases distinct tight balanced exponent* form is a product of coprime square-free positive rational numbers raised to distinct exponents in the interval $(-1/2, 1/2]$, or -1 times that, or 0.

Remark. For example, input #10 is that form, obtained by combining a numerator and denominator factor of input #7.

Unfortunately there is no known way to square-free factor an integer faster than by factoring it then combining bases having identical exponents. For example, to square-free factor 2910600, we can factor it into $2^3\, 3^3\, 5^2\, 7^2 11$, then combine factors having the same exponent to produce $6^3\, 35^2\, 11$ in which 6, 35 and 11 are square free. However, if we are incurring the cost of integer factorization anyway, it is simpler and probably faster to use full factorization for the internal form. Consequently although these coprime square-free distinct-exponent forms are often the most concise *display* forms, we do not recommend them as an *internal* form.

2.3 Forms based on perfect power computation

Our next family of canonical forms uses perfect-power factorization to avoid the cost of integer factorization.

Definition. For integers $k > 1$, $m > 1$ and $n > 1$, n is a k^{th} *perfect power* of m if and only if $m^k = n$.

We are interested in determining the maximum $\hat{k}$ and minimum integer $\check{m}$ for which $\check{m}^{\hat{k}} = n$.

Perfect powers can be determined from a prime factorization, but there is a much faster way: As described by Fitch [2], given integers $n > 1$ and $k > 1$, Newton's method with the help of the floor function can be used to quickly compute an *exact* k^{th} root of n or determine that one does not exist. If n has b bits, then starting with a guess that has $\lceil b/k \rceil$ bits, the number of correct bits of the result doubles from 1 or more with each iteration, which is fast. Since we want the largest such k, we can try successive primes starting with 2, repeating each prime until it no longer works. Each success substantially reduces m from its initial value of n, reducing the work for subsequent trials. We can stop when for the next prime p and the current value of m, $2^p > m$. Here are some extreme examples for large n, ordered from least to most applications of Newton's method:

1. For $n = 6^{2 \cdot 3 \cdot 5 \cdot 7}$, it requires one successful and one unsuccessful application of Newton's method with each of the 4 successive primes 2, 3, 5 and 7 to determine that the 164 digit $n = 6^{210}$.

2. For $n = 2^{2^9}$ it requires 9 successful applications of Newton's method with the first-tried prime 2 to determine that the 155 digit $n = 2^{512}$.

3. For $n = m^2$ with m being the largest prime less than 2^{256}, it requires 1 successful application of Newton's method followed by 53 unsuccessful applications to determine that the 154 digit $n = m^2$.

4. For $n = 2^{509}$, it requires 96 unsuccessful applications with successive primes followed by one successful application with the prime 509 to determine that the 154 digit $n = 2^{509}$.

5. For n being the largest prime less than 2^{512}, it requires 97 unsuccessful applications with successive primes to determine that the 154 digit n is not a perfect power.[3]

For both the mean case and worst case this method is much faster than factoring large integers.

Definition. A positive rational number is a *perfect* k^{th} *power* if it is a perfect k^{th} power of an integer, or the reciprocal of such a perfect power, or its numerator is a perfect j^{th} power, its denominator is a perfect ℓ^{th} power, and k evenly divides $\gcd(j, \ell)$.

Definition. A positive rational number is an *imperfect power* if it is not a perfect power.

We are interested in determining the maximum $\hat{k}$ and minimum rational number $\check{r}$ for which $\check{r}^{\hat{k}} = r > 1$. For a reduced positive fraction r that is neither an integer nor a reciprocal, we can compute the $\hat{k}_1$ for whichever of the numerator and denominator is smaller, then if $\hat{k}_1 > 1$, restrict the choice of primes for applying Newton's method to the other part to primes that exactly divide $\hat{k}_1$.

2.3.1 A positive rational power of a positive rational number that is an imperfect power

Definition. *Single rational imperfect power base positive exponent* form is a positive rational power of a positive rational number that is an imperfect power, or -1 times that, or 0.

Remark. For example, input #11 in Table 4 has this form.

To transform a positive rational power α of a positive rational number r to this form, maximally perfect-power factor $r \to \check{r}^{\hat{k}}$, then return $\check{r}^{\hat{k}\alpha}$.

