Linear differential, difference and q-difference homogeneous equations having no rational solutions

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Abstract

Some changes of the traditional scheme for finding rational solutions of linear differential, difference and q-difference homogeneous equations with rational coefficients are proposed. In many cases these changes allow one to predict the absence of rational solutions in an early stage of the computation.

1 Introduction

Finding rational solutions (i.e., rational-function solutions) of linear differential, difference and q-difference equations is a part of various algorithms. Investigations of new ways to construct such solutions are quite valuable for computer algebra. The first algorithms for constructing rational solutions were proposed in [12] (the differential case) and in [2], [3] (the difference and q-difference cases). Later, further algorithms were proposed ([6], [10], [7] etc). The algorithms fit in the following scheme that we call RS:

RS1: Construct a rational function $R(x)$ such that any rational solution of the original equation can be represented as $R(x)f(x)$ with polynomial $f(x)$.

RS2: Transform the original equation into such an equation that a polynomial $f(x)$ is a solution of the transformed equation iff $R(x)f(x)$ is a solution of the original equation.

RS3: Construct all polynomial solutions of the transformed equation.

Many equations (even a "majority" of them) have no (non-zero) rational solutions. However if one uses the above scheme then the absence of such solutions will be recognized only in the last step of computation when steps RS1, RS2 are completely executed (strictly speaking in the differential case the algorithms from [6], [7] are an exception; we will say more about this in the end of Section 3). Below we discuss some changes in the scheme RS for the case of homogeneous equation $L(y) = 0$ with rational coefficients. In any case these changes do not increase the computation cost, but allow one quite often to predict the absence of rational solutions in an early stage of computation and to stop the work.

We suppose that the original linear operator L (differential, difference or q-difference) has coefficients which are rational functions over a field k of characteristic 0, and consider non-zero polynomial and rational solutions of $L(y) = 0$, i.e., solutions belonging to $k[x] \setminus \{0\}$ and

$k(x) \setminus \{0\}$. For short, we will refer to these solutions as the polynomial and rational solutions of the operator L.

The so-called indicial equation at infinity $I(\lambda) = 0$ is associated with the operator L. In the difference and differential case $I(\lambda)$ is a polynomial in λ over k , and $I(\lambda)$ is a polynomial in q^λ in the q-difference case. The known algorithms use the indicial equations to find a bound for degrees of all polynomial solutions of an operator and finds all polynomial solutions of the operator using this bound ([1], [4], [11], [8] etc). However this indicial equation gives more information on the operator L than just a bound for degrees of polynomial solutions. The fact is that if L has a solution $S(x) = \frac{s_1(x)}{s_2(x)}$, $s_1(x), s_2(x) \in k[x]$, then the integer number

$$\mathrm{val}_\infty S(x) = \deg s_1(x) - \deg s_2(x)$$

(the *valuation* of $F(x)$ at infinity), is a root of the indicial equation. For the differential case this was noted in [6, Lemma 1]. It is possible that this fact is (well) known for the difference and q-difference cases as well, but our attempts to find corresponding references have not met with success. That is why we incorporate to this paper the preliminary Section 2 where a proof of this fact (in a unified way for three types of operators) as well as a formal definition of the indicial equation are given. The mentioned fact is the main property of the indicial equation in the context of our paper. Of course, if $f(x)$ is a polynomial solution of L then it follows from this property that $\deg f(x)$ is a root of the indicial equation, since $\mathrm{val}_\infty f(x) = \deg f(x)$ in this case. But moreover this property allows us to improve the traditional scheme RS (Section 3). In Section 4 some additional changes related to the difference case are proposed.

2 Indicial equations at ∞

We will suppose that the operator is of the form

$$a_n(x)\delta^n + \cdots + a_1(x)\delta + a_0(x), \tag{1}$$

where $a_1(x), \ldots, a_{n-1}(x) \in k(x)$, $a_0(x), a_n(x) \in k(x) \setminus \{0\}$, $\delta = D = \frac{d}{dx}$ in the differential case, $\delta = \Delta$ in the difference case ($\Delta(y(x)) = y(x+1) - y(x)$), and $\delta = Q$ in the q-difference case ($Q(y(x)) = y(qx)$). If $F(x) \in k(x)$ and

$$F(x) = c\,\frac{f(x)}{g(x)}, \tag{2}$$

where $c \in k$ and $f(x), g(x)$ are monic polynomials then we write $c = \mathrm{lc}\,F(x)$.

