Series crimes

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Abstract

Puiseux series are power series in which the exponents can be fractional and/or negative rational numbers. Several computer algebra systems have one or more built-in or loadable functions for computing truncated Puiseux series. Some are generalized to allow coefficients containing functions of the series variable that are dominated by any power of that variable, such as logarithms and nested logarithms of the series variable. Some computer algebra systems also have built-in or loadable functions that compute *infinite* Puiseux series. Unfortunately, there are some little-known pitfalls in computing Puiseux series. The most serious of these is expansions within branch cuts or at branch points that are incorrect for some directions in the complex plane. For example with each series implementation accessible to you:

Compare the value of $(z^2+z^3)^{3/2}$ with that of its truncated series expansion about $z=0$, approximated at $z=-0.01$. Does the series converge to a value that is the negative of the correct value?

Compare the value of $\ln(z^2+z^3)$ with its truncated series expansion about $z=0$, approximated at $z=-0.01+0.1i$. Does the series converge to a value that is incorrect by $2\pi i$?

Compare $\operatorname{arctanh}(-2+\ln(z)z)$ with its truncated series expansion about $z=0$, approximated at $z=-0.01$. Does the series converge to a value that is incorrect by about πi?

At the time of this writing, most implementations that accommodate such series exhibit such errors. This article describes how to avoid these errors both for manual derivation of series and when implementing series packages.

1 Introduction

This article is a companion to reference [5]. That article describes how to overcome design limitation that make many current Puiseux-series implementations unnecessarily inconvenient, such as not providing the order that the user requests or not allowing requests for negative or fractional orders.

In contrast, this article describes how to overcome the more serious problem of results that are incorrect on branch cuts and at branch points for most current implementations. This article is relevant to both truncated and infinite series of almost any type, including hierarchical, Fourier, Dirichlet and Poisson series. However, for concreteness the discussion is specific to Puiseux series that are generalized to permit in the coefficients sub-polynomial functions of the expansion variable, such as logarithms.

Section 2 discusses branch bugs for logarithms. Section 3 discusses branch bugs for fractional powers. Section 4 discusses branch bugs for inverse trigonometric and inverse hyperbolic functions.

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Let z be a complex variable, and let x, y, r, and θ be real variables. An appropriate substitution can always transform any expansion point including ∞, $-\infty$ and the complex circle at radius ∞ to $z = x + iy = re^{i\theta} = 0$. Therefore without loss of generality the discussion assumes that 0 is the expansion point.

To test an implementation for branch bugs in a truncated series $U(z)$ for an expression $u(z)$:

1. Evaluate $|u(0) - U(0)|$. If there is a singularity at $z = 0$, then the result might be undefined even if the series is correct. Otherwise this absolute error should be 0 or very nearly so.
2. Do a high-resolution 3D plot of $|u(z) - U(z)| \mid z \to x + iy$ centered at $z = 0$ with enough terms to span an exponent range of at least 4. Try zooming in from a moderate initial box radius. An initial box radius of 0.5 works well for most examples in this article.
3. 3D plots can easily miss discontinuities that are ribs, crevasses, or thin cusps emanating from $z = 0$ – particularly if an edge doesn't lie along a grid line. Therefore if step 2 doesn't reveal an incorrect result, then
   (a) Do a 2-D plot of $|u(z) - U(z)| \mid z \to r_0 e^{i\theta}$ for $\theta = (-\pi, \pi]$ and various fixed r_0 that are well within the estimated radius of convergence.
   (b) For each critical direction θ_c defined in Section 2.1.1, plot $|u(z) - U(z)| \mid z \to re^{i\theta_c}$ for $r = [-R, R]$ and various fixed $R > 0$ that are well within the estimated radius of convergence.
4. If the result of step 1 is undefined because of a singularity, then exclude $z = 0$ or clip the plot magnitude to a positive value $\ll 1$.
5. Within rounding error and the radius of convergence, the surface and the curves should converge to 0.0 as the number of terms increases. If instead any of these plots converge to an obvious jump touching $z = 0$, then the formula is almost certainly incorrect. If there is a hint of a jump that grows from magnitude 0.0 at $z = 0$, then try instead plotting the relative error $|(u(z) - U(z))/u(z)|$, excluding $z = 0$.

For a real variable x, it suffices instead to evaluate $u(0) - U(0)$ and to plot $|u(x) - U(x)|$ and

$$\left| \frac{u(x) - U(x)}{u(x)} \right|.$$

2 Branch bugs for ln

"*Spare the branch and spoil the child.*"
– adapted from King Solomon's proverbs.

Table 1 gives several logands together with one or more correct alternatives for their dominant 0-degree terms of the logarithm series expanded about complex $z = 0$ or real $x = 0$.

If an implementation gives a degree-0 term that isn't equivalent to these correct alternatives at $x = 0$ and as $x \to 0$ from both directions or at $z = 0$ and as $z \to 0$ from all directions, then the computer algebra result is incorrect. For example, most implementations currently give the following generalized infinite generalized Maclaurin series or a truncated version of it for $\ln(z^2+z^3)$:

$$2\ln z+\sum_{k=0}^{\infty}\frac{(-1)^k z^{k+1}}{k+1}. \tag{1}$$

Within the radius of convergence 1, this series converges to values that are too large by $2\pi i$ wherever

$$y\geq 0\ \wedge\ x<\frac{\sqrt{1+3y^2}-1}{3},$$

or too small by $2\pi i$ wherever

$$y<0\ \wedge\ x\leq\frac{\sqrt{1+3y^2}-1}{3},$$

with $z=x+iy$. For example, at $z=0.1i$, $\ln(z^2+z^3)\simeq 4.6002-\mathbf{3.04192}\boldsymbol{i}$, whereas series (1) truncated to $o(z^4)$ gives approximately $4.6002+\mathbf{3.24126}\boldsymbol{i}$. Series (1) is also incorrect at $z=-0.1$.

If an implementation gives a degree-0 term that is equivalent to one of the results listed in Table 1 but more complicated, then there is room for improvement of the simplification in ways described below.

One way to avoid returning an incorrect result is to refuse attempting series expansions on branch cuts and on the branch points at their ends, either returning an error indication or an unsimplified result such as "series$(\cdots)$". However, this precludes useful results for many examples of frequent interest, such as

- fractional powers and logarithms of many expressions whose dominant exponent is non-zero or whose dominant coefficient isn't positive,
- $\arcsin(u(z))$, $\arccos(u(z))$, $\operatorname{arccosh}(u(z))$, and $\operatorname{arctanh}(u(z))$ at $u(z)=1$ or $u(z)=-1$.

Another way to avoid returning an incorrect result is to force the user to specify a numeric direction θ_0 for the series expansion variable $z=re^{i\theta}$, compute

$$\text{series}\left(f\left(re^{i\theta_0}\right),\ r=0^+,\ o(r^n)\right),$$

substitute $r\to ze^{-i\theta_0}$ into the result, then preferably attach to the result the constraint "$\mid z=re^{i\theta_0}\wedge r>0$". The result is then guaranteed only for direction θ_0. This is a reasonable approach when the only purpose of the series is to determine a uni-directional limit of an expression via that of its dominant term, and for a bi-directional limit we can invoke series$(\ldots)$ twice with two different values of θ_0. However, this approach isn't appropriate for omni-directional limits.

