A computer study of symmetry in Celtic interlace patterns

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April 13, 2011

Abstract: I have written software in the MATHEMATICA† computer language to construct interlace patterns and endless knots that occur in geometric art. Application of this software to Celtic designs highlights their symmetry properties that complement those studied in related art of other cultures. The input to my software is mnemonic and the commands can be extended to support many further kinds of geometical construction, of use in the description of art objects and "discovery" in mathematics education.

Keywords: interlace, endless knots, symbolic calculation, computer algebra, mathematics education.

Introduction

Endless knots, interlace borders, key patterns and other geometrical designs occur in manuscripts, textiles, monuments and other art forms of several cultures. Many articles and books have been published on these topics. Budny's survey that includes explanations of several descriptive notations and provides an extensive bibliography [1] and Bain's educational text [2] have been of particular help to me. The manual operations can be mechanized, using software that performs symbolic calculations and provides graphic output. I have started to write MATHEMATICA scripts for this purpose, that I call ELCON, to further the exchange of ideas between workers in areas of art, mathematics and computer science. Successive sections of this paper (1) display the diagrams produced in the course of constructing an illustrative knot with 4-fold rotational symmetry, (2) explain the input, (3) give another example with 4-fold symmetry, (4) extend to other rotational symmetries, (5) develop an interlace by translation, (6) extend the variety of patterns, and (7) put the work in perspective. Appendixes 1 and 2 summarize the coding. Appendix 3 suggests follow up work.

1 An example of an endless knot

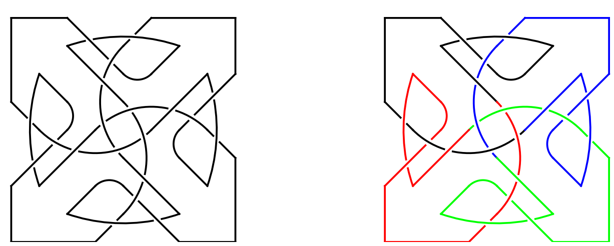

Figure 1: Knot from ENCYCLOPEDIA OF IRISH AND WORLD ART website.

Fig. 1 shows an endless knot that is topologically the same as the main part of the "Example of Celtic Art" near the top of the web site [3]. The colored version emphasises the rotational symmetry, which I use to construct the diagrams. Most of the endless knots I have seen in facsimiles and popular accounts have this symmetry. Fig. 2 shows the geometry of the top left quarter of the knot. To depict the alternation of underpass and overpass along the knot, the points where the top left quarter is crossed by itself and by the other quarter pieces (actually just the top right and lower left) are found by MATHEMATICA statements based on elementary geometry. The results are labeled Q1 to Q9 in Fig. 3. The first quarter piece is redrawn with gaps, to depict underpasses, in Fig. 4. Rotating this by successive quarter turns, and superimposing the results, constructs the endless knot in Fig. 1.

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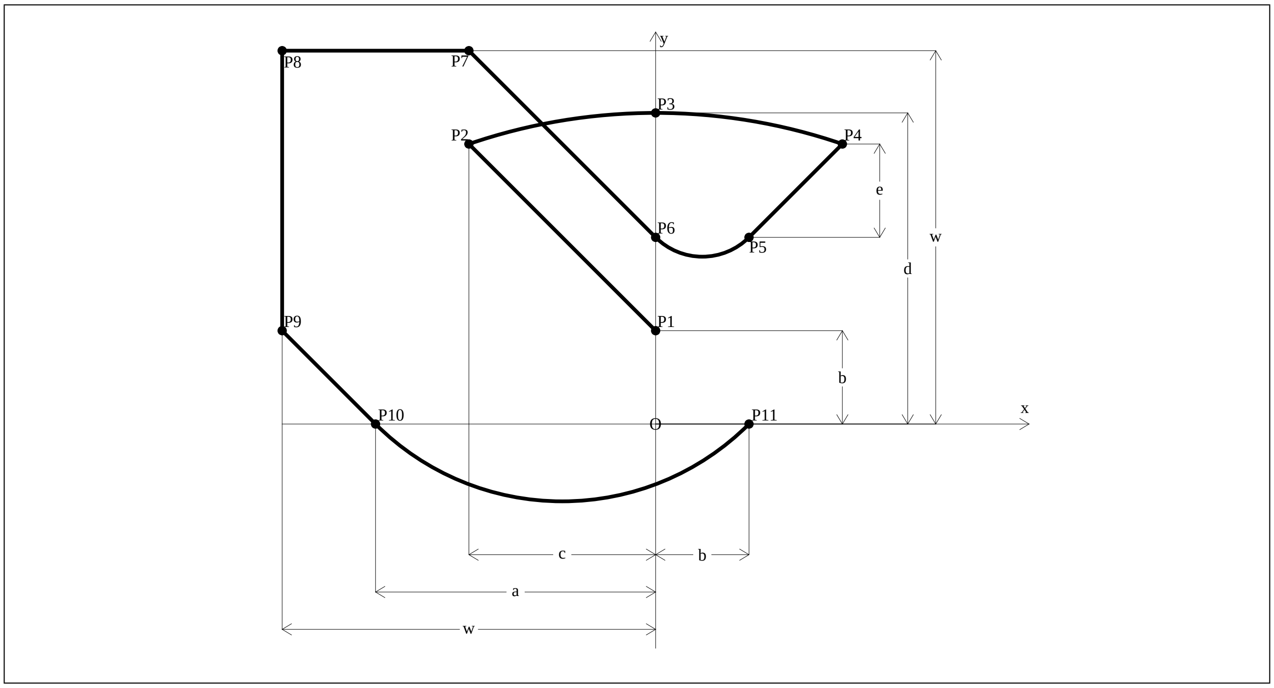

Figure 2: The geometrical construction of the basic unit of repeat.

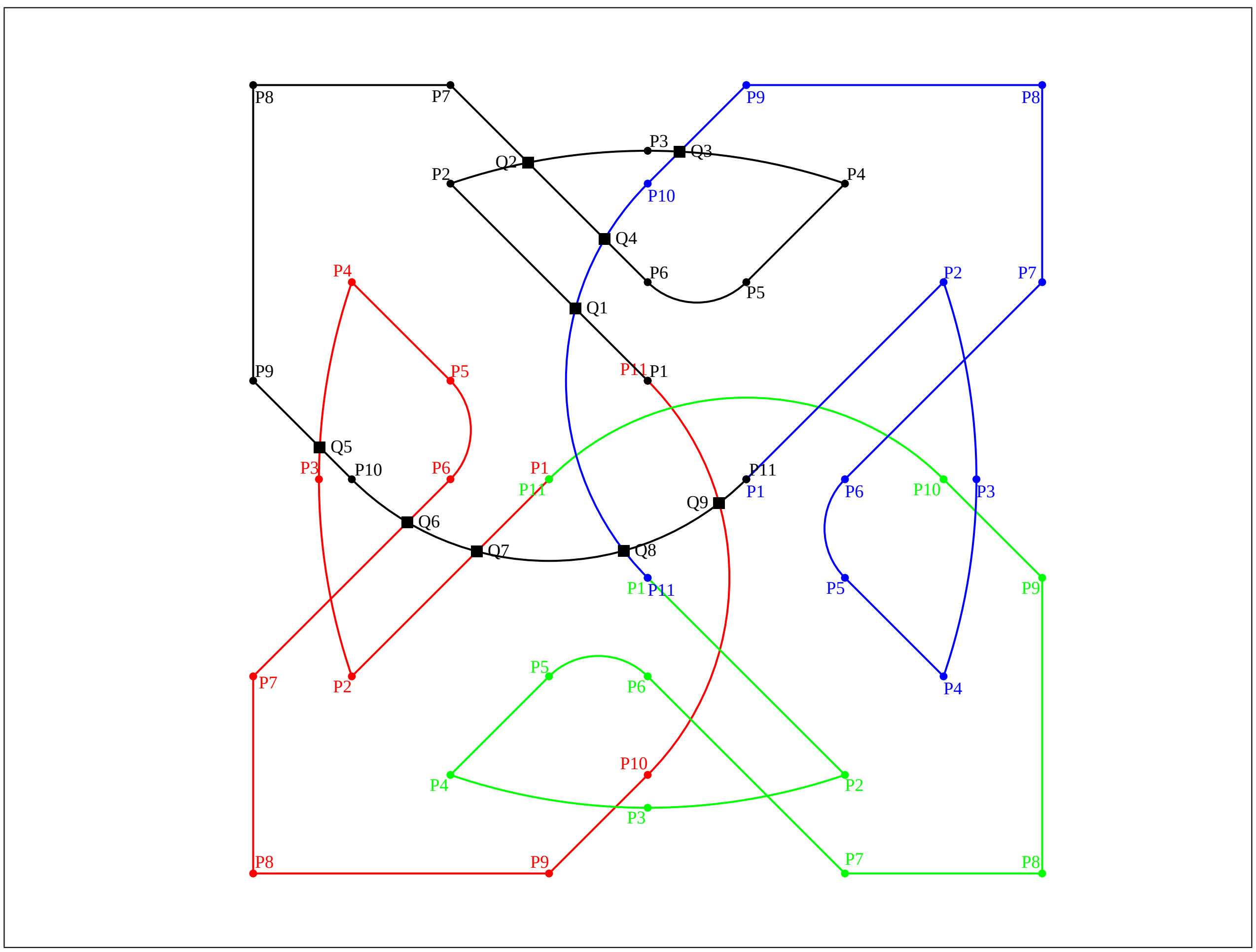

Figure 3: Crossing points.