Proposition 1. *Let L be as in (1). Let in differential and difference cases the number ω and the polynomial $I(\lambda)$ be*

$$\omega = \max_{0 \leqslant j \leqslant n} (\mathrm{val}_\infty a_j - j), \qquad I(\lambda) = \sum_{\substack{0 \leqslant j \leqslant n \\ \mathrm{val}_\infty a_j - j = \omega}} \mathrm{lc}(a_j)\lambda^{\underline{j}} \tag{3}$$

($\lambda^{\underline{j}} = \lambda(\lambda-1)\ldots(\lambda-j+1)$), and in the q-difference case

$$\omega = \max_{0 \leqslant j \leqslant n} \mathrm{val}_\infty a_j, \qquad I(\lambda) = \sum_{\substack{0 \leqslant j \leqslant n \\ \mathrm{val}_\infty a_j = \omega}} \mathrm{lc}(a_j) q^{\lambda j}. \tag{4}$$

Then $\mathrm{val}_\infty L(F) \leqslant \mathrm{val}_\infty F(x) + \omega$ for any $F(x) \in k(x) \setminus \{0\}$, and strict inequality takes place iff $I(\mathrm{val}_\infty F(x)) = 0$.

Proof. Let $F(x)$ be as in (2) and $\deg f = u$, $\deg g = v$, $m = u-v$. One can prove by induction on j that

$$D^j(F(x)) = c\,\frac{m^{\underline{j}}\,x^{u+j(v-1)} + \dots}{g^{j+1}(x)} = c\,\frac{m^{\underline{j}}\,x^{u+nv-j} + \dots}{g^{n+1}(x)},$$

$$\Delta^j(F(x)) = c\,\frac{m^{\underline{j}}\,x^{u+j(v-1)} + \dots}{g(x)g(x+1)\dots g(x+j)} = c\,\frac{m^{\underline{j}}\,x^{u+nv-j} + \dots}{g(x)g(x+1)\dots g(x+n)},$$

$$Q^j(F(x)) = c\,\frac{f(q^j x)}{g(q^j x)} = c\,\frac{q^{\frac{n(n+1)}{2}v+jm}x^{u+nv} + \dots}{g(x)g(qx)\dots g(q^n x)}$$

for $j = 0, 1, \dots$ (dots in the numerators hide lower terms). Suppose that all $a_j(x)$ are polynomials. We have then for $j = 0, 1, \dots, n$:

$$a_j(x)D^j(F(x)) = c\,\frac{\mathrm{lc}(a_j)m^{\underline{j}}\,x^{u+nv+\deg a_j-j} + \dots}{g^{n+1}(x)},$$

$$a_j(x)\Delta^j(F(x)) = c\,\frac{\mathrm{lc}(a_j)m^{\underline{j}}\,x^{u+nv+\deg a_j-j} + \dots}{g(x)g(x+1)\dots g(x+n)},$$

$$a_j(x)Q^j(F(x)) = c\,\frac{\mathrm{lc}(a_j)q^{\frac{n(n+1)}{2}v+jm}x^{u+nv+\deg a_j} + \dots}{g(x)g(qx)\dots g(q^n x)}.$$

The corresponding expression for $L(F(x))$ has the numerator $cI(m)x^{u+nv+\omega} + \dots$, in the differential and difference cases and $cq^{\frac{n(n+1)}{2}v}I(m)x^{u+nv+\omega} + \dots$ in the q-difference case. This proves the case of polynomial coefficients. Let $a_j(x) = \frac{\tilde{a}_j(x)}{w(x)}$, where $w(x)$ all $\tilde{a}_j(x)$ are polynomials, $w(x)$ is a monic polynomial, $\tilde{L} = \tilde{a}_n(x)\delta^n + \dots + \tilde{a}_1(x)\delta + \tilde{a}_0(x)$, and $\tilde{\omega}, \tilde{I}(\lambda)$ correspond to $\tilde{L}$. The statement holds for $\tilde{L}, \tilde{\omega}, \tilde{I}(\lambda)$. Since $\mathrm{val}_\infty L(F(x)) = \mathrm{val}_\infty \tilde{L}(F(x)) - \deg w(x) \leqslant \mathrm{val}_\infty F(x) + \tilde{\omega} - \deg w(x) = \mathrm{val}_\infty F(x) + \omega$ and $I(\lambda) = \tilde{I}(\lambda)$, it holds also for $L, \omega, I(\lambda)$. $\square$

Definition 1. *Let L be as in (1) and $I(\lambda)$ as in (3), (4). Then the equation $I(\lambda) = 0$ is the* indicial equation *of L at infinity.*

Let $S(x) \in k(x)$. Then $S(x)$ is non-zero iff $\mathrm{val}_\infty S(x) \in \mathbb{Z}$ (by definition $\mathrm{val}_\infty 0 = -\infty$). Therefore $L(F(x)) = 0$ iff $\mathrm{val}_\infty L(F(x)) \notin \mathbb{Z}$. As a consequence of the proven proposition we get the main statement of this section:

Proposition 2. *Let L be of the form (1) and $L(F(x)) = 0$ for $F(x) \in k(x) \setminus \{0\}$. Let $I(\lambda) = 0$ be the indicial equation of L at ∞. Then $I(\mathrm{val}_\infty F) = 0$.*

Remark 1. *Let $K = LF$, where F is a non-zero rational function, and $I_L(\lambda) = 0, I_K(\lambda) = 0$ be the indicial equations of L and K. Then one can prove that up to a constant non-zero factor*

$$I_L(\lambda + \mathrm{val}_\infty F(x)) = I_K(\lambda). \tag{5}$$

In the q-difference case this follows directly from the formula for $a_j(x)Q^j(F(x))$ given above. In the differential and difference cases the corresponding version of Leibniz rule together with the equality $(\lambda+\mu)^{\underline{n}} = \sum_{i=0}^n \binom{n}{i}\lambda^{\underline{i}}\,\mu^{\underline{n-i}}$ is used for this. Evidently $L(F) = 0$ iff $K(1) = 0$. So if $L(F) = 0$ then the equation $K(y) = 0$ has a polynomial solution of degree 0, and therefore $I_K(0) = 0$. By (5) we have $I_L(\mathrm{val}_\infty F(x)) = 0$. This is another proof of Proposition 2.

Example 1. *Let $L = (x+2)\phi - x$. We have $I(\lambda) = \lambda + 2$. Therefore if L has a non-zero rational solution $y(x)$ then $\mathrm{val}_\infty y(x) = -2$. One can check that any rational function $\frac{C}{x(x+1)}$ with constant C is a solution of L. Since $\mathrm{ord}\, L = 1$ this operator has no extra rational solution.*

3 A new scheme

Step RS2 results in the equation with the operator $M = LR$ (the product of L and the zero order operator R). The polynomial solutions of M have to be found in step RS3. As we have mentioned, this search is as follows. First an upper bound $d \geqslant 0$ for degrees of all polynomial solutions has to be found using the indicial equation of M, and, second, one finds all polynomial solutions of M using this bound.

In the sequel we will denote the indicial equations of L and $M = LR$ at ∞ by $I_L(\lambda) = 0$ and $I_M(\lambda) = 0$, respectively.

Proposition 3. *If $I_L(\lambda) = 0$ has no integer roots then M has no polynomial solutions and L has no rational solutions. If this indicial equation has integer roots and λ_0 is the maximal one then the inequality $\deg f(x) \leqslant \lambda_0 - \mathrm{val}_\infty R(x)$ is valid for any polynomial solution $f(x)$ of M (if $\lambda_0 - \mathrm{val}_\infty R(x) < 0$ then there is no polynomial solutions).*

Proof. If $I_L(\lambda) = 0$ has no integer roots then, by the main property of the indicial equation, M has no rational solutions and hence no polynomial solutions. Assume that $y(x)$ is a rational solution of L. Then it follows from RS1 and RS2 that $y(x) = R(x)f(x)$ for some polynomial $f(x)$, and so $M(f(x)) = (LR)(f(x)) = L(R(x)f(x)) = L(y(x)) = 0$. But M has no polynomial solutions, a contradiction. So we conclude that L has no rational solutions.

Now let us prove the second part. If M has a polynomial solution $f(x)$ then L has the rational solution $R(x)f(x)$. Thus

$$\deg f(x) + \mathrm{val}_\infty R(x) = \mathrm{val}_\infty f(x) + \mathrm{val}_\infty R(x) = \mathrm{val}_\infty (R(x)f(x)) \;\leqslant\; \lambda_0.$$

The inequality $\deg f(x) \leqslant \lambda_0 - \mathrm{val}_\infty R(x)$ follows. □

This proposition makes feasible an improvement of the scheme RS given in Introduction. One can start with constructing the indicial equation $I_L(\lambda) = 0$ of the original operator L. We get the scheme RS′:

RS′0: Construct the equation $I_L(\lambda) = 0$; if it does not have integer roots then STOP otherwise define λ_0 as its maximal integer root.

RS′1: Construct a rational function $R(x)$ such that any rational solution of L can be represented as $f(x)R(x)$ with polynomial $f(x)$; set $d = \lambda_0 - \mathrm{val}_\infty R(x)$; if $d < 0$ then STOP.

RS′2: Construct $M = LR$.

RS′3: Construct all polynomial solutions of M taking into account that degree of each of them is $\leqslant d$.