Moreover, with this approach it is important for θ_0 to have *no* default, such as the most likely choice 0. Otherwise many users won't realize that their formula might not be correct for non-positive or non-real z.

In contrast, this article presents formulas that are correct for all θ_0 that aren't precluded by any constraints provided by the user, such as "$\ldots\mid z>0$" or "$\ldots\mid -\pi/3<\arg z\leq 2\pi/3$". Moreover, even for the approach of requiring a numeric θ_0, the formulas presented in the remainder of this article are helpful for determining the correct behavior for that θ_0.

2.1 Incorrect extraction of the dominant term

"*The devil is in the details.*"
– after Gustave Flaubert.

Most computer-algebra systems use a particular branch when a multiply-branched function is simplified for numeric arguments. This branch is most often the principal branch. However, some computer algebra systems offer the option or the default of using the real branch for fractional powers having odd reduced denominators together with real radicands. Either way, for consistency the same branch should be used for expressions and their series.

One source of incorrect ln series is omitting the Υi term in the following universal principal-branch formula for the distribution of logarithms over products:

$$\ln(uv) \equiv \ln(u) + \ln(v) + \Upsilon i, \tag{2}$$

where

$$\begin{aligned} \Upsilon &= \arg(uv) - \arg(u) - \arg(v) &(3)\\ &= \begin{cases} 2\pi & \text{if } \arg(u)+\arg(v) \le -\pi, \\ -2\pi & \text{if } \arg(u)+\arg(v) > \pi, \\ 0 & \text{otherwise.} \end{cases} &(4) \end{aligned}$$

This can be proved from

$$\begin{aligned} \ln(|uv|) &\equiv \ln(|u|) + \ln(|v|), &(5)\\ \ln(|w|) &\equiv \ln(w) - \arg(w)\, i, &(6)\\ \arg(uv) &\equiv \operatorname{mods}(\arg(u)+\arg(v),\, 2\pi). &(7) \end{aligned}$$

Here $\operatorname{mods}(u,v)$ is the residue of u of mod v in the near-symmetric interval $(-v/2, v/2]$ for v positive.[1]

Remark 1. These formulas require the useful but non-universal definition

$$\arg(0) := 0, \tag{8}$$

as is done in *Mathematica*®. If a built-in $\arg(\ldots)$ function does anything else, then an implementer should prepend here and throughout this article appropriate cases for each possible combination of an argument of $\arg(\ldots)$ being 0. For example,

$$\begin{aligned} \Upsilon &= \begin{cases} 0 & \text{if } u=0 \vee v=0, \\ \arg(uv)-\arg(u)-\arg(v) & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0 & \text{if } u=0 \vee v=0 \vee -\pi < \arg(u)+\arg(v) \le \pi, \\ 2\pi & \text{if } \arg(u)+\arg(v) \le -\pi, \\ -2\pi & \text{otherwise.} \end{cases} \end{aligned}$$

[1] The name mods is inspired by that built-in Maple function.

Many of the formulas involving $\arg(\ldots)$ in this article can be expressed more concisely using the unwinding number described in[1] and later redefined more conveniently with the opposite sign in [2]. But alas, unwinding numbers aren't yet built-into the mathematics curriculum and most computer-algebra systems.

Here and throughout this article, Boolean expressions and braced case expressions are assumed to be done using short-circuit evaluation from left- to-right within top-to-bottom order to avoid evaluating ill-defined sub-expressions and to avoid the clutter of making the tests mutually exclusive.

Alternative (3) is more compact than alternative (4) and reveals that jumps in Υ can occur only where one of $\arg(u)$, $\arg(v)$ or $\arg(uv)$ is π. However, alternative (4) is more candid because it makes the piecewise constancy manifest rather than cryptic. Moreover, approximate values are often substituted into expressions for purposes such as plotting, and the conditional alternative (4) avoids having the magnitude of the imaginary part of a result be several machine ε when it should be 0: Unlike the unconditional alternative, the conditional alternative never subtracts two approximate angles from approximately π, giving approximately 0.

For series $(\ln(\ldots), z=0, o(z^n))$ with negative n, the result is $0+o(z^n)$ if the logand doesn't contain an essential singularity. In contrast, for a non-negative requested n, the usual algorithm for computing the logarithm of a series entails converting the dominant term of the logand series to 1 by factoring out the dominant term then distributing the logarithm over the resulting product:

$$\begin{aligned}\ln\left(c(z)\,z^{\alpha}+g(z)\right) &\rightarrow \ln\left(c(z)\,z^{\alpha}\left(1+\frac{g(z)}{c(z)\,z^{\alpha}}\right)\right)\\ &\rightarrow \left(\Omega i+\ln\left(c(z)z^{\alpha}\right)\right)+\ln\left(1+\frac{g(z)}{c(z)\,z^{\alpha}}\right). \qquad (9)\end{aligned}$$

Here $c(z)\,z^{\alpha}$ is the dominant term and $g(z)$ is the sum of all the other terms, with alternatives (3) and (4) giving

$$\Omega = \arg\left(c(z)z^{\alpha}+g(z)\right)-\arg\left(1+\frac{g(z)}{c(z)\,z^{\alpha}}\right)-\arg\left(c(z)\,z^{\alpha}\right) \qquad (10)$$

$$= \begin{cases} 2\pi & \text{if } \arg\left(1+\frac{g(z)}{c(z)z^{\alpha}}\right)+\arg\left(c(z)\,z^{\alpha}\right)\le-\pi,\\ -2\pi & \text{if } \arg\left(1+\frac{g(z)}{c(z)z^{\alpha}}\right)+\arg\left(c(z)\,z^{\alpha}\right)>\pi,\\ 0 & \text{otherwise.}\end{cases} \qquad (11)$$

Always $\Omega=0$ at $z=0$.

It is important to simplify Ω as much as is practical for each particular logand series. Not only is the result more intelligible – it is usually also more accurate for approximate computation: There can be catastrophic cancellation between terms of $g(z)$. Therefore without good algebraic simplification, Ω can be dramatically incorrect along and near branch cuts when evaluated with approximate arithmetic.

Remark 2. For example, $\Omega\equiv0$ if $g(z)\equiv0$, because then $\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right)\equiv0$ and $\arg\left(c(z)\,z^{\alpha}\right)$ must be in the interval $(-\pi,\pi]$.

Even if we can't determine a simpler formula that is equivalent to formula (11) throughout the entire complex plane or the entire real line for real z, Ω might be equivalent to a simpler expression ω throughout the radius of convergence or the useful portion thereof. It is especially important to exploit this for examples such as

$$\text{series}\left(z^{-1}+e^{z}\ln(-2-z),\, z=0,\, o\left(z^{5}\right)\right)$$

where Ω or ω infects all but one result term, giving a bulky result that is difficult to comprehend.