Proposition 3. *Single rational imperfect power base positive exponent form is canonical.*

Proof. To convert pure primal representation $P = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ to this form, let $\gamma \leftarrow \gcd(\alpha_1, \alpha_2, \ldots)$, which is positive, then let $n_1 \leftarrow \alpha_1/\gamma$, $n_2 \leftarrow \alpha_2/\gamma$, etc. All of the n_j are coprime *integers*. Consequently P can be represented as r^γ where r is the expanded rational number $p_1^{n_1} p_2^{n_2} \cdots$. Then use Newton's method to express r as an imperfect power base raised to a positive exponent: $r \to \check{r}^{\hat{k}}$ giving $P \to \check{r}^{\hat{k}\gamma}$. The pure primal representation together with γ, n_1, n_2, r, $\check{r}$ and $\hat{k}$ are all unique, therefore this single power form for P is unique. Now consider the other direction: Factor $\check{r}$, distribute $\hat{k}\gamma$, then sort the factors into canonical order, giving the canonical pure primal form. □

To *multiply* two such forms $u_1 r_1^{\alpha_1}$ and $u_2 r_2^{\alpha_2}$ with $u_1, u_2 \in \{1, -1\}$:

1. Let $\gamma \leftarrow \gcd(\alpha_1, \alpha_2)$, $n_1 \leftarrow \alpha_1/\gamma$, $n_2 \leftarrow \alpha_2/\gamma$, making n_1 and n_2 integer
2. Use Newton's method to compute $r_1^{n_1} r_2^{n_2} \to \check{r}^{\hat{k}}$, then return $u_1 u_2 \left(\check{r}\right)^{\hat{k}\gamma}$.

To *add* two such forms:

1. Let $m_1 \leftarrow \lfloor \alpha_1 \rfloor$, $m_2 \leftarrow \lfloor \alpha_2 \rfloor$, $\beta_1 \leftarrow \alpha_1 - m_1$, $\beta_2 \leftarrow \alpha_2 - m_2$.
2. If $\beta_1 \neq \beta_2$, then the ratio of the two inputs is irrational, so their sum cannot be represented as a single absurd number.

[3] It might be worth using a primality test in perfect power factorization.

3. Otherwise, let $g \leftarrow \gcd(r_1, r_2)$, $\bar{r}_1 \leftarrow r_1/g$, $\bar{r}_2 \leftarrow r_2/g$, $n \leftarrow \text{numerator}(\beta_1)$, $d \leftarrow \text{denominator}(\beta_1)$.

4. If $\bar{r}_1 \not\to \breve{r}_1^d$ or $\bar{r}_2 \not\to \breve{r}_2^d$ then the sum cannot be represented as a single absurd number.

5. Otherwise let $\rho \leftarrow u_1 r_1^{m_1} \breve{r}_1^n + u_2 r_2^{m_2} \breve{r}_2^n$.

6. If $\rho = 0$ then return 0.

7. Use the multiplication algorithm to return the result of ρg^γ as a positive rational power of a positive rational number that is an imperfect power − or as −1 times that.

This is a canonical form that avoids the cost of integer factorization!

However, for strict consistency, even a rational number might have to be represented as a perfect power such as $256/81 \to (4/3)^4$. The frequent use of Newton's method to maintain this would unacceptably slow down rational arithmetic. Therefore in practice whenever a resulting exponent is an integer, then it is better to expand the power to the more typical representation for a rational number.

Although fractional exponents are often merely half-integer or thirds of an integer, this form can result in large radicands. For example,

$$\frac{29^{31/10}}{2^{1/10}} \to \left(\frac{29^{31}}{2}\right)^{1/10} \to \left(\frac{2159424054808578564166497528588784562372597429}{2}\right)^{1/10}. \tag{7}$$

Extracting a maximal rational factor from the pure primal form made it much faster to add absurd numbers, so it is natural to wonder if the same is true for single rational imperfect power base positive exponent form. For example,

$$\left(\frac{576}{25}\right)^{2/3} \to \left(\frac{24}{5}\right)^{4/3} \to \frac{24}{5}\left(\frac{24}{5}\right)^{1/3}.$$

Unfortunately, this isn't a canonical form, because $24 = 2^3 3$, so this number also has a different representation in this form as

$$\frac{48}{5}\left(\frac{3}{5}\right)^{1/3}, \tag{8}$$

and it requires square-free integer factoring to obtain this form, which requires integer factoring. One way to make it canonical is: Whenever an input or a tentative result is a rational number times a fractional power of a rational number:

1. Transform the product to single rational imperfect power base positive exponent form.

2. Then use the floor function to extract a rational factor if the positive fractional power exceeds 1, making the fractional power proper.