Note that on the step RS1 of the scheme RS some of algorithms related to the differential case $L = a_n(x)\frac{d^n}{dx^n} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$ (the coefficients are polynomials) one can recognize that the rational solutions do not exists, due to the absence of integer roots of the indicial equations of L of another kind, i.e., the indicial equations at irreducible factors of $a_n(x)$ ([6]). This can be combined with the step RS′1. (For example the indicial equations at those irreducible factors and at ∞ can be considered in a random order.)

However the known algorithms for the difference case always produce a rational function $R(x)$ on the step RS1. In the next section we propose another additional trick for the differential case.

4 Additional STOPs in the difference case

Now we turn to the step RS′1. On occasion in the difference case the conclusion that rational solutions of L do not exist may be made in the beginning of constructing $R(x)$. The operator L of the form (1) with $\delta = \Delta$ can be transformed into $a'_n(x)\phi^n + \cdots + a'_1(x)\phi + a'_0(x)$, where $\phi(y(x)) = y(x+1)$, and $a'_1(x), \dots, a'_{n-1}(x) \in k(x)$, $a'_0(x), a'_n(x) \in k(x) \setminus \{0\}$. After left multiplication of L by a suitable polynomial we get an operator

$$u_n(x)\phi^n + \cdots + u_1(x)\phi + u_0(x), \quad \phi(y(x)) = y(x+1),$$

with polynomial coefficients. Numerous algorithms ([2], [3], [8], [9], [5] etc) in a preliminary stage of constructing $R(x)$ compute the *dispersion* $\operatorname{dis}(u_n(x-n), u_0(x))$ of polynomials $u_n(x-n)$ and $u_0(x)$, i.e., the maximal non-negative integer h such that $u_n(x-n)$ and $u_0(x+h)$ are not co-prime. If there are no such non-negative integers then $h = -\infty$. It is proven in [2] that if $h = -\infty$ then L has no solutions in $k(x) \setminus k[x]$ (one can use 1 as $R(x)$ in RS′3). Otherwise it is possible to construct $R(x)$ in the form $\frac{1}{U(x)}$ with a polynomial $U(x)$ which is a *universal denominator* (but in some cases other functions $R(x)$ can also be taken).

Proposition 4. *Let $h = \operatorname{dis}(u_n(x-n), u_0(x)) \geqslant 0$, and let λ_0 be as in Proposition 3. In this case, if the inequality*

$$\lambda_0 + (h+1)\min\{\deg u_0(x), \deg u_n(x)\} \geqslant 0 \tag{6}$$

is not valid then L has no rational solutions.

Proof. Considering the algorithm from [3] it is easy to see that this algorithm results in the universal denominator $U(x)$ such that $\deg U(x) \leqslant (h+1)\min\{\deg u_0(x), \deg u_n(x)\}$. We do not suppose that $R(x)$ is computed by the algorithm from [3], but nevertheless the rational function $R(x) = \frac{1}{U(x)}$ *may be* used in RS′2. By Proposition 3 the inequality (6) has to be valid if L has rational solutions. □

Therefore in the difference case the step RS′1 can be used in the following form:

RS′1: Execute a preliminary stage of constructing $R(x)$ which includs the computation $h = \operatorname{dis}(u_n(x-n), u_0(x))$; if $h = -\infty$ and $\lambda_0 \geqslant 0$ then set $M = L$, $d = \lambda_0$ and go to RS′3; if $h = -\infty$ and $\lambda_0 < 0$ then STOP; if (6) is not valid then STOP; terminate constructing $R(x)$; set $d = \lambda_0 - \operatorname{val}_\infty R(x)$; if $d < 0$ then STOP.

Example 2. *For $L = 2(x+2)\phi + (2x+3)$ the indicial equation is $2\lambda + 1 = 0$. There are no integer roots. Thus L has no rational solutions.*

For $L = (x+1)(x^2+1)\phi - x(x^2-4x+1)$ the indicial equation is $\lambda + 5 = 0$. We have $\lambda_0 = -5$, $h = 0$. Inequality (6) is not valid. Thus L has no rational solutions.

For $L = (x+2)\phi - x$ (the same operator as in Example 1) we have $\lambda + 2 = 0$, $\lambda_0 = -2$, $h = 1$. Inequality (6) is valid. If the algorithm from [3] is used, then we get $R(x) = \frac{1}{x(x+1)}$. We have $-2 - \operatorname{val}_\infty R(x) = 0$. Thus 0 is an upper bound for degrees of polynomial solutions of $LR = \frac{1}{x+1}\phi - \frac{1}{x+1}$. Any constant is a polynomial solution of this operator.

Note that the situation is similar in the q-difference case.

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