Let $R > 0$ be the classic radius of convergence computed disregarding any closer branch cuts, or let R be the "radius of computational utility" for divergent series. Let $\underline{R} > 0$ be the largest radius from $z = 0$ within which Ω and a simpler ω give identical values. Radius $\underline{R}$ can be an arbitrarily small portion of R because a branch cut can pass arbitrarily close to $z = 0$. However, it is almost always justifiable to use ω in place of Ω because:

- If our purpose is to determine the local behavior of the series at $z = 0$, such as for computing a limit by computing the limit of the dominant term, then any $\underline{R} > 0$ is sufficient justification for using ω.
- It seems pointless to use Ω rather than a simpler ω for any purpose if $\underline{R} > R$ or if $\underline{R}$ is greater than the percentage of R beyond which convergence is impractically slow or subject to unacceptable catastrophic cancellation.
- If we don't expect a generalized Puiseux series to capture infinite magnitudes associated with singularities not at $z = 0$, then why should we expect such series to capture the less severe finite-magnitude jumps associated with branch cuts that don't touch $z = 0$?
- We can take the view that the *generalized* radius of convergence is the distance to the nearest singularity *or jump* in Ω that we can't account for with ω, and there should be no expectation that a series is truthful beyond its generalized radius of convergence.

Here is one such opportunity for computing an ω that is significantly simpler than Ω:

Proposition 3. *Let ω_{real} denote ω for the special case of real z. If all of the terms in the truncated logand series have real coefficients and integer exponents, then $\omega_{\text{real}} \equiv 0$.*

Proof. If the truncated logand series has all integer powers of real z and all real coefficients, then $1 + g(z) / (c(z) z^\alpha)$ is real for all real z, and $\arg(1 + g(z) / (c(z) z^\alpha)) = 0$ for all real z such that $g(z) / (c(z) z^\alpha) \geq -1$. There is a singularity wherever $g(z)/(c(z)z^\alpha) = -1$, providing an upper bound on the radius of convergence. Also, $-\pi < \arg(c(z)z^\alpha) \leq \pi$. Therefore throughout the radius of convergence

$$-\pi < \arg\left(1 + \frac{g(z)}{c(z)\, z^\alpha}\right) + \arg(c(z)\, z^\alpha) \leq \pi$$

in equation (11), making $\omega_{\text{real}} \equiv 0$. □

Remark 4. If we are in a mode that consistently uses the real branch for fractional powers having odd denominators, such as the TI-Nspire[2] real mode, then more generally $\omega_{\text{real}} \equiv 0$ if all of the coefficients are real and none of the reduced exponents have even denominators.

2.1.1 The non-zero dominant exponent case

"*Beware of geeks bearing formulas.*"
– Warren Buffett.

Here is a useful easy simplification test for real or complex z when the dominant exponent $\alpha \neq 0$:

[2]The computer algebra used in Texas Instruments products has no name separate from the variously named calculators, Widows and Macintosh products that contain it. The most recent such product is named TI-Nspire.

Proposition 5. *If all of the coefficients in a truncated logand series U are real and all of the exponents of z in U are integer multiples of a non-zero dominant exponent, then $\omega \equiv 0$.*

Proof. Let the truncated logand series be $U = c\,(z)\, z^{\alpha} + g(z)$ with dominant term $c\,(z)\, z^{\alpha}$ having non-zero α. Also, let "**near** $z = 0$" denote all

$$z \;=\; re^{i\theta} \mid r>0 \wedge \left|\frac{g(z)}{c(z)z^{\alpha}}\right|<1, \tag{12}$$

which bounds the radius of convergence. From equation (10), it is apparent that Ω can jump only where one of its three $\arg(\ldots)$ terms jumps. Term $\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right)$ can't jump near $z = 0$ because $c(z)z^{\alpha}$ dominates $g(z)$. Every term in $c(z)z^{\alpha} + g(z)$ can be expressed as $k(z)\,(z^{\alpha})^{m}$ with real $k(z)$ and integer m, because the transformation $z^{\gamma} \to (z^{\alpha})^{m}$ with $m = \gamma/\alpha$ is always valid for integer m, as discussed in Section 3.1. Thus $c\,(z)\, z^{\alpha} + g(z)$ is real where

$$\arg\left(c\,(z)\, z^{\alpha}\right) \;=\; \pi, \tag{13}$$

which are the only places that $\arg\left(c\,(z)\, z^{\alpha}\right)$ jumps. Since $c\,(z)\, z^{\alpha}$ is negative there and dominates $g(z)$, adding the relatively small-magnitude real $g(z)$ to negative $c\,(z)\, z^{\alpha}$ leaves

$$\arg\left(c\,(z)\, z^{\alpha} + g(z)\right) \;=\; \pi \tag{14}$$

near $z = 0$. In equation (10), $\arg\left(c\,(z)\, z^{\alpha} + g(z)\right)$ and $\arg\left(c\,(z)\, z^{\alpha}\right)$ have opposite signs. Therefore these jumps cancel within the radius of convergence. $\square$

Propositions 3 and 5 don't identify all opportunities for dramatically simplifying ω. Consider equation (10) in the neighborhood of $z = 0$. Let

$$\eta\,(\theta) \;=\; \lim_{r\to 0^{+}} \arg\left(c\left(re^{i\theta}\right)\right). \tag{15}$$

Then in the punctured neighborhood of $z = 0$, the term $\arg\left(c\,(z)\, z^{\alpha}\right)$ is

$$\lim_{r\to 0^{+}} \arg\left(c\left(re^{i\theta}\right)\left(re^{i\theta}\right)^{\alpha}\right) \;=\; \operatorname{mods}\left(\eta\,(\theta) + \alpha\theta, 2\pi\right). \tag{16}$$

Let $\hat{\eta}$ denote $\eta\,(\theta)$ in formula (15) in the common case when $\eta\,(\theta)$ is independent of θ for all $-\pi < \theta \le \pi$ not precluded by constraints provided by the user or introduced by the computer algebra system. Solving $\operatorname{mods}\left(\hat{\eta} + \alpha\theta_c, 2\pi\right) = \pi$ for the critical angles θ_c gives

$$-\pi \;<\; \theta_c = \frac{(2n+1)\,\pi - \hat{\eta}}{\alpha} \;\le\; \pi, \tag{17}$$

with integer n. Solving the inequalities in (17) for n gives for $\alpha > 0$

$$\left\lfloor \frac{\hat{\eta}}{2\pi} - \frac{1}{2} - \frac{\alpha}{2} \right\rfloor < n \le \left\lfloor \frac{\hat{\eta}}{2\pi} - \frac{1}{2} + \frac{\alpha}{2} \right\rfloor, \tag{18}$$

versus for $\alpha < 0$

$$\left\lceil \frac{\hat{\eta}}{2\pi} - \frac{1}{2} + \frac{\alpha}{2} \right\rceil \le n < \left\lceil \frac{\hat{\eta}}{2\pi} - \frac{1}{2} - \frac{\alpha}{2} \right\rceil. \tag{19}$$