For example,

$$\frac{48}{5}\left(\frac{3}{5}\right)^{1/3} \to \left(\frac{48^3\,3}{5^3\,5}\right)^{1/3} \to \left(\left(\frac{24}{5}\right)^4\right)^{1/3} \to \left(\frac{24}{5}\right)^{4/3} \to \frac{24}{5}\left(\frac{24}{5}\right)^{1/3}.$$

Gcds can be used to simplify sums of absurd numbers in this form without doing this canonicalizing reabsorption. For example, $\gcd(24/5, 3/5) \to 3/5$, so

$$\frac{24}{5}\left(\frac{24}{5}\right)^{1/3} - \frac{48}{5}\left(\frac{3}{5}\right)^{1/3} \to \frac{24}{5}8^{1/3}\left(\frac{3}{5}\right)^{1/3} - \frac{48}{5}\left(\frac{3}{5}\right)^{1/3} \to \frac{48}{5}\left(\frac{3}{5}\right)^{1/3} - \frac{48}{5}\left(\frac{3}{5}\right)^{1/3} \to 0.$$

However, the loss of canonicality for irrational absurd numbers is still troublesome. For example, for any function f – including a generic one with no current definition – we would like

$$f\left(\frac{24}{5}\left(\frac{24}{5}\right)^{1/3}\right) - f\left(\frac{48}{5}\left(\frac{3}{5}\right)^{1/3}\right) \to 0.$$

However, that won't happen with this non-canonical form unless every time a subexpression of the form $f(u) - f(v)$ is encountered during all transformations we check to see if $u - v$ can be simplified to 0. This is time consuming, a programming nuisance, and unlikely to enjoy 100% programmer compliance.

A way to partially overcome this dilemma is to use the proper variant only temporarily during a sequence operations with irrational absurd numbers, then represent the result of this sequence as a rational number if it is one, or as a unit times a single power otherwise. However, such context-dependent departure from pure locally self-contained bottom-up simplification is extra programming work, hence an invitation to inconsistent behavior.

In any event, it is definitely worthwhile overall to represent rational absurd numbers as rational numbers.

2.3.2 A ratio of positive rational powers of positive integers that are imperfect powers

Definition. *Ratio of two imperfect power integer bases raised to positive exponents* form is a ratio of two positive rational powers of positive integers that are imperfect powers, or −1 times that, or 0.

Input #12 in Table 4 has this form. For this example the resulting two exponents are identical, but that might not be so. For example, this form can help reduce or avoid radicand growth by transforming the right side of (7) to the left side.

2.3.3 Maximal positive reciprocal power of a positive rational number form

Definition. *Maximal positive reciprocal-exponent form* is the largest possible positive reciprocal power of a positive rational number, or −1 times that, or 0.

Remark. For example, input #13 in Table 4 has this form. As another example, $(9/4)^{1/4}$ does not have this form, but the equivalent expression $(3/2)^{1/2}$ does. Although 8/27 is a perfect cube, $(8/27)^{1/2}$ has this form because the exponent in $(2/3)^{3/2}$ is not a reciprocal.

To convert a positive reduced fractional power of a positive reduced rational number $r^{n/d}$ to this form:

1. Use Newton's method to find $\bar{d}$, the largest divisor of d such that $r \to \bar{r}^{\bar{d}}$.

2. Expand $\bar{r}^n$ giving $\hat{r}$.

3. Return the form $\hat{r}^{\bar{d}/d}$. Note that this generally requires fewer applications of Newton's method than to determine the *maximal* perfect root of r.

Proposition 4. *A maximal positive reciprocal power of a positive rational number is canonical.*

Proof. To convert pure primal representation $P = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ to this form, let $\gamma \leftarrow \gcd(\alpha_1, \alpha_2, \ldots)$, which is positive, then $m_1 \leftarrow \alpha_1/\gamma$, $m_2 \leftarrow \alpha_2/\gamma$, etc. (The gcd of two fractions is the gcd of their numerators divided by the least common multiple of their denominators. Therefore all of the multiplicities m_j are coprime *integers*.) Consequently P can be represented as $r^{1/\text{denominator}(\gamma)}$ where r is the expanded rational number $(p_1^{m_1} p_2^{m_2} \cdots)^{\text{numerator}(\gamma)}$. The pure primal representation together with γ, m_1, m_2 and r are all unique. Therefore this single power form for P is unique. Now consider the other direction: Factor r, distribute $1/\text{denominator}(\gamma)$, then sort the factors into canonical order, giving the canonical pure primal form. □

