"Some definite integrals containing the Tree T function" by Corless, Hu and Jeffrey

Review:

At first I was excited by this paper because it was great to see a connection between the Lambert W function and identities involving definite integrals. However, this paper needs considerable revisions and additions before it is acceptable for publication. A reproach often made to the authors Corless and Jeffrey is that they do not sufficiently recognize the huge body of work outside of their periphery. This is reflected by the following. Consider their integral of equation (15):

$$\int_0^\infty \exp(-\alpha v) \left(\frac{v}{1-e^{-v}}\right)^n \, dv \tag{1}$$

where $n = \alpha - \beta + 1$ where it is understood that n might not be an integer. Their integral I_0 of their equation (11) and I_1 of their equation (14) can also be put decomposed into a slight generalization of the above:

$$\int_0^\infty \exp(-av) \frac{v^b}{(1-e^{-v})^m} \, dv \tag{2}$$

Just below their section 3 entitled "Curiosities", they state:

> "As just discussed, the paper[3] notes that the integral is a trigamma function if $\alpha = \beta$. Neither Maple or Wolfram Alpha can identify the trigamma function from the integrals presented here. However, both systems can evaluate the integrals for a variety of integer values of α and β."

1. Firstly, it should be noted that neither Maple nor Mathematica can solve *all* definite integrals and that includes even definite integrals which

are indeed solvable in closed form (albeit in terms of special functions). $\Rightarrow$*Did the authors consult a book of integrals like Gradshteyn and Ryzhik or Prudnikov, Brychkov and Marichev? It's not enough to show solutions beyond what is current in computer algebra systems: are the solutions new with respect to the literature?*

2. It turns out that with some basic variable transformations, not only can Maple reveal solutions of the integral in eq. (1) in terms of the trigamma functions, the integrals can be brought into a canonical form which first the patterns for which these books offer at least some solutions. Both snapshots of these Prudnikov *et al* solutions and Maple worksheets are enclosed to demonstrate this.

    - One worksheet shows that their integral is equivalent to:

    $$\int_0^\infty \exp(-\alpha v)\left(\frac{v}{1-e^{-v}}\right)^{\alpha-\beta+1} dv \;=\; \int_0^\infty \frac{\exp(-(\beta-1)v)\; v^{\alpha-\beta+1}}{(e^v-1)^{\alpha-\beta+1}}\, dv$$

    The Right-Hand-side of the above fits into the pattern:

    $$\int_0^\infty \frac{x^a \exp(-px)}{(e^{qx}-z)^\lambda}\, dx$$

    for which solutions can be found in Prudnikov *et al.* for a number of special cases in section 2.3.13 on pages 337-340, in particular 2.3.13.6 on page 337. Also note that I_0 of their equation (11) and I_1 of their equation (14) can also be put decomposed into combinations of the above.

    - Moreover, if you make the variable transformation

    $$\begin{aligned} y &= 1-\exp(-v) \\ z &= 1-y \end{aligned} \qquad (3)$$

Their integral can be transformed to:

$$\int_0^\infty \exp(-\alpha v)\left(\frac{v}{1-e^{-v}}\right)^n dv = \int_0^1 z^{\alpha-1}\left(\frac{\ln(z)}{z-1}\right)^n dz$$
$$= \int_0^1 (1-y)^{\alpha-1}y^{-n}(-\ln(1-y))^n\, dy$$

where $n = \alpha - \beta + 1$. Now for $n = 0$, Maple can solve the the above integral directly:

$$\int_0^1 (1-y)^{\alpha-1}y^{-n}\, dy = \frac{\Gamma(\alpha)\Gamma(-n+1)}{\Gamma(-n+\alpha+1)}$$

Successive differentiation of both sides with respect to α generates the logarithmic terms for n integer only but by using the limit process, the solutions in terms of the trigamma function are obtained as shown in one of the enclosed worksheets. Moreover, The integral on the right side of the above has a canonical form which can be found in Prudnikov et al. volume 1, for example, eqs. 2.6.5.2 and 2.6.5.2 on page 490.

3. There is an extra $\exp(-\alpha v)$ in their equation (21): once inside the summation and in front of the summation. This puts the derivation of their equation (22) into doubt.

4. There is an issue of numerical convergence and stability in using Maple's `evalf/int` for the integral in eq. (1), in particular near the origin where $v \approx 0$. However, by setting

$$x = \frac{v}{1 - exp(-v)}$$

and solving for v in terms of x, we get:

$$v = x + W(-xe^{-x})$$

we can transform the integral to:

$$\int_0^\infty \exp(-\alpha v)\left(\frac{v}{1-e^{-v}}\right)^n dv \;=\; \int_1^\infty x^{n-1}\left(-\frac{W(-xe^{-x})}{x}\right)^\alpha \frac{(x+W(-xe^{-x})}{(1+W(-xe^{-x}))}\,dx$$

The argument:

$$-\frac{W(-xe^{-x})}{x}$$

starts at unity for $x = 1$ and asymptotically becomes e^{-x} and so the integral on the right-hand-side of the above is very stable numerically as shown in yet another Maple worksheet. This form is closer to the original for of the integrals by Gautschi, making me wonder what is it that the authors think they actually proven regarding the numerical challenge(?) Some clarification is needed by the authors here.

5. There are only 4 references and the only reference to the Lambert W function is their joint paper with D. Knuth. Yes, that was an important paper but there is a huge body of work on the W apart from their own. Also, which convention are they using for their generalized hypergeometric function? I recommend citing a standard book like Prudnikov *et al* or Y.L Luke's book "The Special Functions and their Approximations".

6. Related to this last point, the very transformations they use are not exactly original. Their transformations are similar to the manipulations in a paper by William B, Jordan and M.L. Glasser which appeared in SIAM Review, vol. 12. no. 1 (Jan. 1970), pp. 153-154. An electronic copy of this paper has been enclosed.

What saves the situation is that Prudnikov *et al.* does not provide solutions for *arbitrary* α and *arbitrary* β. I suspect that their solution in terms of the generalized hypergeometric function is the most general of all. Gradshteyn and Ryzhik have a 10% error rate. The integral tables of Prudnikov *et al* have less errors but also contain errors nonetheless. **However, it is the responsibility of the authors to check these solutions in Prudnikov and demonstrate that.** Their Tables 1 and 2 are all well and fine but they need to show at least a few integrals whose solutions are definitely NOT in Prudnikov *et al.* and therefore *beyond* the literature. Some numerical vindication as in the type shown in the enclosed Maple worksheets would be convincing.

In particular, the authors need to clarify what is the bottom-line of their paper i.e. *what is really new?* especially with respect to their concluding remarks. Their paper also needs to be more self-contained. This is only a suggestion but once done, I suggest that they make a Maple worksheet available online. If the authors can address these issues, this will be a stronger paper worthy of SIGSAM.