There are no such angles if $|\alpha| < 1$ and $\hat{\eta}$ is sufficiently close to 0, in which case $\omega \equiv 0$. More generally let $\underline{\eta}$ be a lower bound and $\overline{\eta}$ be an upper bound on $\eta(\theta)$ over $-\pi < \theta \leq \pi$ not excluded by any constraints. Then there are *no* critical angles if

$$(|\alpha| - 1)\pi < \underline{\eta} \ \wedge \ \overline{\eta} < (1 - |\alpha|)\pi. \tag{20}$$

Here is how we can proceed when none of the above tests are beneficial: As $z \to 0$,

$$c(z)z^{\alpha} + g(z) \ = \ c(z)z^{\alpha}\left(1 + \frac{g(z)}{c(z)z^{\alpha}}\right) \ \to \ c(z)z^{\alpha}. \tag{21}$$

Therefore the solution curves to equations (13) and (14) pair to form cusps where they don't coincide to cancel. Along a critical angle θ_c, $\arg(c(z)z^{\alpha}) = \pi$. If also $\arg(1 + g(z)/(c(z)z^{\alpha})) = 0$ along θ_c near $z = 0$, then $\theta = \theta_c$ is also the companion solution to equation (14) near $z = 0$. The $\arg(\ldots)$ terms for equations (13) and (14) have opposite signs in formula (10), so these jumps cancel within the radius of convergence. Since $\arg(1 + g(z)/(c(z)z^{\alpha}))$ can't jump near $z = 0$, ω is then identically 0 in the angular neighborhood of θ_c near $z = 0$.

If $\omega = 0$ in the angular neighborhood of every critical angle, then $\omega \equiv 0$. For example, this is true for $\ln(-z^2 + z^3)$ and $\ln(-z^{-2} + z)$, which don't satisfy Proposition 5 or inequality (20). Even if there are some non-zero cusps, constraints might exclude those cusps for a non-infinitesimal distance from $z = 0$. Even if there are some included cusps, it might be possible to omit either the 2π case or the -2π case as follows:

For $\alpha > 0$, as θ increases through θ_c, $\arg\left(c \cdot \left(re^{i\theta}\right)^{\alpha}\right)$ increases with θ on both sides of a jump down from π to $(-\pi)^+$. If $\arg(1 + g(z)/(cz^{\alpha})) > 0$ along θ_c near $z = 0$, then from equation (11), $\omega = -2\pi$ along and clockwise of θ_c until but excluding the curved solution to equation (14). If instead $\arg(1 + g(z)/(cz^{\alpha})) < 0$ along θ_c near $z = 0$, then $\omega = 2\pi$ counter-clockwise of θ_c through the curved solution to equation (14).

Similarly for $\alpha < 0$, for which $\arg\left(c \cdot \left(re^{i\theta}\right)^{\alpha}\right)$ decreases on both sides of a jump up from $(-\pi)^+$ to π as θ increases through θ_c: If $\arg(1 + g(z)/(cz^{\alpha})) > 0$ along θ_c near $z = 0$, then $\omega = -2\pi$ along and counter-clockwise of θ_c until but excluding the curved solution to equation (14). If instead $\arg(1 + g(z)/(cz^{\alpha})) < 0$ along θ_c near $z = 0$, then $\omega = 2\pi$ clockwise of θ_c through the curved solution to equation (14).

Thus we can omit the 2π case if for all critical directions $\arg(1 + g(z)/(cz^{\alpha})) \geq 0$, or we can omit the -2π case if for all critical directions $\arg(1 + g(z)/(cz^{\alpha})) < 0$. For example, we can omit the 2π case for $\ln(iz^2 + iz^4)$ and we can omit the -2π case for $\ln(z + iz^2)$.

Notice that although ω can be 2π either clockwise or counter-clockwise of a critical direction, ω can't be 2π *along* a critical direction.

For real t let

$$\mathrm{signum}(t) \ := \ \begin{cases} -1, & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases}$$

To determine whether expression $\arg(1 + g(z)/(cz^{\alpha}))$ is zero, positive or negative along θ_c near (but not at) $z = 0$, we could attempt computing

$$\lim_{r \to 0^+} \mathrm{signum}\left(\arg\left(1 + \frac{g\left(re^{i\theta_c}\right)}{c \cdot \left(re^{i\theta_c}\right)^{\alpha}}\right)\right).$$

However, such limits are beyond the capabilities of most computer algebra systems if g has more than one term – particularly if any of the exponents are fractional and/or any of the coefficients depend on z. A simpler surrogate within the radius of convergence is to instead compute

$$\lim_{r\to 0^+} \text{signum}\left(\Im\left(\frac{g\left(re^{i\theta_c}\right)}{c\cdot\left(re^{i\theta_c}\right)^{\alpha}}\right)\right). \tag{22}$$

However, this limit is also often beyond the capabilities of most computer algebra systems. Fortunately there is an easily-computed surrogate for formula (22): Let $s\left(re^{i\theta}\right)$ be the coefficient and β be the exponent of the lowest-degree term of $g(z)/\left(c(z)z^{\alpha}\right)$ for which

$$\left(I_c\left(\frac{g(z)}{cz^{\alpha}}, \theta_c\right) := \lim_{r\to 0^+} \Im\left(s(re^{i\theta_c})e^{i\beta\theta_c}\right)\right) \neq 0, \tag{23}$$

if any such term exists. Let $I_c\left(\ldots,\theta_c\right) := 0$ if no such $s\left(re^{i\theta}\right)$ exists. The imaginary part of $s\left(re^{i\theta_c}\right)\left(re^{i\theta_c}\right)^{\beta}$ dominates the imaginary parts of all subsequent terms of $g(z)$ along θ_c if any exist. Therefore I_c is a more-easily computed surrogate for determining whether $\arg\left(1+g(z)/\left(cz^{\alpha}\right)\right)$ is zero, positive or negative along θ_c near $z=0$. The term limits for computing I_c are trivial in the common case when a coefficient is a numeric constant, and easy even if a coefficient is a typical sub-polynomial function of z. We can use Ω given by formula (10) or (11) if a coefficient contains an indeterminant other than z that isn't sufficiently constrained to decide the sign of the imaginary part and if no imaginary parts were non-zero for previous terms.

The order to which the logand series is computed might not reveal the lowest-degree term for which one of the $I_c\left(\ldots,\theta_c\right)$ isn't 0, thus affecting the resulting value of Ω. For example, if the series for $\ln\left(z^2 - iz^3 - iz^6\right)$ is computed to $o\left(z^n\right)$ with $n \geq 4$, then the dominant term is

$$\ln\left(z^2\right) + \begin{cases} 2\pi & \arg\left(1 - iz - iz^4 + \cdots + o\left(z^n\right)\right) + \arg\left(z^2\right) \leq -\pi, \\ 0 & \text{otherwise.} \end{cases}$$

However, the piecewise term is absent if the series is computed to $o\left(z^3\right)$, making it incorrect by about $2\pi i$ for values such as $z = i/10 - 10^{-6}$ and $z = -i/10 + 10^{-6}$. It is disturbing that computing additional terms of a logand can thus change the zero-degree term of the logarithm series. Moreover, it is awkward for algorithms that compute additional terms incrementally, such as described by Norman [4]. Such is the nature of partial information along or near a branch cut.