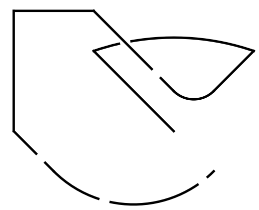

Figure 4: The unit of repeat with gaps.

The choice of the "unit of repeat" that comprises the part shown in black in the right hand knot of Fig. 1 could be changed to any other portion between equivalent points. My choice made the bounding box close to a square. It is rotated by one, two and three quarter turns to produce the parts shown in red, green and blue, counterclockwise.

Each of the straight line segments and circular arcs that make up the unit of repeat are specified by their end points. Another detail is given for each arc, such as a 3^{rd} point through which it passes. Eleven points are needed. These are labeled P1 to P11. Their positions depend on 6 distances, labeled a, b, c, d, e, w, that can be varied slightly without altering the pairs of lines that cross. Units of repeat are taken to be minimal.

Each endless knot displayed in this paper can be changed into another endless knot by switching the overpasses with the underpasses. This pairing corresponds to closing a book on an inked cord that embodies the knot and leaves imprints on facing pages, that are related by reflection about a vertical plane through the spine of the book.

2 The input for the knot from the Encyclopedia of Irish and World Art

The path that joins the points P1 and P11 in Fig. 2 leads to an endless knot, when it is rotated three successive quarter turns about the origin, because P1 and P11 are the same distance from the origin (along the positive y and x axes). Also, the knot is smooth where the basic unit and its successive rotations join, because the undrawn direct line from P1 to P11 is on a diameter of the quarter circle that joins P10 to P11 and, consequently, meets it at a right angle.

The line from P1 to P2 slopes diagonally to the NW. The circular arc from P2 to P4 is adjusted by varying the position of P3. The line from P4 to P5 slopes diagonally to the SW. The arc that joins P5 to P6 continues smoothly from the line that joins P4 to P5. This, in turn, is continued smoothly by the line from P6 to P7, sloping diagonally to the NW. The lines from P7 to P8, and P8 to P9, are horizontal and vertical, respectively. The line from P9 to P10 slopes diagonally to the SE. The arc from P10 to P11 is a smooth continuation of the line from P9 to P10. The arcs from P5 to P6, and from P10 to P11, are quarter circles, but this property is not used explicitly in the current construction.

Elementary geometry provides the following coordinates for the points, that are input as `p[1]` to `p[11]`.

```
points[cw] =
{p[1] -> {0,b}, p[2] -> {-c,b+c}, p[3] -> {0,d}, p[4] -> {c,b+c},
 p[5] -> {c-e,b+c-e}, p[6] -> {0,b+c-e}, p[7] -> {b+c-e-w,w}, p[8] -> {-w,w},
 p[9] -> {-w,-a+w}, p[10] -> {-a,0}, p[11] -> {b,0}};
```

The successive straight and curved pieces of the path are specified mnemonically as follows.

```
lines[cw] =
{seg[1,2] -> Line[{p[1], p[2]}] ,
 arc[2,4] -> arc[through[p[2], p[3], p[4]] ] ,
 seg[4,5] -> Line[{p[4], p[5]}] ,
 arc[5,6] -> arc[continuationOfLineThrough[p[4],p[5]], through[p[6]]] ,
 seg[6,7] -> Line[{p[6], p[7]}] ,
 seg[7,8] -> Line[{p[7], p[8]}] ,
 seg[8,9] -> Line[{p[8], p[9]}] ,
 seg[9,10] -> Line[{p[9], p[10]}] ,
 arc[10,11] -> arc[continuationOfLineThrough[p[9], p[10]], through[p[11]]]};
```

The `Line` expressions are elementary MATHEMATICA objects. The `arc` expressions are interpreted by scripts in the ELCON package. The constants that lead to Figs. 1–4 are

```
constants[cw] = {w -> 200, a -> 3w/4, b -> w/4, c -> w/2, d -> 5w/6, e -> w/4};
```

3 An endless knot associated with the Book of Kells

Fig. 5 (left) shows the topology and threading of the knot that is embossed on the cover of a facsimile reproduction of the Book of Kells [4]. This "is an artist's interpretation of a Book of Kells motif [that] bears a very close resemblance to the roundel that appears second from the left at the base of the canon table on folio 5r (p.9 in the facsimile)" [5].

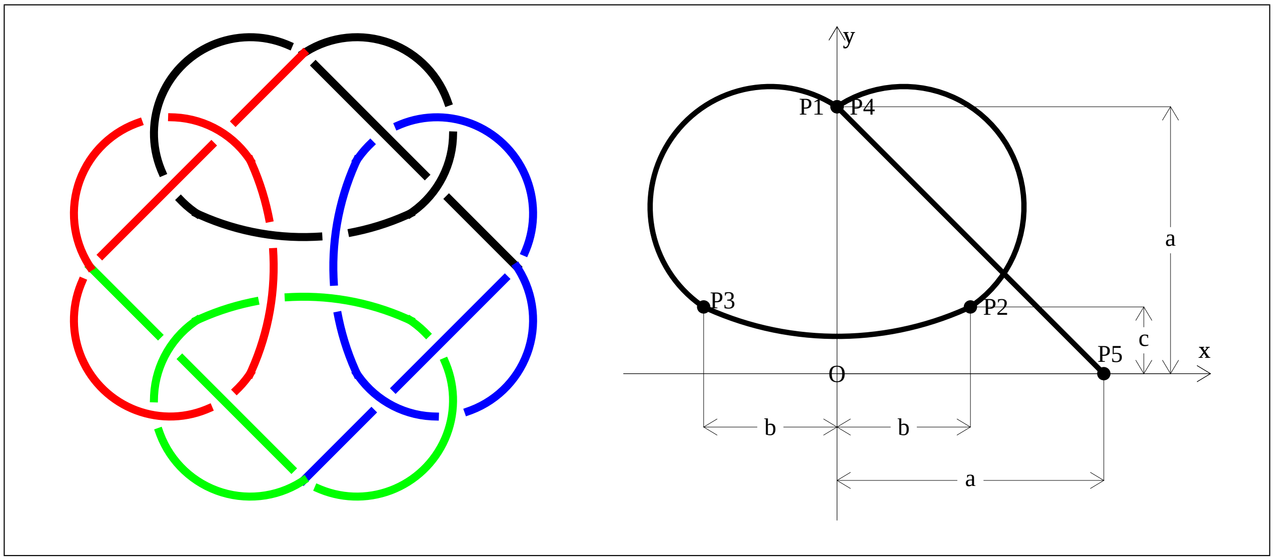

Figure 5: A knot associated with the Book of Kells topology and geometry.

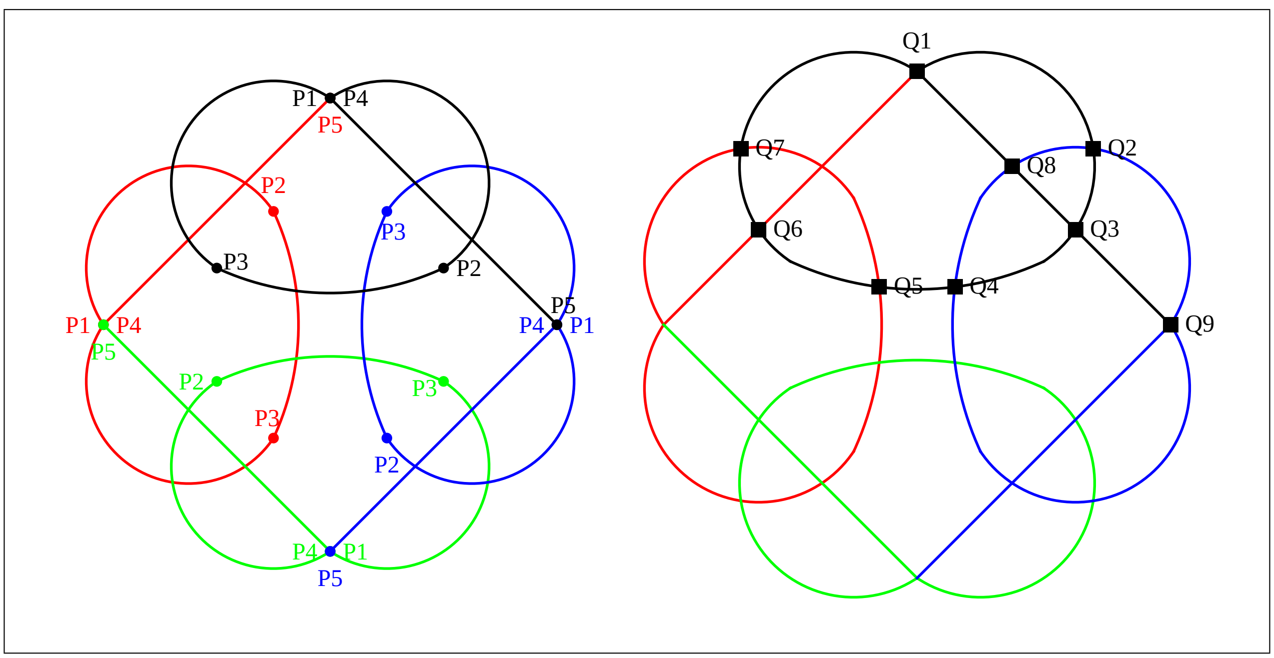

Figure 6: The Book of Kells knot with labeled end and crossing points shown separately.