As illustrated by comparing inputs #11 and #13 in Table 4, this form can be less concise than imperfect power form. However, the arithmetic is faster:

To *multiply* two such forms $u_1 r_1^{\alpha_1}$ and $u_2 r_2^{\alpha_2}$ with $u_1, u_2 \in \{1, -1\}$:

1. Let $\gamma \leftarrow \gcd(\alpha_1, \alpha_2)$, which is a reciprocal because α_1, and α_2 are both reciprocals.
2. Let $m_1 \leftarrow \alpha_1/\gamma$, $m_2 \leftarrow \alpha_2/\gamma$, making m_1 and m_2 integer.
3. Expand $u_1 u_2$ giving u and expand $r_1^{m_1} r_2^{m_2}$ giving r.
4. Convert $r^\gamma \to \hat{r}^{1/\check{d}}$ by the algorithm at the beginning of this sub-subsection.
5. Return $u\hat{r}^{1/\check{d}}$.

To *add* two such forms:

1. If $\alpha_1 \neq \alpha_2$, then the sum cannot be represented as a single absurd number.
2. Otherwise, let $g \leftarrow \gcd(r_1, r_2)$, $n_1 \leftarrow r_1/g$, $n_2 \leftarrow r_2/g$, $d \leftarrow \text{denominator}(\alpha_1)$.
3. If $n_1 \not\rightarrow \bar{n}_1^d$ or $n_2 \not\rightarrow \bar{n}_2^d$, then the sum cannot be represented as a single absurd number. (These perfect root computations require only one or two applications of Newton's method for one specific d, making them faster than determining the maximal perfect powers.)
4. Otherwise let $\rho \leftarrow u_1 \bar{n}_1 + u_2 \bar{n}_2$.
5. If $\rho = 0$ then return 0.
6. Otherwise use the multiplication algorithm to transform ρg^γ into a maximal reciprocal power of a positive rational number − or −1 times that.

This is another canonical form that avoids the cost of integer factorization, and requires fewer applications of Newton's method than imperfect power form. Moreover, rational numbers are a special case wherein γ is the reciprocal of 1. However, the radicand can become quite large if the numerator of the given exponent is large. For example,

$$29^{31/10} \to \left(29^{31}\right)^{1/10} \to 215942405480857856416649752858878456237259742 9^{1/10}.$$

2.3.4 A ratio of two maximal reciprocal powers of positive integers

Definition. *Ratio of two maximal reciprocal powers of positive integers* form is a ratio of two maximally positive reciprocal powers of positive integers, or -1 times that, or 0.

Input #14 in Table 4 has this form. For this example the resulting two exponents are identical, but that need not be so. For example,

$$\frac{2^{3/4}\,7^{1/4}}{3^{2/3}5^{1/3}} \rightarrow \frac{(8\times 7)^{1/4}}{(9\times 5)^{1/3}} \rightarrow \frac{56^{1/4}}{45^{1/3}}.$$

In contrast, the radicand for unification into a single fractional power can be significantly larger:

$$\frac{56^{1/4}}{45^{1/3}} \rightarrow \frac{56^{3/12}}{45^{4/12}} \rightarrow \frac{175616^{1/12}}{4100625^{1/12}} \rightarrow \left(\frac{175616}{4100625}\right)^{1/12}.$$

However, arithmetic using separate single numerator and denominator radicals is more complicated, because we must use gcds to insure that numerator radicands and denominator radicands are relatively prime and contend with possibly different exponents of their gcd if it isn't 1.

2.4 Proper power of an integer forms

People often like to have absurd numbers displayed as a rational number times *one* rationalized proper fractional power of the smallest possible positive integer because:

- It is proper.
- The denominator is rationalized, which students are taught to overvalue.
- It has only one fractional power and the radicand is an integer.
- The maximum possible amount of rational coefficient is factored out, so the one radicand is as simple as possible for such a form.

Definition. *Single minimal integer base raised to a proper exponent* form is a rational number or a rational number times the smallest possible positive integer raised to an exponent in the interval $(0, 1)$.