A way to avoid this annoyance is to use the original logand expression in the $\arg(\ldots)$ sub-expressions of alternative (10) or (11) rather than using a truncated series for that logand. However, the original logand isn't always available: Perhaps as the logand we are given a truncated series, not knowing a closed-form expression that it approximates. Even if we did know, using the original expression can make computing a series in one step give a different result than series composition. Such compositional inconsistency is an undesirable property.

More seriously, including the original non-series expression as a proper sub-expression of a series result is worse than simply returning the original expression rather than a series: Presumably the user requested the series to obtain insight about the behavior of the function, or to enable symbolic operations that otherwise couldn't be done, or to enable a numeric approximation. Returning a result that contains the original expression as a proper sub-expression thwarts all of these objectives.

The annoyance of order-dependent coefficients is vastly preferable. A consolation is that a zero-degree term that differs so dramatically from that of the infinite series is at least appropriate for a nearby problem across the branch cut.

Remark 6. If the truncated logand series is simple enough so that for remaining cusps we can solve

$$\arg\left(c\left(x+iy\right)\left(x+iy\right)^{\alpha}+g\left(x+iy\right)\right) \;=\; \pi \tag{24}$$

for $x\left(y\right)$ or for $y\left(x\right)$, or else solve

$$\arg\left(c\left(re^{i\theta}\right)\left(re^{i\theta}\right)^{\alpha}+g\left(re^{i\theta}\right)\right) \;=\; \pi \tag{25}$$

for $\theta\left(r\right)$ or for $r\left(\theta\right)$, then we can construct a more candidly explicit representation of ω. For example, with $\ln\left(z^2+z^3\right)$,

$$\Omega \;=\; \begin{cases} 2\pi & \text{if } \Im\left(z\right)<0 \wedge 0<\Re\left(z\right)\leq \dfrac{\sqrt{1+3\,\Im\left(z\right)^2}-1}{3}, \\ -2\pi & \text{if } \Im\left(z\right)>0 \wedge 0\leq\Re\left(z\right)< \dfrac{\sqrt{1+3\,\Im\left(z\right)^2}-1}{3}, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

There is catastrophic cancellation here computing $\sqrt{1+3\Im\left(z\right)^2}-1$ for $\left|\Im\left(z\right)\right|\ll 1$. This can be avoided by expanding the expression into the series $\Im\left(z\right)^2/2-3\Im\left(z\right)^4/8+\ldots$, which is also more consistent with the user's request for a *series* result. Perhaps generalized series reversion could be used to obtain an explicit truncated series solution to equation (24) or (25) even when we can't solve them exactly – at least when the coefficients are all numeric and the exponents are all non-negative integers.

Remark 7. When z is real, we can almost always simplify ω_{real} to either 0 or a two-piece result that is -2π for negative z and/or positive z, but 0 everywhere else as follows:

If 0 isn't a critical angle or $I_c(g(z)/\left(cz^{\alpha}\right),0)\leq 0$, then $\omega_{\text{real}}=0$ for $z\geq 0$. Otherwise $\omega_{\text{real}}=-2\pi$ for $z>0$.

If π isn't a critical angle or $I_c(g(z)/\left(cz^{\alpha}\right),\pi)\leq 0$, then $\omega_{\text{real}}=0$ for $z\leq 0$. Otherwise $\omega_{\text{real}}=-2\pi$ for $z<0$.

2.1.2 The degree-0 case

If $\alpha=0$ and $\overline{\eta}<\pi$, then $\omega\equiv 0$ because for formula (11), $\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right)\to 0$. If instead $\alpha=0$ and $\arg\left(c\left(z\right)\right)\equiv\pi$ throughout all z near $z=0$ that aren't excluded by any constraints, then $\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right)+\arg\left(c\left(z\right)\right)$ can't be less than $-\pi$, because $\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right)\to 0$. Therefore we can at least omit the 2π case. Thus using $\omega_{\pi,0}$ to denote ω for this special case of $z=0$ being *in* the branch cut,

$$\omega_{\pi,0} \;=\; \begin{cases} -2\pi & \text{if } \arg\left(1+\dfrac{g(z)}{c(z)}\right)>0, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

There are singularities wherever $g(z_s)/c(z_s) = -1$, and those singularities aren't at the expansion point $z = 0$. Consequently the radius of convergence doesn't exceed the least of those $|z_s|$. Therefore

$$\omega_{\pi,0} \quad = \quad \begin{cases} -2\pi & \text{if } 0 < \arg\left(\dfrac{g(z)}{c(z)}\right) < \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Formula (28) avoids adding a small magnitude quantity to 1.0 near $z = 0$, so this formula is more likely to be correct than formula (27) with approximate arithmetic.

Whenever we are comparing $\arg(f(z))$ with $-\pi/2$, 0, $\pi/2$, or π, we can often simplify the test further and make it more accurate for approximate numbers by writing the comparison in terms of $\Re(f(z))$ and/or $\Im(f(z))$ – particularly when $f(z)$ has more than one term. Thus

$$\omega_{\pi,0} \quad = \quad \begin{cases} -2\pi & \text{if } \Im\left(\dfrac{g(z)}{c(z)}\right) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

This test can often be simplified further: Let $b(z)z^{\sigma}$ be the dominant term of $g(z)/c(z)$, and let

$$\tau(\theta) \quad = \quad \lim_{r\to 0^+} \arg\left(b\left(re^{i\theta}\right)\right). \tag{30}$$

Let $\hat{\tau}$ denote τ in the common case where it is independent of θ for all $0 < \theta \le \pi$ that aren't excluded by any constraints.

In the neighborhood of $z = 0$, as θ increases from $(-\pi)^+$ through π, $\arg(g(z)/(c(z)z^{\alpha}))$, $\arg(b(z)z^{\sigma})$ and $\arg(\hat{\tau}z^{\alpha})$ increase from $(-\sigma\pi)^+ + \hat{\tau}$ through $\sigma\pi + \hat{\tau}$, but jumping down from π to $(-\pi)^+$ as θ increases past every critical angle where $\operatorname{mods}(\sigma\theta_c + \hat{\tau}, 2\pi) = \pi$. Expression $\Im(g(z)/c(z))$ changes sign at those places and also at every critical angle where $\operatorname{mods}(\sigma\theta_c + \hat{\tau}, 2\pi) = 0$. There are no critical angles of either type if $0 < \sigma < 1/2$ and τ is sufficiently close to $\pi/2$ or $-\pi/2$. More specifically if $\underline{\tau}$ is a lower bound on $\tau(\theta)$ and $\overline{\tau}$ is an upper bound, then $\omega_{\pi,0} \equiv 0$ when

$$(\sigma - 1)\pi < \underline{\tau} \wedge \overline{\tau} < -\sigma\pi. \tag{31}$$

If instead

$$\sigma\pi < \underline{\tau} \wedge \overline{\tau} < (1-\sigma)\pi, \tag{32}$$

then

$$\omega_{\pi,0} \quad = \quad \begin{cases} 0 & \text{if } z = 0, \\ -2\pi & \text{otherwise.} \end{cases} \tag{33}$$

As examples, $\omega_{\pi,0} \equiv 0$ for $\ln\left(-1 + iz^{1/4}\right)$, whereas equation (33) applies to $\ln\left(-1 - iz^{1/4}\right)$.