I split the knot under consideration into the unit of repeat shown by Fig. 5 right. I specify five points, P1 to P5, although P1 and P4 coincide.The software requires this. The coordinates depend on four constants, w, a, b, c. The input uses the `elcon` functions to draw semicircles and quarter circles counter-clockwise, between specified points.

```
constants[bk2] = {w -> 200, a -> w/2, b -> a/2, c -> a/4};
```

```
points[bk2] =
 {p[1] -> {0, a},  p[2] -> {b,c}, p[3] -> {-b,c}, p[4] -> {0,a}, p[5] -> {a,0}};
```

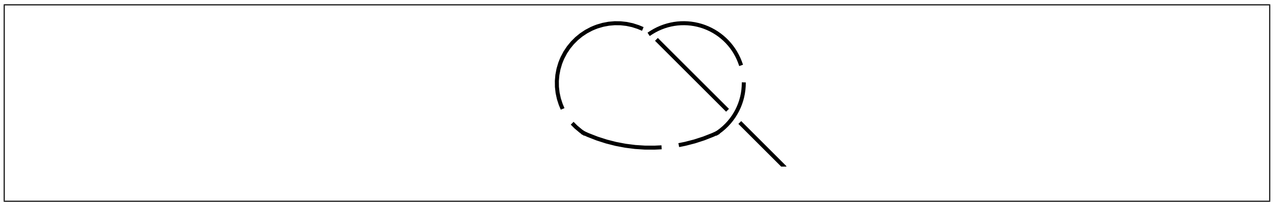

Figure 7: The basic unit of the Book of Kells knot with gaps.

```
lines[bk2] =
{arc[1,2] -> semicircleOn[p[2], p[1]], arc[2,3] -> quarterCircleOn[p[3], p[2]],
 arc[3,4] -> semicircleOn[p[4], p[3]], seg[4,5] -> Line[{p[1], p[5]}]};
```

Fig. 6 (left) shows the end points in the combined unit of repeat and its rotations through three successive quarter turns. Fig. 6 (right) shows the crossing points separately, to avoid clutter. Fig. 7 shows the basic unit with gaps.

4 Other rotational symmetries

Endless knots with 5, 6, 7 and 9 fold rotational symmetry are drawn on pages 30 and 31 of [2]. Fig. 8 shows a pair of interlaced endless paths with 4- and 5-fold rotational symmetry that have a simple relationship.

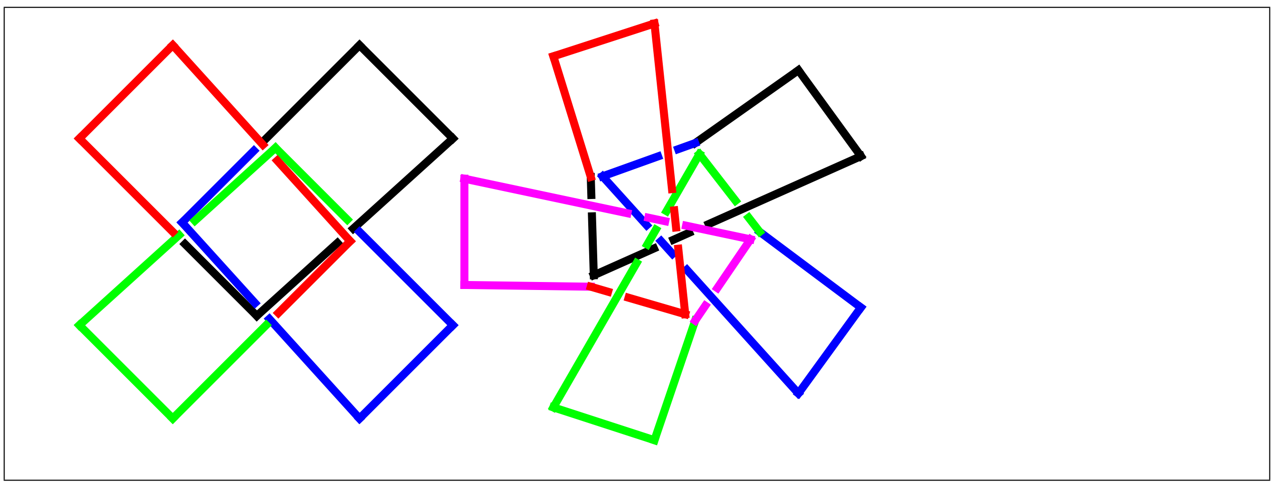

Figure 8: A pair of related interlaced endless paths with 4- and 5-fold rotational symmetry.

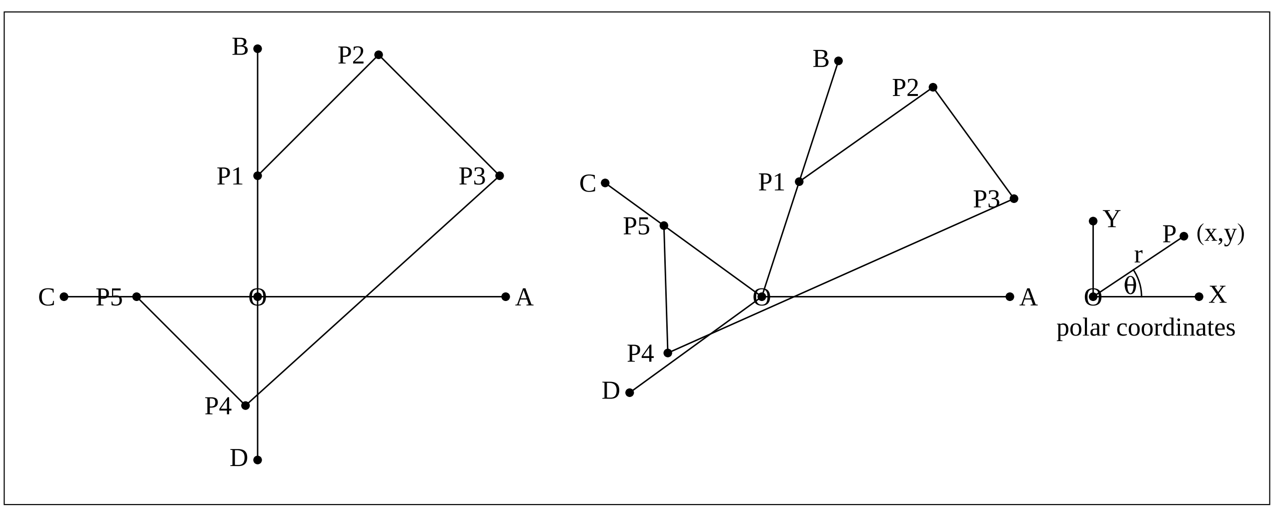

Figure 9: Units of repeat for the “sails”, and the polar coordinate notation.

Fig. 9 shows the units of repeat of the knots in Fig. 8. These units are interconverted using polar coordinates, depicted at the right of Fig. 9. I have taken the use of (x, y) coordinates for granted in earlier sections. These measure the distance of a point from a pair of perpendicular Cartesian axes. The polar coordinates (r, θ) are, respectively, the distance of a point from the origin O, and the angle that it subtends at O with the x-axis of the Cartesian system.

The following rule produced the middle part of Fig. 9.

Construction 1 *The transformation (r, θ) to $(r, 4/5\theta)$ converts the unit of repeat of an endless knot with 4-fold rotational symmetry into the unit for an endless knot with 5-fold symmetry.*

Other rules give units of repeat that provide 5-fold symmetry but are quite different in other respects. I chose Construction 1 because it is very simple to describe and to use. It compresses the angles AOB, AOC and AOD (measured counter clockwise) from $\pi/2$, π and $3\pi/2$ to $2\pi/5$, $4\pi/5$ and $6\pi/5$, and preserves the colinearity of points on the same line through the origin O.