Input #15 in Table 4 has this form. Proper exponent primal form can be converted to this form by unifying all its fractional powers into one. For example,

$$\frac{2}{15}2^{1/3}3^{1/3}5^{1/3}7^{2/3} \rightarrow \frac{2}{15}\left(2\times 3\times 5\times 7^2\right)^{1/3} \rightarrow \frac{2}{15}1470^{1/3}.$$

Unfortunately we do know how to guarantee this form canonical without integer factorization. However, the following form is similar and avoids integer factorization, but can result in larger integer radicands:

Definition. *Single integer imperfect power base* form is derived from the ratio of two imperfect power integer bases with positive exponents form as follows:

1. Extract a rational factor by independently making the numerator and denominator exponents proper,
2. Rationalize the denominator.
3. Unify the resulting two numerator fractional powers into a single fractional power of an integer.

For example, starting with input #12,

$$\frac{28^{2/3}}{15^{2/3}} \rightarrow \frac{28^{2/3}\,15^{1/3}}{15^{2/3}\,15^{1/3}} \rightarrow \frac{(784 \times 15)^{1/3}}{15} \rightarrow \frac{11760^{1/3}}{15}$$

giving input #16 in Table 4. This is not as nice as input #16, but depending on taste and the application, it is arguably nicer than inputs #11 through #13, which are the only other listed ones based only on imperfect powers with no need for integer factorization.

Although it is a good default display form for implementations that totally avoid integer factorization, this form is not very convenient as an internal form.

2.5 A hybrid of proper-exponent primal and imperfect power forms

- Primal internal forms entail integer factorization time that is unacceptable to many users if the second largest prime factor is larger than about, say, 30 digits. However, these forms are the only points of departure for generating many display forms that users might value. Of the various primal forms, arithmetic is the fastest and easiest to implement for proper-exponent primal form, with tight balanced primal form being a close second.
- coprime square-free distinct exponent internal forms also occasionally entail unacceptable integer factorization time, but they tend to require less display space than primal forms. However, coprime forms are a helpful point of departure for fewer display forms, and the arithmetic is somewhat slower and harder to implement.
- Internal forms based on imperfect power factorization do not entail integer factorization, but the radicands can become large. Moreover, although input #11 in Table 4 is among the most concise for this particular absurd number, this display form would not be liked by many users on some other examples, and these internal forms are not helpful as a point of departure for most of the display forms that users might want.

Thus, the advantages of proper primal internal form subsume those of coprime square-free distinct exponent internal forms, and in comparison the disadvantages of canonical forms based solely on perfect power factorization don't make up for its better worst-case computing time. These considerations suggest the following hybrid internal representation and algorithmic ideas:

1. Use radicand factorization and proper-exponent primal form up through some prime $\hat{p}$. Composite integer factors exceeding $\hat{p}^2$ are merely perfect-power factored, and the exponents are made proper. Let's call any consequent integer radicand exceeding $\hat{p}^2$ a *quasi-prime*. Tight balanced exponents could be used instead of proper exponents.

2. Treat the quasi-prime factors the same as prime factors, except compute gcds between each new quasi prime and any other quasi primes and a rational denominator and/or numerator whose magnitude exceeds $\hat{p}$. Any resulting non-trivial gcd splits the radicand or radicands and might enable extracting more rational coefficient. This process is illustrated in computations (3) through (6).

3. The resulting form isn't necessarily canonical if it contains a quasi-prime radicand, which will be rather infrequent if $\hat{p}$ is set rather large. However:

   (a) The radicands are always coprime to each other and the numerator and denominator of any rational factor.

   (b) Addition, multiplication and rational powers of absurd numbers always yield a single absurd number if the result can be so represented, and therefore 0 is always recognized in such results.

3 Conclusions

Some major computer algebra systems currently produce erroneous results that could be prevented by transforming absurd numbers to a canonical internal form. There are reasonably efficient canonical forms that avoid the potential cost of factoring large integers, and they are not difficult to implement. However the radicands can become large, the arithmetic is slower when there are no large prime factors, and these forms are not good points of departure for popular display forms.

Thus for an *internal* representation we recommend using the proper or tight balanced exponent primal form up to some particular prime base $\hat{p}$, beyond which only perfect power factorization is used, together with gcds to assure 0-recognition and that all factors are coprime.

We think the default display form should be concise. No one form will be the most concise for all examples, but the most concise will often be in the set:

1. a rational number times one minimal integer raised to a proper exponent, as exemplified by input #15;

2. coprime square-free bases raised to distinct exponents − perhaps times a rational number − as exemplified by inputs #5 through #10.

