It is to be noted that for the case $e = 0$, $n = b + 2c + 3d$, the end binomial coefficient reduces to $\binom{-1}{-1}$ which is to be taken as 1.

Problem 68-17, *A Definite Integral*, by B. F. LOGAN, C. L. MALLOWS and L. A. SHEPP (Bell Telephone Laboratories).

Evaluate the integral

$$I = \int_0^\infty e^{-w/2}\sqrt{w}\,du,$$

where

$$w = \frac{u}{1 - e^{-u}}.$$

The integral arose in a probability problem.

Solution by WILLIAM B. JORDAN (G. E. Knolls Atomic Power Laboratory).

Letting $w = u + v$, it follows that

$$e^u = w/v, \qquad we^{-w} = ve^{-v},$$

$$du = dw/w - dv/v, \qquad u]_0^\infty = w]_1^\infty = v]_1^0.$$

Whence,

$$I = \int_{u=0}^\infty \{we^{-w}\}^{1/2}\left\{\frac{dw}{w} - \frac{dv}{v}\right\}$$

or

$$I = \int_1^\infty \{we^{-w}\}^{1/2}\frac{dw}{w} - \int_1^0 \{ve^{-v}\}^{1/2}\frac{dv}{v}$$

$$= \int_0^\infty t^{-1/2}e^{-t/2}\,dt = \sqrt{2\pi}.$$

M. L. GLASSER (Battelle Memorial Institute) obtains the following generalization by means of the generating function for the generalized Laguerre polynomials:

$$\int_0^\infty \{ye^{-y}\}^\nu\,dx = \int_0^\infty \{ye^{-y}\}^\nu\,dy, \qquad \nu > 0,$$

where $y = x/(e^x - 1)$ or $x/(1 - e^{-x})$.

In the solution by the proposers, it was noted that Logan and Shepp obtained the result $(2\pi)^{-1/2}I = 1$ as the expression of the probabilistic fact that a standard Wiener process is almost certain to meet a boundary curve. Additionally, Mallows obtains the still further generalization:

Suppose (i) $h(x)$ and its derivative $h'(x)$ are positive and monotone for $-\infty < x < \infty$, with $h(x) = h(-x) + x$ and $h(-\infty) = 0$. Also suppose (ii) $g(w)$ (defined for $0 \leqq w < \infty$) satisfies $g(h(x)) \equiv g(h(-x))$ for $-\infty < x < \infty$. Then $\int_0^\infty g(h(x))\,dx = \int_0^\infty g(w)\,dw$ (with slight additional generality, for any f we have $\int_0^\infty f(g(h(x)))\,dx = \int_0^\infty f(g(w))\,dw$). The proof is straightforward. In the present case we have $g(w) = (we^{-w})^{1/2}$, $h(x) = x/(1 - e^{-x})$.

In general, if the function $g(h(x))$ is given, it may not be easy to see how to choose h so that (i) and (ii) are satisfied. If h is given satisfying (i) (there are many such functions; it is only necessary that $h'(x) - \frac{1}{2}$ is odd and monotone, with $h(-\infty) = 0$), then it is easy to construct many functions $g(w)$ satisfying (ii). Any function of $|h^{-1}(w)|$ will do; alternatively we may write $g(w) = G(w, w - h^{-1}(w))$ where $G(w, w')$ is any symmetric function. (Then $g(w) = G(h(x), h(-x))$, where $w = h(x)$.) In the present case $G(w, w') = (|w - w'|\min(w, w')/ww')^{1/2}$.