Note that in all the knots of 4-fold symmetry that are displayed in this paper, the starting and ending points of the unit of repeat are the same distance from the origin. Also, these points subtend a right angle $\pi/2$ at the origin. Correspondingly, the end points of the unit of repeat for the 5-fold knot in this section are equidistant from the origin and subtend the angle $2\pi/5$.

In mathematical terms, an endless knot with n-fold rotational symmetry belongs to the cyclic group of order n. This group is denoted by C_n. A necessary (but **not** sufficient) condition for an open path to be a unit of repeat for a C_n knot is that the end points (P_1, P_n) satisfy $OP_1 = OP_n$ and $n \times \angle P_1OP_2/\pi$ is even. This leads immediately to the following pair of rules. (The explanation of the diagrams in §§5,6 can be understood without reading these.)

Construction 2 *The transformation (r, θ) to $(r, m/n\theta)$ converts the unit of repeat of a C_m knot into the unit for a C_n knot, for integer $m, n \geq 3$.*

Let $\mathcal{R}_{n,k} \bullet v$ stand for the rotation of a geometrical object v by $2k\pi/n$ about the origin.

Construction 3 *To produce the C_n interlaced knot from a unit of repeat that satisfies the conditions stated above:*

1. Let $\tau_\ell^{(0)}$, $1 \leq \ell \leq \bar{\ell}$ stand for the $\bar{\ell}$ successive pieces of the unit of repeat. This range of ℓ is implied hereafter.
2. Construct the $\tau_\ell^{(k)} = \mathcal{R}_{n,k} \bullet \tau_\ell^{(0)}$, $0 < k < n$.
3. Let χ_ℓ stand for the list of intersections of $\tau_\ell^{(0)}$ with every $\tau_{\ell'}^{(k)}$ except itself, *i.e.* with $1 \leq \ell, \ell' \leq \bar{\ell}, 0 \leq k < n$ but $\vee(k = 0 \wedge \ell = \ell')$. Construct the χ_ℓ. This counts each intersection of the unrotated unit with itself twice.
4. Sort each list χ_ℓ into the order of occurrence of the intersection points along the unit of repeat.
5. Construct the list $\bar{\chi}$ by concatenating the results of the previous step, and deleting items at **EITHER** every even position or at every odd position. The choice is arbitrary and produces two different knots (see end of §1).
6. Construct η_ℓ from χ_ℓ by deleting any items that came from χ_ℓ and were deleted from $\bar{\chi}$.
7. Construct ω_ℓ by splitting $\tau_\ell^{(0)}$ at the positions listed in η_ℓ if it is not empty, and by copying $\tau_\ell^{(0)}$ if η_ℓ is empty. Omit resulting pieces that would have a zero or negative length.
8. The set $\xi = \{\omega_1, \ldots, \omega_{\bar{\ell}}\}$ is the unit of repeat with gaps.
9. The set$\{\mathcal{R}_{n,k} \bullet \xi \mid 0 \leq k < n\}$ is the interlaced knot.

The τ_ℓ are straight line segments and circular arcs in the present paper, but this restriction can be relaxed.

5 An example of knotwork

Figs. 10 replicates the structure of the prototype "knotwork" on p. 29 of [2]. Figs. 11–12 show the steps in its construction. Movement parallel to the axes is called "translation". Define the "base line" of an open path as the line through its ends. Let d stand for the distance between them. An open path can be used as the unit of repeat if its projection onto the base line closes the gap between the ends (*i.e.* the path bulges). The pattern is constructed by moving the unit along the base line by distance d, repeatedly, as in a simple frieze.

Construction 3 can be converted to the corresponding **Construction 4** for the knotwork by simple text editing. Let $\mathcal{T}_{d,k}$ denote the single translation by distance $k \times d$. Then the following changes produce the new construction:

1. replace "*the C_n interlaced knot*" by "*knotwork*",
2. replace "$\mathcal{R}_{n,k}$" by "$\mathcal{T}_{d,k}$",
3. replace "$0 \leq k < n$" by "$k_1 \leq k \leq k_2$ for some $k_1 < k_2$".

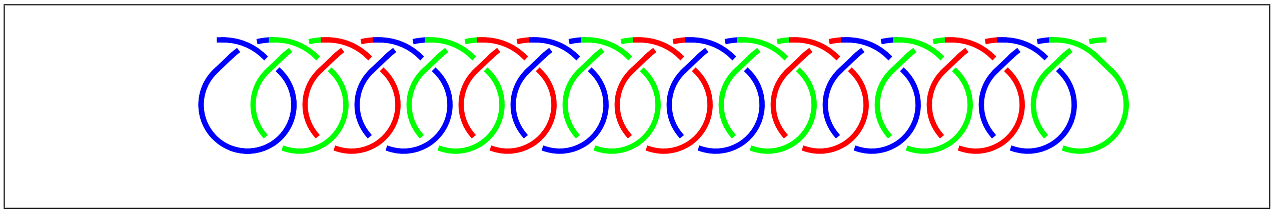

Figure 10: An interlace pattern with translational symmetry.

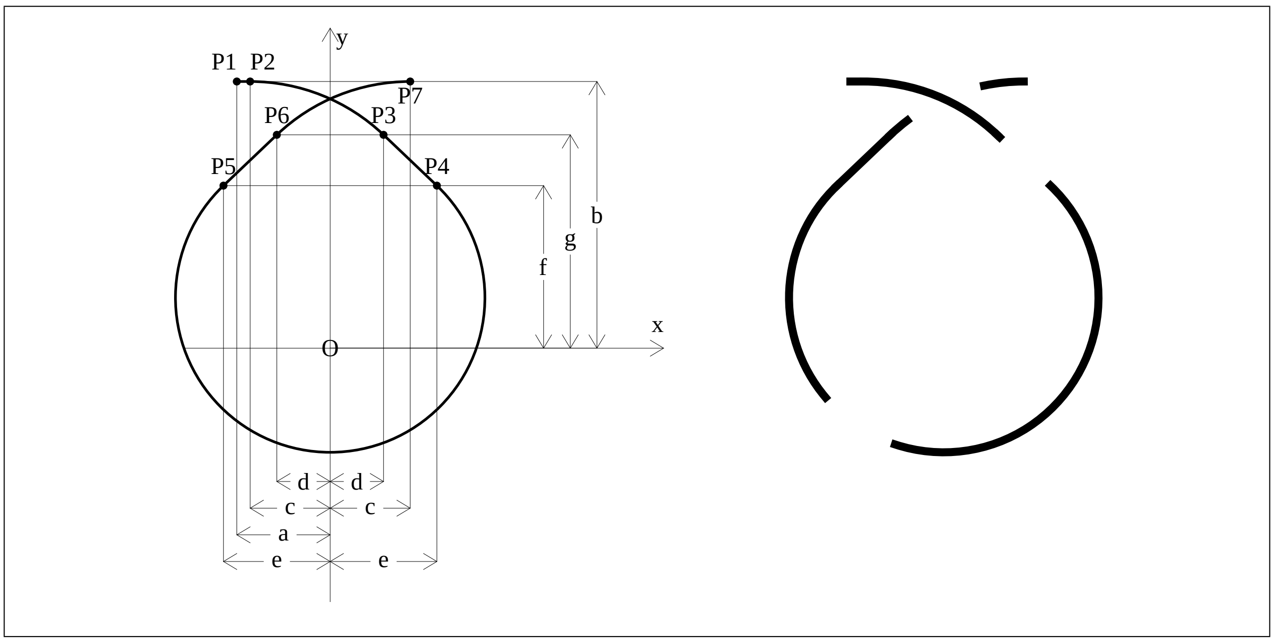

Figure 11: The geometry of the unit of repeat.