Remark 8. If critical angles occur only at the edges of regions not excluded by constraints, then we can compute $I_c(g(z)/c(z), \ldots)$ at those critical angles and at one included non-critical angle to attempt simplifying formula (29). For example with $\ln\left(-1 + iz^{1/2} + z\right)$, the one critical angle is π, along which I_s is non-positive. I_c is also non-positive along the included non-critical angle $-\pi/2$, so $\omega_{\pi,0} \equiv 0$.

In contrast for $\ln\left(-1 - iz^{1/2} + z\right)$, the one critical angle is π, along which I_c is non-positive. However, I_c is positive along the included non-critical angle $\pi/2$. Therefore ω is 0 for the ray $z \leq 0$ but -2π everywhere else.

If a critical angle is interior to an included region, then we can simply use formula (29). Remark 7 is also applicable to $\omega_{\pi,0}$ for real z.

Remark 9. For an example such as $\ln\left(c + z^2 + z^3\right)$ where a value such as 1, -1 or 0 could subsequently be substituted for the literal constant c, a completely correct result should piecewise account for all possible cases.

2.2 Incorrect extraction of the dominant coefficient

A logarithm of a product is more concise and more efficient to approximate numerically than a sum of logarithms of the factors. For this reason, users often prefer a result that contains a logarithm of a product rather than an equivalent sum of logarithms.

However, at this time many existing generalized Puiseux-series implementations always distribute the logarithm of the dominant term over its coefficient and cofactor. This distribution is justified for a hierarchical series if the coefficient is itself a series in logarithms or nested logarithms depending on z – at least when this distribution unifies two otherwise different logarithms. For example, it is more consistent to have $\ln(z)$ throughout a series than to have $\ln(z)$ in some places and $\ln(-2z)$ in other places.

However, this distribution provides an additional opportunity for incorrect results caused by ignoring the Υi term in equation (2). Applying this transformation twice and simplifying gives

$$\ln\left(c\left(z\right)z^{\alpha} + g(z)\right) \;\; \rightarrow \;\; \left(\Phi i + \ln\left(c\left(z\right)\right) + \ln\left(z^{\alpha}\right)\right) + \ln\left(1 + \frac{g(z)}{c\left(z\right)z^{\alpha}}\right), \tag{34}$$

where

$$\begin{aligned}
\Phi &= \Omega + \arg\left(c(z)z^{\alpha}\right) - \arg\left(c(z)\right) - \arg\left(z^{\alpha}\right), && (35)\\
&= \Omega + \begin{cases} 2\pi & \text{if } \arg\left(c(z)\right) + \arg\left(z^{\alpha}\right) \leq -\pi, \\ -2\pi & \text{if } \arg\left(c(z)\right) + \arg\left(z^{\alpha}\right) > \pi, \\ 0 & \text{otherwise,} \end{cases} && (36)\\
&= \arg\left(c\left(z\right)z^{\alpha} + g(z)\right) - \arg\left(1 + \frac{g(z)}{c\left(z\right)z^{\alpha}}\right) - \arg\left(c(z)\right) - \arg\left(z^{\alpha}\right), && (37)\\
&= \begin{cases} 2\pi & \text{if } \arg\left(c(z)\right) + \arg\left(z^{\alpha}\right) + \arg\left(1 + \dfrac{g(z)}{c(z)z^{\alpha}}\right) \leq -\pi, \\ -2\pi & \text{if } \arg\left(c(z)\right) + \arg\left(z^{\alpha}\right) + \arg\left(1 + \dfrac{g(z)}{c(z)z^{\alpha}}\right) > \pi, \\ 0 & \text{otherwise.} \end{cases} && (38)
\end{aligned}$$

Alternatives (35) and (36) are advantageous when ω is a constant, because only one term occurs in the arguments of $\arg(\ldots)$. Otherwise alternative (38) is more concise and candid.

Let $\underline{\upsilon}$ be a lower bound and $\overline{\upsilon}$ be an upper bound on $\operatorname{mods}(\alpha\theta, 2\pi)$ for all angles $-\pi < \theta \leq \pi$ that aren't excluded by constraints, with $\underline{\eta}$ and $\overline{\eta}$ being corresponding bounds on $\eta(\theta)$ defined by equation (15). In equation (36) the 2π case can be omitted if

$$\underline{\eta} + \underline{\upsilon} \;\; > \;\; -\pi, \tag{39}$$

and/or the -2π case can be omitted if

$$\overline{\eta}+\overline{\upsilon} \quad < \quad \pi. \tag{40}$$

The same is true for alternative (38) because $\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right) \rightarrow 0$.

We can use $\underline{\eta}=\overline{\eta}=\hat{\eta}$ for the common case where η is independent of θ. For unrestricted complex z we can use $\overline{\upsilon}=\min\left(1,|\alpha|\right)\pi$ and $\underline{\upsilon}=-\overline{\upsilon}$. For unrestricted real z we can use the intervals

$$[\underline{\upsilon},\overline{\upsilon}] \quad = \quad \begin{cases} [\alpha\pi,0] & \text{if } -1<\alpha<0, \\ [0,0] & \text{if } \alpha=0, \\ [0,\alpha\pi] & \text{if } 0<\alpha\leq 1, \\ [-\pi,\pi] & \text{otherwise.} \end{cases} \tag{41}$$

Alternative (37) reveals that jumps in ω near $z=0$ can occur only where $\arg\left(z^{\alpha}\right)=\pi$, where $\arg\left(c(z)\right)=\pi$, and where $\arg\left(c\left(z\right)z^{\alpha}+g(z)\right)=\pi$. When $\arg\left(c(z)\right)\neq 0$, the three solution sets for these three equations typically don't pair to cancel or form cusps. Instead they typically form non-cusp wedges of values -2π, 0 and 2π. Therefore, if there are critical angles interior to the included directions, then:

- Non-zero local ϕ occur in infinitely more directions than non-zero ω. Therefore it is less likely that any constraints will preclude all of the non-zero wedges or even all of the positive ones or all of the negative ones.
- It is impossible for both edges of every wedge to coincide and thereby cancel to give $\phi\equiv 0$ for all directions near $z=0$.

Thus we often pay dearly for distributing the logarithm over a non-positive coefficient and its co-factor, which is unnecessary in many applications.