A system could compute all these forms and perhaps others, and then display the most concise one. However, when displaying an expression containing multiple absurd numbers, consistency in the form used for those numbers is also important. Also the prime factorization provided by the pure, loose balanced-exponent, and proper primal forms is particularly informative. Therefore systems should provide a convenient mechanism for users to set the default form used to display absurd numbers. Ideally this default display form setting would be done using a transformation dialog box such as that shown in Section 2.

References

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Table 4: Some alternative forms for the same absurd number:

#	**Ratio form**	**Product form**	**Coef**	**Name of form**
				(Bases are primes:)
1	$\dfrac{2^{4/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$2^{4/3}\,3^{-2/3}\,5^{-2/3}\,7^{2/3}$	± 1	pure primal
2	$\dfrac{2\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}}{15}$	$\frac{2}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{2/3}$	$\mathbb{Q}$	proper-exponent primal
3	$\dfrac{14\cdot 2^{1/3}\,3^{1/3}\,5^{1/3}}{15\cdot 7^{1/3}}$	$\frac{14}{15}\,2^{1/3}\,3^{1/3}\,5^{1/3}\,7^{-1/3}$	$\mathbb{Q}$	tight balanced-exponent primal
4	$\dfrac{2\cdot 2^{1/3}\,7^{2/3}}{3^{2/3}\,5^{2/3}}$	$2\cdot 2^{1/3}\,3^{-2/3}\,5^{-2/3}\,7^{2/3}$	$\mathbb{Q}$	loose balanced-exponent primal
				(Bases are coprime and square-free:)
5	$\dfrac{2^{4/3}\,7^{2/3}}{15^{2/3}}$	$2^{4/3}\,7^{2/3}\,15^{-2/3}$	± 1	coprime square free integer bases, distinct exponents
6	$\dfrac{2\,7^{2/3}\,30^{1/3}}{15}$	$\frac{2}{15}\,7^{2/3}\,30^{1/3}$	$\mathbb{Q}$	coprime square free integer bases, distinct proper exponents
7	$\dfrac{14\,30^{1/3}}{15\,7^{1/3}}$	$\frac{14}{15}\,7^{-1/3}\,30^{1/3}$	$\mathbb{Q}$	coprime square free integer bases, distinct tight balanced exponents
8	$\dfrac{2\cdot 2^{1/3}\,7^{2/3}}{15^{2/3}}$	$2\cdot 2^{1/3}\,7^{2/3}\,15^{-2/3}$	$\mathbb{Q}$	coprime square free integer bases, distinct loose balanced exponents
9	$2\cdot 2^{1/3}\left(\frac{7}{15}\right)^{2/3}$	$2\cdot 2^{1/3}\left(\frac{7}{15}\right)^{2/3}$	$\mathbb{Q}$	coprime square free rational bases, distinct proper exponents
10	$\frac{14}{15}\left(\frac{30}{7}\right)^{1/3}$	$\frac{14}{15}\left(\frac{30}{7}\right)^{1/3}$	$\mathbb{Q}$	coprime square free rational bases, distinct tight balanced exponents
				(Bases are imperfect powers:)
11	$\left(\frac{28}{15}\right)^{2/3}$	$\left(\frac{28}{15}\right)^{2/3}$	± 1	single rational imperfect power base, positive exponent
12	$\dfrac{28^{2/3}}{15^{2/3}}$	$15^{-2/3}\,28^{2/3}$	± 1	ratio of two imperfect power integer bases, positive exponents
				(Maximal reciprocal exponents:)
13	$\left(\frac{784}{225}\right)^{1/3}$	$\left(\frac{784}{225}\right)^{1/3}$	± 1	single rational base, maximal positive reciprocal exponent
14	$\dfrac{784^{1/3}}{225^{1/3}}$	$225^{-1/3}\,784^{1/3}$	± 1	ratio of two integer bases, maximal positive reciprocal exponents
				(One integer base:)
15	$\dfrac{2\,1470^{1/3}}{15}$	$\frac{2}{15}\,1470^{1/3}$	$\mathbb{Q}$	single minimal integer base, proper exponent
16	$\dfrac{11760^{1/3}}{15}$	$\frac{1}{15}\,11760^{1/3}$	$\mathbb{Q}$	single integer imperfect power base, proper exponent