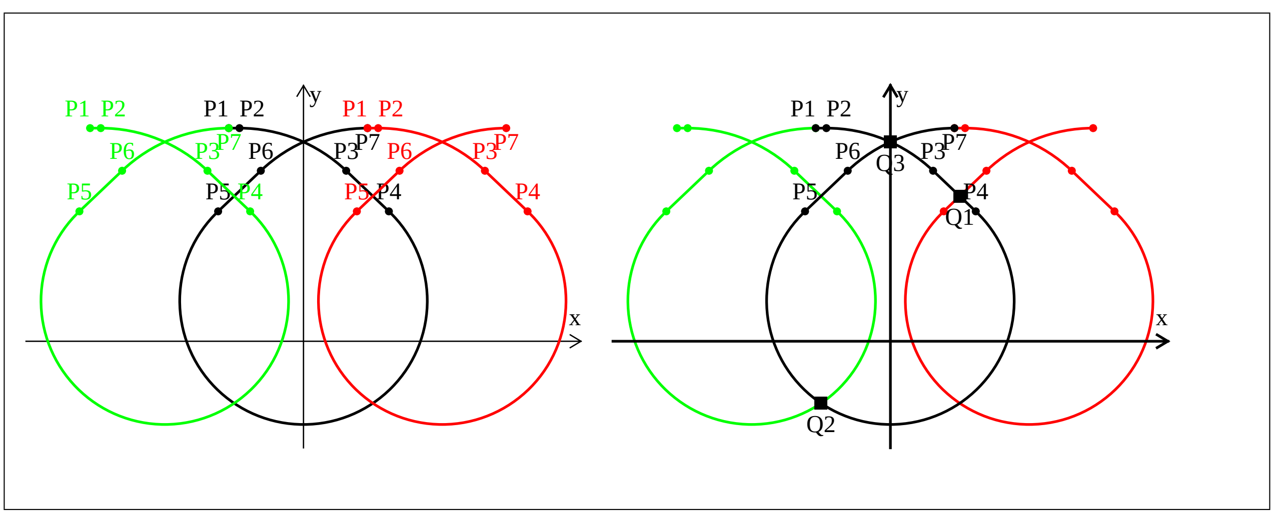

Figure 12: Three consecutive loops.

6 Branching out

The unit of repeat in every example that has been given, so far, is an open path. Changing this prescription leads to major kinds of pattern that have been studied extensively. The unit of repeat in Fig. 13 is just a complete circle, split to allow circles to be linked to form a chain. The Borromean knot that links three rings, and related clusters, occur in Buddhist, Norse and Renaissance art, but I have not found examples in [2], which I assume covers the range of Celtic work. The unit in Fig. 14 consists of two lines that cross each other diagonally. This is the prototype of a vast variety of patterns that resemble braiding and plaiting. Figs. 13 and 14 are very simple patterns that are considered in knot theory. This branch of mathematics is applied to material that overlaps the content of this paper in [8, 9].

Translation in two directions produces borders and the carpet and key patterns that cover an unrestricted area of the plane. The unit of repeat in Fig. 15 consists of four pairs of intersecting lines. An instance, picked arbitrarily, is colored red for explanation. Any other square of the same area could have been chosen anywhere in the plane. The pattern is both an interlacing and a tiling. Texts on tilings include [10].

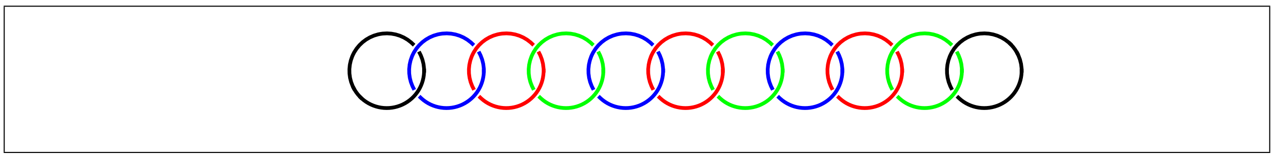

Figure 13: A chain of linked circles.

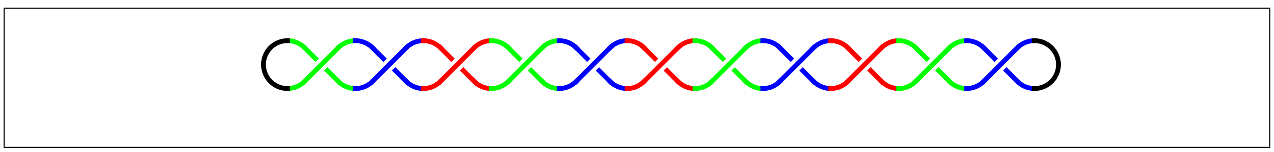

Figure 14: An interlace pattern based on a 2-strand unit of repeat.

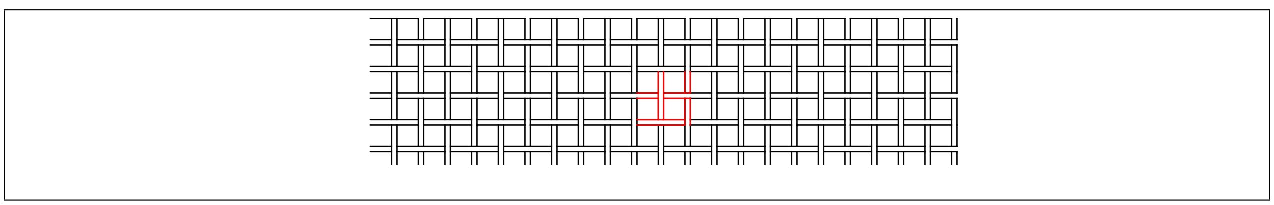

Figure 15: A carpet pattern.

7 Conclusion

I hope that the ELCON software will contribute to the ongoing dialogue between the arts and the sciences that is an important part of the present scholarly scene. My material here is quite close to the consideration of frieze patterns that led to an important paper in the field of "nuclear magnetic resonance" (the basis of MRI) [11], the collection of essays in *The Visual Mind: Art and Mathematics* [12] and du Sautoy's recent study of symmetry [13].

Computer graphics contributes to the study and practice of visual art, in the development of databases of pictures and indexes. The development of software for particular kinds of constructive geometry, using algebraic formulas, can help analyze digitized records of art objects. I plan to extend ELCON to handle spirals, and to support constructions that are written in the style that Stevick uses for a wealth of material (see [14] and earlier work cited therein).

The most detailed studies of the symmetry of interlace has focused on Islamic art. This includes floor mosaics, carpets and other extended surfaces where tiling theory provides a powerful approach [15]. Triskelions, swastikas and other motifs with rotational symmetry have been staples of many visual arts since ancient times. It is reasonable to assume that the mediaeval artists knew that many of the endless knots that they drew have this property. Was this property noted in writing and, if so, how it was described? This question is significant to the history of mathematics.

From an art history viewpoint, might mediaeval artists have used discs of different radii as templates for arcs and complete circles? How might they have transferred shapes from templates?

The use of rotational symmetry to construct the paths of endless knots is implicit in several constructions used by Bain [2], that include the design on the Hilton of Cadbol Stone (plate 14, p.51). However, he consistently leaves the breaking of lines, to depict underpass and overpass, until the final stage of the construction, making the work unnecesarily repetitive. But, as he points out for the Shandwick Stone (plate 13, p.51) the Pictish artist departed from a strict algorithmic design.

The precision of the 16-fold rotational symmetry of Leonardo's "Concatenations" and Durer's "Sechs Knoten", both reproduced in [2], suggest some method of mechanical positioning and copying of the respective units of repeat. Here, too, contemporary written comment would be of interest. Coomaraswamy discusses the symbolism of these designs and comparable materials, in a mid-20$^{\text{th}}$ paper that has been reproduced in a recent collection of essays [16].

Developing an awareness of symmetry is considered important in mathematics education [17]. The software that is reported here can be used to construct a variety of workbook materials that lead into solid and computational geometry and aspects of topology, graph theory and group theory that relate to knots and to tilings. A systematic exploration and classification of the material in [2] is one possible starting point. Beyond this, I hope to build examples of the mechanized use of analogy on the constructions in §§4,5, by extending the methods in [6].

Acknowledgements

I thank Colum Hourihane, Mildred Budny and Robert Stevick for introducing me to aspects of Insular, Celtic and related art and encouraging my work on the subject, and to Felicity O'Mahony for information about the Book of Kells knot.

Appendix 1. The Mathematica infrastructure

The recent versions of MATHEMATICA contain hundreds of graphics commands and options, that are volatile and inadequately documented. I work with a very small subset. Fig. 16 illustrates the very simple conventions of the 5 elementary graphics objects and 3 graphics directives that were used to construct all the diagrams in this note. These are `Point`, `Line`, `Circle`, `Polygon`, `Text`, and `AbsolutePointSize`, `AbsoluteThickness` and `RGBColor`. The PDF file for this figure was written by the statement `export["mma01",`$\{\ell_1, \ell_2, \ell_3, \ell_4\}$`]` where $\ell_1, \ldots, \ell_4$ are short lists of expressions that produced the successive parts of the display. The `export` function, explained at the end of this section, uses parameters that are set in advance, to configure the output. Here, they were set to treat the horizontal median of the screen and the left edge as the x and y axes of a coordinate system that made 1 unit in a graphics expression correspond to 1 printer's point on the typeset page.