Remark 10. For real z, $\lim_{z\rightarrow 0^{+}}\arg\left(z^{\alpha}\right)=0$ and $\lim_{z\rightarrow 0}\arg\left(1+g(z)/\left(c(z)z^{\alpha}\right)\right)\rightarrow 0$. Therefore equation (38) reveals that $\phi=0$ for $z\geq 0$ if $\lim_{z\rightarrow 0^{+}}\arg\left(c(z)\right)\notin\{-\pi,\pi\}$. When $\lim_{z\rightarrow 0^{+}}\arg\left(c(z)\right)\in\{-\pi,\pi\}$, then we can compute $I_{c}\left(g(z)/\left(c(z)z^{\alpha}\right),0\right)$ from definition (23) to decide whether ϕ is 2π, -2π or 0 for $z>0$. Similarly $\lim_{z\rightarrow 0^{-}}\arg\left(z^{\alpha}\right)=\operatorname{mods}\left(\alpha\pi,2\pi\right)$. Therefore if

$$\lim_{z\rightarrow 0^{-}}\left(\operatorname{mods}\left(\alpha\pi,2\pi\right)+\arg\left(c(z)\right)\right)\notin\{-\pi,\pi\},$$

then $\phi=0$ for $z\leq 0$. Otherwise we can compute $I_{c}\left(g(z)/\left(c(z)z^{\alpha}\right),\pi\right)$ to determine whether ϕ is 2π, -2π or 0 for $z<0$. We can then combine these results to obtain ϕ_{real}.

2.3 Incorrect extraction of the dominant exponent

If the dominant exponent is neither 0 nor 1 and either the coefficient of the dominant term is 1 or we have distributed the logarithm over it, then we can consider extracting the exponent of the dominant power.

2.3.1 Principal branch

The relevant universally correct principal-branch identity for extracting the exponent of $\ln(z^{\alpha})$ is

$$\ln(u^{\alpha}) \equiv \alpha\ln(u) + \xi i, \tag{42}$$

where

$$\xi = \arg(u^{\alpha}) - \alpha\arg(u), \tag{43}$$
$$= 2\pi\left\lfloor\frac{1}{2} - \frac{\alpha\arg(u)}{2\pi}\right\rfloor. \tag{44}$$

These alternatives can be derived from identity (6) together with $\ln(|u^{\alpha}|) = \alpha\ln(|u|)$ and the fact that $\arg(u^{\alpha}) = \operatorname{mods}(\alpha\arg(u), 2\pi)$. Alternative (44) is more accurate for approximate arithmetic and more candid because it is manifestly piecewise-constant integer multiples of 2π.

Expression ξ can be simplified to a single constant $2n\pi$ if

$$(2n-1)\pi < \alpha\arg(u) \leq (2n+1)\pi, \tag{45}$$

for an integer n together with the smallest and largest $\arg(u)$ in $(-\pi, \pi]$ that aren't excluded by any constraint. For example, $\xi \equiv 0$ if

$$-\pi < \alpha\arg(u) \leq \pi. \tag{46}$$

If instead $(2n-1)\pi < \alpha\arg(u) \leq (2n+3)\pi$ for all included $\arg(u)$, then ξ can be more candidly represented as

$$\begin{cases} 2n\pi & \text{if } \alpha\arg(u) \leq (2n+1)\pi, \\ (2n+2)\pi & \text{otherwise.} \end{cases} \tag{47}$$

Extraction of a fractional power has the benefit of converting a troublesome fractional power into a benign multiplication by a fraction.

Also, there is more justification for extraction of the dominant exponent for a hierarchical series wherein the coefficient can be a series in a logarithm or nested logarithm of z.

In our case, $u = z$, and combining this transformation with distribution over the dominant term and the coefficient thereof gives the overall transformation

$$\ln(c(z)z^{\alpha} + g(z)) \rightarrow (\Psi i + \ln(c(z)) + \alpha\ln(z)) + \ln\left(1 + \frac{g(z)}{c(z)z^{\alpha}}\right), \tag{48}$$

where

$$\begin{aligned}
\Psi &= \Phi + \arg\left(z^{\alpha}\right) - \alpha \arg\left(z\right), & (49)\\
&= \Phi + 2\pi\left\lfloor \frac{1}{2} - \frac{\alpha\arg(z)}{2\pi}\right\rfloor, & (50)\\
&= \Omega + \arg\left(c(z)z^{\alpha}\right) - \arg\left(c(z)\right) - \alpha\arg\left(z\right), & (51)\\
&= \Omega + 2\pi\left\lfloor \frac{1}{2} - \frac{\alpha\arg(z)}{2\pi}\right\rfloor + \begin{cases} 2\pi & \text{if } \arg\left(c(z)\right) + \arg\left(z^{\alpha}\right) \le -\pi, \\ -2\pi & \text{if } \arg\left(c(z)\right) + \arg\left(z^{\alpha}\right) > \pi, \\ 0 & \text{otherwise,} \end{cases} & (52)\\
&= \arg\left(c\left(z\right)z^{\alpha} + g(z)\right) - \arg\left(1 + \frac{g(z)}{c\left(z\right)z^{\alpha}}\right) - \arg\left(c\left(z\right)\right) - \alpha\arg\left(z\right), & (53)\\
&= 2\pi\left\lfloor \frac{1}{2} - \frac{\alpha\arg(z)}{2\pi}\right\rfloor + \begin{cases} 2\pi & \text{if } \arg(c(z)) + \arg(z^{\alpha}) + \arg\left(1 + \dfrac{g(z)}{c(z)z^{\alpha}}\right) \le -\pi, \\ -2\pi & \text{if } \arg(c(z)) + \arg(z^{\alpha}) + \arg\left(1 + \dfrac{g(z)}{c(z)z^{\alpha}}\right) > \pi, \\ 0 & \text{otherwise.} \end{cases} & (54)
\end{aligned}$$

Let ψ denote a perhaps simpler local version of Ψ near $z = 0$. Alternative (50) is advantageous when ψ is non-constant but ϕ is simple. Alternative (52) is advantageous when ψ and ϕ are non-constant but ω is simple. Otherwise alternative (54) is most candid and least subject to catastrophic cancellation.

Equation (53) reveals that the critical angles are the union of the solutions to $\arg(z) = \pi$, $\arg(c(z)z^{\alpha}) = \pi$, and (if $c(z)$ depends on z) $\arg(c(z)) = \pi$. The piecewise-constant values of Ψ are also multiples of 2π, but no longer limited to be one of -2π, 0 or 2π. Therefore with or without constraints on $\arg(z)$, the local version ψ is even less likely than ϕ to be identically 0 or otherwise constant, and likewise for ψ_{real}. However, we will see that the use of Ψ rather than Ω is mandatory for the fractional power of a series. Therefore it is important to simplify Ψ as much as is practical: The floor term in alternatives (50), (52) and (54) can be simplified as discussed at the beginning of this sub-subsection. Subsection 2.2 describes how to simplify the conditional term in alternative (52). Simplification of the conditional term in alternative (54) is similar because $\arg\left(1 + g(z)/\left(c(z)z^{\alpha}\right)\right) \to 0$.

