



Problem 68-17

Author(s): William B. Jordan and M. L. Glasser

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It is to be noted that for the case $e = 0, n = b + 2c + 3d$, the end binomial coefficient reduces to $\binom{-1}{-1}$ which is to be taken as 1.

Problem 68-17, A Definite Integral, by B. F. LOGAN, C. L. MALLOWS and L. A. SHEPP (Bell Telephone Laboratories).

Evaluate the integral

$$I = \int_0^\infty e^{-w/2} \sqrt{w} \, dw,$$

where

$$w = \frac{u}{1 - e^{-u}}.$$

The integral arose in a probability problem.

Solution by WILLIAM B. JORDAN (G. E. Knolls Atomic Power Laboratory).

Letting $w = u + v$, it follows that

$$\begin{aligned} e^u &= w/v, & we^{-w} &= ve^{-v}, \\ du &= dw/w - dv/v, & u]_0^\infty &= w]_1^\infty = v]_1^0. \end{aligned}$$

Whence,

$$I = \int_{u=0}^\infty \{we^{-w}\}^{1/2} \left\{ \frac{dw}{w} - \frac{dv}{v} \right\}$$

or

$$\begin{aligned} I &= \int_1^\infty \{we^{-w}\}^{1/2} \frac{dw}{w} - \int_1^0 \{ve^{-v}\}^{1/2} \frac{dv}{v} \\ &= \int_0^\infty t^{-1/2} e^{-t/2} \, dt = \sqrt{2\pi}. \end{aligned}$$

M. L. GLASSER (Battelle Memorial Institute) obtains the following generalization by means of the generating function for the generalized Laguerre polynomials:

$$\int_0^\infty \{ye^{-y}\}^v \, dx = \int_0^\infty \{ye^{-y}\}^v \, dy, \quad v > 0,$$

where $y = x/(e^x - 1)$ or $x/(1 - e^{-x})$.

In the solution by the proposers, it was noted that Logan and Shepp obtained the result $(2\pi)^{-1/2} I = 1$ as the expression of the probabilistic fact that a standard Wiener process is almost certain to meet a boundary curve. Additionally, Mallows obtains the still further generalization:

Suppose (i) $h(x)$ and its derivative $h'(x)$ are positive and monotone for $-\infty < x < \infty$, with $h(x) = h(-x) + x$ and $h(-\infty) = 0$. Also suppose (ii) $g(w)$ (defined for $0 \leq w < \infty$) satisfies $g(h(x)) \equiv g(h(-x))$ for $-\infty < x < \infty$. Then $\int_0^\infty g(h(x)) dx = \int_0^\infty g(w) dw$ (with slight additional generality, for any f we have $\int_0^\infty f(g(h(x))) dx = \int_0^\infty f(g(w)) dw$). The proof is straightforward. In the present case we have $g(w) = (we^{-w})^{1/2}$, $h(x) = x/(1 - e^{-x})$.

In general, if the function $g(h(x))$ is given, it may not be easy to see how to choose h so that (i) and (ii) are satisfied. If h is given satisfying (i) (there are many such functions; it is only necessary that $h'(x) - \frac{1}{2}$ is odd and monotone, with $h(-\infty) = 0$), then it is easy to construct many functions $g(w)$ satisfying (ii). Any function of $|h^{-1}(w)|$ will do; alternatively we may write $g(w) = G(w, w - h^{-1}(w))$ where $G(w, w')$ is any symmetric function. (Then $g(w) = G(h(x), h(-x))$, where $w = h(x)$.) In the present case $G(w, w') = (|w - w'| \min(w, w')/ww')^{1/2}$.