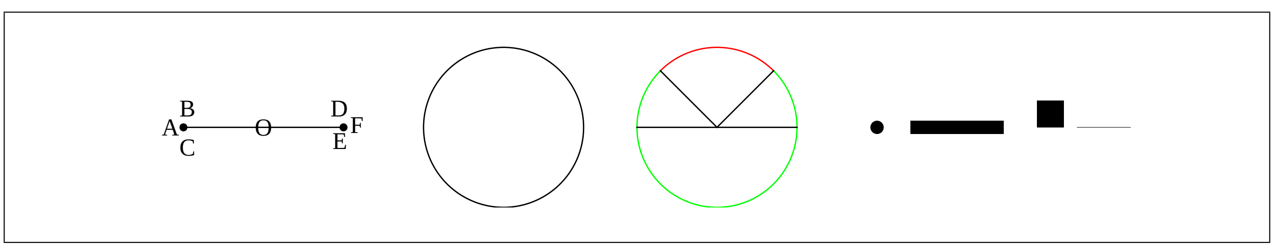

Figure 16: Output of elementary graphics expressions for the explanation of the Mathematica infrastructure. The horizontal line at the left, flanked by letters `A` to `F`, was produced by the expressions

```
Line[{{20, 0}, {80,0}}],           (* draw the horizontal line on the left *)
Point[{20, 0}],                    (* draw the point at its left end *)
Point[{80, 0}],                    (* draw the point at its right end *)
Text["O", {50,0}],                 (* display "O" centered at {50, 0} *)
Text["A", {20, 0}, {1.5, 0}],      (* display "A" offset W of the actual point *)
Text["B", {20, 0}, {-0.5, -1.5}], (* display "B" offset NE *)
Text["C", {20, 0}, {-0.5, 1.5}],  (* display "C" offset SE *)
Text["D", {80, 0}, {0.5,-1.5}],   (* display "D" offset NW *)
Text["E", {80, 0}, {0.5, 1.5}],   (* display "E" offset SW *)
Text["F", {80, 0}, {-2, 0}]       (* display "F" offset E *)
```

In general,

1. `Line[{{`x_1, y_1`},{`x_2, y_2`}}]` draws the straight line segment between the specified positions.
2. `Point[{`x, y`}]` draws the point at the specified position.
3. `Text[`s`,{`x, y`}]` displays the text string s, with its bounding box centered at the specified position.
4. `Text[`s`,{`x, y`},{`σ_x, σ_y`}]` displays the text with the center of its bounding box displaced to the left by σ_x times the width of the box and down by σ_y times the height. The precise dimensions of the bounding box are difficult to determine, and for small scale work, the offsets that meet a local need can be found by trial and error.

The two circles and their contents are drawn by

```
Circle[{140,0}, 30],                            (* empty circle *)
RGBColor[1,0,0],                                (* change to red *)
Circle[{220,0}, 30, {Pi/4, 3 Pi/4}],            (* arc around upper quarter of circle *)
RGBColor[0,1,0],                                (* change to green *)
Circle[{220,0}, 30, {3Pi/4, 9Pi/4}],            (* arc around lower 3/4 of circle *)
RGBColor[0,0,0],                                (* change to black *)
Line[{{190,0}, {250,0}}],                       (* horizontal  diameter *)
Line[{{220,0}, {220+30/Sqrt[2], 30/Sqrt[2]}}],  (* radius pointing NE from center *)
Line[{{220,0}, {220-30/Sqrt[2], 30/Sqrt[2]}}]   (* radius pointing NW from center *)
```

In general,

1. `Circle[{`x, y`},`r`]` draws the outline of a circle centered at x, y with radius r.
2. `RGBColor[`r, g, b`]` switches the hue to the combination of red, green and blue in the ratio r::g::b. It remains in effect until another directive that specifies color is encountered.
3. `Circle[{`x, y`},`r`,{ `θ_1, θ_2`}]` draws the circular arc counterclockwise from the radii that subtend θ_1 and θ_2 with the positive x axis. The system requires $\theta_2 > \theta_1$. When necessary, this is achieved by increasing θ_2 by 2π.

The final part of Fig. 16 is drawn by the statements

```
AbsolutePointSize[5],         (* make diameter of Point objects 5 points *)
Point[{280,0}],               (* draw the large dot *)
AbsoluteThickness[5],         (* make lines 5 points thick *)
Line[{{295,0}, {325,0}}],                          (* draw the thick line *)
Polygon[{{340,0}, {350,0}, {350,10}, {340,10}}], (* draw the square *)
AbsoluteThickness[.1],        (* make lines .1 points thick *)
Line[{{355, 0}, {375, 0}}]}]  (* thin line at right *)
```

In general,

1. `AbsolutePointSize[`n`]` makes subsequent `Point` expressions draw solid circles with diameter n printer's points. These are centered on the specified coordinates.
2. `AbsoluteThickness[`n`]` makes subsequent `Line` expressions draw lines with thickness n printer's points. The x, y values in these expressions specify the end points of the medians of these lines.
3. `Polygon[`$list\, of\, pairs\, of\, coordinates$`]` draws a solid polygon with the specified points as apices.

The leftward extension of the final horizontal line in Fig. 16 goes through the base of the solid square and through the horizontal median lines of the thick dot and line, in accordance with the conventions just listed.

I wrote the `export` function as a wrapper for the built in MATHEMATICA `Export` function. The expression `export[`f, ℓ`]` is interpreted as `Export["`f`".pdf",Show[Graphics[{`ℓ`}, window]]]` where

```
window =
 Sequence[PlotRange->{{0,450},{-320,320}}, AspectRatio->640/450, ImageSize->{450,640}]
```

In general, 1 unit in the `Point`, `Line`, `Circle`, `Polygon` and other graphics primitives corresponds to 1 printer's point of linear measure in a LaTeX document, when the `Export` command contains

$$\texttt{PlotRange->\{\{}x_\ell, x_h\texttt{\},\{}y_\ell, y_h\texttt{\}\}, AspectRatio->}\frac{y_h - y_\ell}{x_h - x_\ell}\texttt{,ImageSize->\{}x_h - x_\ell, y_h - y_\ell\texttt{\}}$$

When an `\includegraphics` command brings the PDF file, written under the control of these parameters, into a LaTeX document, the width and height of its bounding box are $x_h - x_\ell$ and $y_h - y_\ell$. I discuss the sizing of MATHEMATICA graphics, and their inclusion in LaTeX files in [18].

Appendix 2. The elcon infrastructure

I call the file of MATHEMATICA functions that I am writing to perform elementary geometrical constructions ELCON. It is very mnemonic and can be extended in many directions by functions to perform further geometrical actions that provide mnemonic usage and simple coding.

In developing the mnemonics, a compromise is needed between everday usage and precise mathematical terminology. Technically, the objects drawn by the MATHEMATICA `Line` expressions in the preceding section are straight line segments. Mathematically, a line has infinitesimal width and infinite extent, and it is not necessarily straight. I represent a straight line by its classical "normal" equation.

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y + \frac{c}{\sqrt{a^2+b^2}} = 0 \tag{1}$$

This is given *e.g.* as Definition 1 in §5.6 of [19]. Then the triple (a', b', c') for the corresponding specification of the normal to this line through (x_0, y_0) is

$$(a', b', c') = \begin{cases} (b, -a, ay_0 - bx_0) & \text{if } a \neq 0 \,\&\, b \neq 0, \\ (1, 0, -x_0) & \text{if } a = 0 \,\&\, b \neq 0, \\ (0, 1, -y_0) & \text{if } a \neq 0 \,\&\, b = 0. \end{cases} \tag{2}$$

The distance of (x_0, y_0) from the original line is

$$d = \left| \frac{ax_0 + b_0 + c}{\sqrt{a^2+b^2}} \right| . \tag{3}$$

Other useful formulas based on Eq. 2 are given in [19]. Geometrical formulas can be taken for granted by users, once ELCON functions have been written to apply them, but I thought it worth mentioning Eqs. 1–3. This information will be useful for the implentation of corresponding functions in other languages.

Many geometrical tests are needed. Often, it is necessary to determine whether two points A and B coincide, *e.g.* when A is in the explicit specification of one geometrical object, and an ELCON function computes the coordinates of B as the intersection of two lines. Usually, the coordinates are not completely precise, and their internal representations are created in different ways, *e.g.* from a square root and by solving a pair of equations, respectively. Then the software will not find them equal, even though they appear to be, when displayed. Accordingly, I specify a tolerance by `nearEnoughToZero = 10^-10` and use it in

```
samePoint[{x1_,y1_}, {x2_,y2_}] :=
 Abs[x1-x2 // Expand] <= nearEnoughToZero && Abs[y1-y2 // Expand] <= nearEnoughToZero
```

The same numerical tolerance, and the same tactic of expanding and taking the modulus, is used in the corresponding functions with the explicit names `isBetween`, `isOnLine`, `isOnSeg`, `isOnCircle`, `isOnArc`. Many details of ELCON, such as the need to expand, were not anticipated, and learned during the development. Some relate to the inconvenience of certain built-in functions of MATHEMATICA. For example, `ArcTan[x,y]` is discontinuous, going from π to $-\pi + \epsilon$ when (x, y) go from $(-1, 0)$ to $(-1, -\epsilon)$. I wrap it in the ELCON function `aTanf` to give values that increase monotonically around the unit circle.

The following ELCON functions construct the (a, b, c) parameters of Eq. (2) of a mathematical straight line, and the MATHEMATICA `Line` expressions for a straight line segment. They begin `abcOfLine` and `seg`, respectively.

1. `abcOfLine[through[{x1,y1},{x2,y2}]]`,
2. `abcOfLine[normalTo[{a,b,c}], through[{x,y}]]`,
3. `abcOfLine[normalTo[{a,b,c}], through[x]]`,
4. `abcOfLine[perpendicularBisectorOf[{x1,y1},{x2,y2}]]`,
5. `abcOfLine[tangentToCircle[{{x,y},r}], through[{xt,yt}]`,
6. `seg[{x1,y1},{x2,y2}]`,
7. `seg[{a,b,c}, x->{x1,x2}]`,
8. `seg[{a,b,c}, y->{y1,y2}]`,
9. `seg[continuationOfArcThrough[{{x1,y1},{x2,y2},{x3,y3}}, through[x->x0]]]`,
10. `seg[continuationOf[Circle[{x,y},r,{t1,t2}], through[x->x0]]]`

These are largely self-explanatory. Items 2 and 3 return normals through a point off the reference line, and on it, respectively. Item 7 returns the `Line` expression for the segment of the specified line with ends at $x = x_1, x_2$. Item 8 returns the segment with ends at $y = y_1, y_2$. Item 9 and 10 return the expressions for the line segments that continue the arcs smoothly. Item 9 uses item 10. Different contexts make the use of one or other of these more convenient.

The following ELCON functions construct complete circles and circular arcs.

1. `xyrOfCircle[through[{x1,y1},{x2,y2},{x3,y3}]]`,
2. `arc[through[{x1,y1},{x2,y2},{x3,y3}]]`,
3. `semicircleOn[Line[{{x1,y1},{x2,y2}}]]`,
4. `semicircleOn[{x1,y1},{x2,y2}]`,
5. `quarterCircleOn[Line[{{x1,y1},{x2,y2}}]]`,
6. `quarterCircleOn[{x1,y1},{x2,y2}]`,
7. `arc[continuationOfLineThrough[{x1,y1],{x2,y2}], through[{x3,y3}]]`,
8. `arc[continuationOf[Circle[{xc,yc},r,{t1,t2}], through[{x,y}]]]`,
9. `complement[Circle[{x,y},r,{t1,t2}]]`.

Item 1 returns the coordinates of the center and the radius. All the other items return `Circle[{`x,y`},`r`,{`t_ℓ, t_h`}]` expressions. The construction of item 2 shows how to handle the determination of $\{t_\ell, t_h\}$. Make P_1, P_2, P_3 the

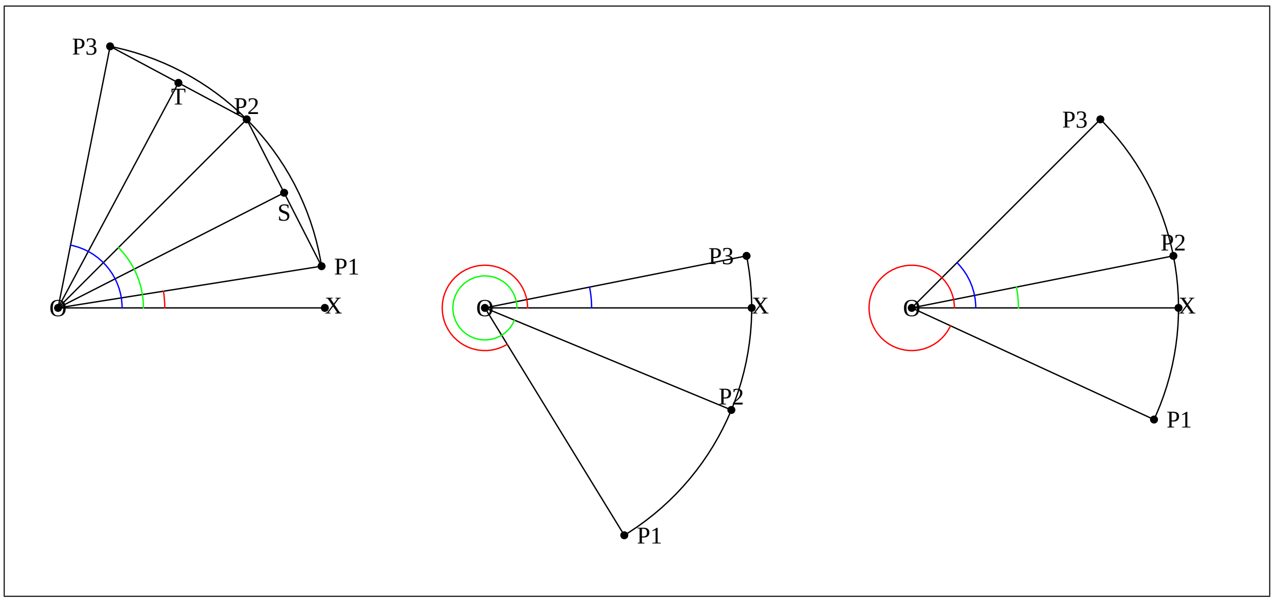

Figure 17: Constructing the arc through 3 points.

names of the points with coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). The center of the arc lies at the intersection of the perpendicular bisectors of the chords P_1P_2 and P_2P_3. This is shown in the first construction in Fig. 17. It is labeled O. The angles $\angle XOP_1$, $\angle XOP_2$ and $\angle XOP_3$ that the radii to P_1, P_2 and P_3 make with right pointing horizontal radius are marked in red, green and blue, respectively. They satisfy the relationship $\angle XOP_1 < \angle XOP_2 < \angle XOP_3$. This corresponds to traveling counter clockwise when going from P_1 to P_2 to P_3 around O. Consequently, $t_\ell = \angle XOP_1$ and $t_h = \angle XOP_3$.

In the second construction, the arc crosses the axis between P_2 and P_3. Consequently, $\angle XOP_3 < \angle XOP_1 < \angle XOP_2$. This is also consistent with the counter clockwise progression from P_1 to P_2 to P_3 around O. However, the MATHEMATICA requirement that $t_\ell < t_h$ is met in this case by setting $t_\ell = \angle XOP_1$ and $t_h = \angle XOP_3 + 2\pi$. In the third construction, the arc crosses the axis between P_1 and P_2. Consequently, $\angle XOP_2 < \angle XOP_3 < \angle XOP_1$. This, too, is consistent with the counter clockwise progression from P_1 to P_2 to P_3 around O. The requirement that $t_\ell < t_h$ is met, again, by setting $t_\ell = \angle XOP_1$ and $t_h = \angle XOP_3 + 2\pi$.

If P_1 and P_3 are interchanged, then none of the three tests for counter clockwise passage from P_1 to P_3 is satisfied. Then $t_\ell = \angle XOP_3$ and $t_h = \angle XOP_1$ or $\angle XOP_1 + 2\pi$, when the arc does not cross the x axis and when it does, The angles t_ℓ and t_h are found by similar reasoning for items 2–6. Item 7 constructs the arc that continues smoothly from the specified straight line segment. The center of the arc is at the intersection of (i) the normal to the line segment P_1P_2

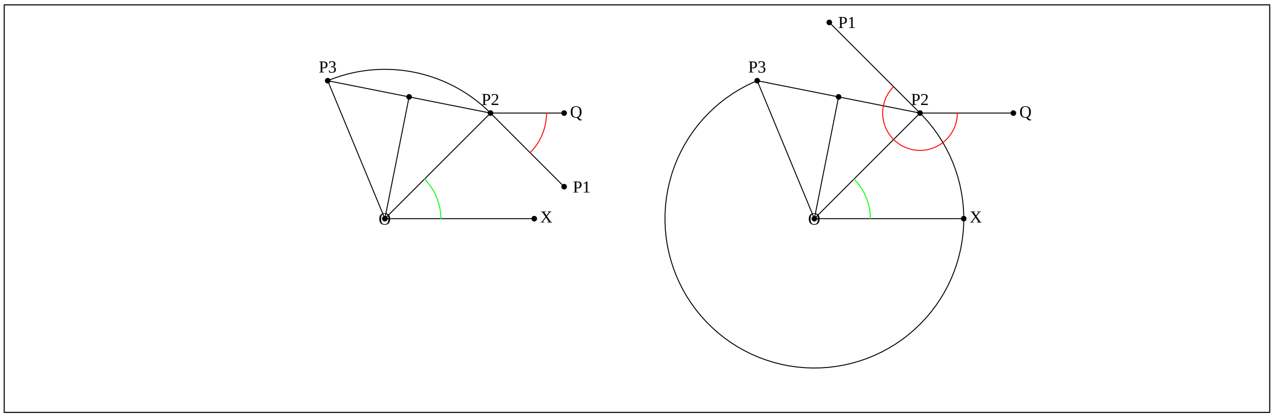

Figure 18: Continuing an arc with a straight line.

at P_2 and (ii) the perpendicular bisector of P_2P_3. In the construction on the left of Fig. 18, the line P_1P_2 continues counter clockwise from P_2 to P_3. The converse holds for the construction on the right. The two cases are distinguished by computing $\angle XOP_2 + \angle P_1P_2Q$. The two angles are measured counter clockwise around O and P_2, respectively. They are marked in green and red. The sum of the angles is $\pi/2$ and $3\pi/2$ in the two cases. Another simple criterion distinguishes the corresponding cases for item 8.

The following ELCON functions deal with points of intersection.

1. `crossPoint[{a1,b1,c1},{a2,b2,c2}]`,
2. `crossPoint[Line[{{xs1,ys1},{xe1,ye1}}],Line[{{xs2,ys2},{xe2,ye2}}]]`,
3. `crossPoint[Line[{{xs,ys},{xe,ye}}],Circle[{xc,yc},r,{tlo,thi}]`,
4. `crossPoint[Circle[{xc1,yc1},r1,{tlo1,thi1}],Circle[{xc2,yc2},r2,{tlo2,thi2}]`,
5. `sortOnSeg[{xs,ys},{xe,ye}][{xyPairs}]`,
6. `sortOnSeg[{Line[{{xs,ys},{xe,ye}}],{xyPairs}}]`,
7. `sortOnArc[Circle[{xc,yc},r,{ts,te}],[{xyPairs}]`.

Items 1–4 give the coordinates of the points of intersection (if any) of two lines specified by the (a, b, c) values of their normal equations, two line segments, a line segment and a circular arc, and two circular arcs, respectively. Item 2 finds the intersection point of the (infinite) lines on which the segments lie (using item 1), and finding if this point is on both the segments. Items 3 and 4 work correspondingly, first finding any intersections of the infinite line and complete circle, or two complete circles, and then testing if these lie on the specified line segment and arc, or on both arcs, respectively. Items 5–7 sort the lists of (x, y) coordinates denoted by `{xyPairs}` into order along the line segment or around the arc, between the ends specified by `{xstart,ystart}` or `{xend,yend}`, respectively.

Another small set of functions rotate `Point`, `Line`, `Polygon`, `Circle` and `Text` objects around the origin by a given angle. Further functions scale by a given factor, and translate parallel to the x or y axis, or both.

Appendix 3. Suggestions for further enquiry.

History of mathematics: Françoise Henry states that the painters used rulers, set squares and compasses and, perhaps, “French curves” ([4], p. 157). Holes made by compass spikes at the centers of circles are common, but so are circles without these. Might circular templates with different radii have been used at times? Joining circular arcs by common tangents produces a smooth curve. Often, a circular arc is needed that (1) goes through two end points determined by existing pieces of an illustration, and (2) must be fitted aesthetically in a predetermined area. A simple way to do this, given a set of templates, picks an intermediate point, in the given area, and uses the template that fits most closely. Henry discusses the trade and other exchange between the Insular monasteries and the Middle East (p. 213). This makes the transmittal of mathematical knowledge of the Arabs feasible when the Book of Kells was produced. Its geometry is sophisticated mathematically for its time. At a later date, Hugh of St. Victor was a geometer of influence [20]. Might it be worth looking for evidence of earlier contributions by monks to geometry?

Mathematics education: What conditions are sufficient and necessary for a unit of repeat to generate a knot by rotation and by translation, for the different kinds of pattern considered in §§1–6? Prove these. What is the condition for the unit of a C_n knot to join its successors smoothly, in general? Find more mappings to interchange the unit of repeat of C_m and C_n knots.

Threadability: Consider Fig. 3. The black arcs and lines from Q_2 to P_4 (through P_3 and Q_3), from P_4 to P_5, P_5 to P_6, and P_6 through Q_4 back to Q_2 define a closed region. The blue line that goes from the blue P_9 to P_{10} to P_{11} penetrates this region from Q_3 to Q_4. The general shape of the unit of repeat is preserved when the distances labeled a, b, c, d, e, w, in Fig. 2, are changed slightly. How much can these be varied without moving the line from Q_3 to Q_4 out of the closed region with top left corner Q_2? Computational geometry provides methods for constructing a set of equations that the six variables must satisfy to preserve the present threading. Recent discoveries of knotted protein molecules give this mathematical problem a relevance to biomedical research.

Analogy plays a key role in a host of mental activities. Taking Construction 3 in §4, that constructs patterns with rotational symmetry, and adapting it to Construction 4 in §5, that constructs patterns with translational symmetry, illustrates a very general kind of process. Let $\mathcal{O} \bullet \mathbf{v}$ denote the application of any operator $\mathcal{O}$ to an operand $\mathbf{v}$. Then the adaptation under consideration can be written

$\mathcal{C}_{3\to 4} =$ **Replace:** (1) "*the C_n interlaced knot*" by "*knotwork*", (2) "$\mathcal{R}_{n,k}$" by "$\mathcal{T}_{d,k}$",
(3) "$0 \leq k < n$" by "$k_1 \leq k \leq k_2$ for some $k_1 < k_2$".

Then **Construction 4** $=$ $\mathcal{C}_{3\to 4} \bullet$ **Construction 3** generates the required prescription.

The practical benefit of formalizing the adaptation is that computer code can be written to perform it mechanically.

In principle, a class of processes that can be inter-converted mechanically are instances of a more abstract process that is parameterized to generate the individual cases. In practice, the discovery of the abstraction depends on prior development of the individual instances, that is facilitated by the use of the inter-conversions. The abstraction, too, is often too unwieldy to apply and explain conveniently. I started to explore mechanized application of analogy as an approach to the derivation of mathematical formulas [6]. In [7], the principle is used to construct a large number of diagrams, for certain chemical experiments, that consist of modules which can be combined with open ended variety. These provide a talking point of interest to chemists, in seeking further chemical topics amenable to mechanized use of analogy. I hope that the formalized analogy discussed here can serve likewise with art scholars.

References

[1] M. Budny, Deciphering the Art of Interlace. Index of Christian Art Occasional Papers no. 4: From Ireland Coming: Irish Art from the Early Christian to the Late Gothic Period and Its European Context, ed. C. Hourihane, 183–210, Princeton University Press, 2001.

[2] G. Bain, Celtic Art: The Methods of Construction. MacLellan, Glasgow, 1973; republished Dover, Mineola, 1973.

[3] Celtic Interlace Designs, Encyclopedia of Irish & World Art: 2010 `http://www.visual-arts-cork.com/cultural-history-of-ireland/celtic-interlace-designs.htm`

[4] The Book of Kells. Reproductions from the Manuscript in Trinity College, Dublin, with a study of the manuscript by Françoise Henry. Alfred A. Knopf, New York, 1974.

[5] F. Mahony, Trinity College Dublin, private communication.

[6] M.P. Barnett, Reasoning in symbolic computation. Communications in Computer Algebra, 42 (1) 1-17, 2008.

[7] M.P. Barnett and I. Pelczer, Pulse sequence editing by symbolic calculation. J. Mag. Reson. 204, 189–195, 2010.

[8] P.R. Cromwell, The henomeric types of 2-sided patterns in the plane. Geometrica Dedicata, 137, 63–84, 2008.

[9] P.R. Cromwell, The distribution of knot types in Celtic interlaced ornament. Journal of Mathematics and the Arts, 2 (2) 61–68, 2008.

[10] B. Grünbaum and G.C. Shepard, Tilings and Patterns. Freeman, NY, 1987.

[11] M.H. Levitt, Symmetry in the design of NMR multi-pulse sequences. J. Chem. Phys. 128, 052205, 1–15, 2008.

[12] M. Emmer, ed. The Visual Mind: Art and Mathematics. MIT Press, Cambridge, Mass, 1993.

[13] M. du Sautoy, Symmetry: A journey into the Patterns of Nature. Harper, New York, 2008.

[14] R.D. Stevick, St. Patrick's Bell Shrine: Form and layout of the plates. J. Roy. Soc. Antiquaries of Ireland, 138, 26–46, 2008.

[15] B. Grünbaum and G.C. Shephard, Interlace patterns in Islamic and Moorish art. In [12] 147–156.

[16] A.K. Coomaraswamy, The Iconography of Dürers "Knots" and Leonardos "Concatenations". Eye of the Heart, Issue 4, 2009, ed. T. Scott, La Trobe University at Bendigo. `http://www.latrobe.edu.au/eyeoftheheart/issue4.html`

[17] G. Pólya, Mathematics and Plausible Reasoning. Princeton University Press, 1954.

[18] M.P. Barnett, Including Mathematica Graphics in LaTeX. TUGBoat, to be published, Fall, 2010.

[19] K. Rektorys, ed. Survey of Applicable Mathematics. MIT Press, Cambridge, Mass. 1969.

[20] P. Rorem, Hugh of St. Victor. Oxford University Press, Oxford, 2009.