For real z, we can compute a local version of Ψ, ψ_{real}, by computing ϕ_{real}, then adding

$$2\pi\left\lfloor \frac{1}{2} - \frac{\alpha}{2}\right\rfloor$$

to the case for negative z.

2.3.2 The real branch for ln of fractional powers

The *Derive* computer algebra system offers a Branch control variable that, if assigned the value Real causes fractional powers having odd denominators to use the real rather than principal branch when a radicand is negative. TI-Nspire bundles this choice into its Real mode.

For example, in Real mode $(-1)^{1/3} \to -1$ rather than $1/2 + i\sqrt{3}/2$, and $(-1)^{2/3} \to 1$ rather than $-1/2 + i\sqrt{3}/2$. This default option is much appreciated by students and faculty who fear or loathe non-real numbers, which is a majority of TI's customers most of the time. However, this choice is incompatible with formulas (42) through (43). For example, these principal-branch formulas give

$$\ln\left(x^{2/3}\right) \quad\to\quad \frac{2}{3}\ln(x) + \left(\arg\left(x^{2/3}\right) - \frac{2}{3}\arg(x)\right) i.$$

If we subsequently substitute -8 for x, then this result simplifies to

$$\frac{2}{3}\ln(-8) + \left(\left(\arg\left((-8)^{2/3}\right) - \frac{2}{3}\arg(-8)\right) i\right) \quad\to\quad 2\ln(2) + \frac{2}{3}\pi i,$$

whereas for the real branch, $\ln\left((-8)^{2/3}\right) \to \ln(4) \to 2\ln(2)$.

The easiest way to simplify logarithms of fractional powers using the real branch is to refrain from extracting reduced fractional exponents having an odd denominator unless the sign of the base can be determined – perhaps with the assistance of a constraint on $\arg(z)$. Moreover, for real-branch mode we should also refrain from extracting integer exponents that are even. For example, we shouldn't extract the exponent 12 from $\ln(x^{12})$ for a real x because

$$\begin{aligned} \ln\left(x^{12}\right) \quad&\to\quad 12\ln(x) + \left(\arg\left(x^{12}\right) - 12\arg(x)\right) i \\ &\to\quad 12\ln(x) - \begin{cases} 12\pi i & \text{if } x < 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Even though this result is real for all real x, this appearance of i in the result is unwelcome to most real-mode customers. Even principal-branch complex-mode customers would rather not see i in an expression if it can as concisely or more concisely be expressed without i – especially if the result is real for all real values of interest for any variables therein. However, to help reduce the number of distinct logands in a result for $x \in \mathbb{R}$, we can extract all of an even exponent but 2:

$$\ln\left(x^{12}\right) \quad\to\quad 6\ln\left(x^2\right).$$

Another alternative for real x is $\ln(x^{12}) \to 12\ln(|x|)$. However, absolute-value functions are troublesome and best avoided where possible. For example, if in the same series we use

$$\ln\left(x^3\right) \quad\to\quad 3\ln(x) - \begin{cases} 2\pi i & \text{if } x < 0, \\ 0 & \text{otherwise,} \end{cases}$$

then we obtain a series that contains both $\ln(x)$ and $\ln(|x|)$.

3 Branch bugs for fractional powers

Some series implementations can give incorrect series results for fractional powers. For example, with $\left(z^2 + z^3\right)^{3/2}$ most implementations currently incorrectly give the equivalent of

$$\sum_{k=0}^{\infty} \frac{3\,(-1)^k\,(2k)!}{(2k-3)(2k-1)k!^2 4^k} z^{k+3}$$

or a truncated version of it. This series is incorrect by a factor of -1 left of $x = \left(\sqrt{3y^2+1}-1\right)/3$ for $z = x + iy$.

Table 2 lists one or more correct alternative multiplicative correction factors for the Puiseux series of some fractional-power expressions expanded about complex $z = 0$ or real $x = 0$.

The probable causes of an incorrect result are related to those for logarithms of series. The usual algorithm for computing a numeric power β of a series requires that the dominant term be 1. Therefore if the series of the radicand is $c\,(z)\,z^{\alpha} + g(z)$, with $c\,(z)\,z^{\alpha}$ being the dominant term, we must:

1. Factor out $c\,(z)\,z^{\alpha}$.
2. Distribute the exponent β over the three factors $(z)\,,\, cz^{\alpha}$ and $1 + g\,(z)\,/\,(c\,(z)\,z^{\alpha})$.
3. Transform $(z^{\alpha})^{\beta}$ to $z^{\alpha\beta}$.
4. Compute the series for $(1 + g\,(z)\,/\,(c\,(z)\,z^{\alpha}))^{\beta}$ by the usual algorithm.
5. Distribute $c\,(z)^{\beta}\,z^{\alpha\beta}$ over the terms computed in step 5.

Steps 2 and 3 can contribute to an angular rotation factor of the form $(-1)^{\cdots}$ that should also be distributed in step 5.

3.1 Principal-branch series of fractional powers

Zippel's formula [6] generalizes to the following universal principal-branch formula for the distribution of real exponents over products:

$$(uv)^{\beta} \equiv (-1)^{\delta}\,u^{\beta}v^{\beta} \tag{55}$$

where

$$\delta = \frac{\beta}{\pi}\left(\arg\left((uv)^{\beta}\right) - \arg(u^{\beta}) - \arg(v^{\beta})\right). \tag{56}$$

For real α and β, a universal principal-branch formula for transforming a power of a power to an unnested power is

$$(w^{\alpha})^{\beta} \rightarrow (-1)^{\zeta}\,w^{\alpha\beta} \tag{57}$$

where

$$\zeta := \frac{\beta}{\pi}\left(\arg\left(w^{\alpha}\right) - \alpha\arg\left(w\right)\right). \tag{58}$$

This can be derived from the identities

$$|p| \equiv (-1)^{-\arg(p)/\pi}p, \tag{59}$$

$$|q^{\alpha}|^{\beta} \equiv |q|^{\alpha\beta}. \tag{60}$$

Applying these formulas to our case gives

$$\begin{aligned}(c(z)\,z^{\alpha}+g(z))^{\beta} &\rightarrow \left(c(z)z^{\alpha}\left(1+\frac{g(z)}{c(z)\,z^{\alpha}}\right)\right)^{\beta}\\ &\rightarrow (-1)^{\Lambda}c(z)^{\beta}\,z^{\alpha\beta}\left(1+\frac{g(z)}{c(z)\,z^{\alpha}}\right)^{\beta} \qquad (61)\end{aligned}$$

where modulo 2, which suffices for $(-1)^{\Lambda}$,

$$\begin{aligned}\Lambda &\equiv \frac{\beta}{\pi}\left(\arg(c(z)\,z^{\alpha}+g(z))-\arg\left(1+\frac{g(z)}{c(z)z^{\alpha}}\right)-\arg(c(z))-\alpha\arg(z)\right)\\ &= \frac{\beta}{\pi}\Psi, \qquad (62)\end{aligned}$$

with Ψ defined by alternatives (49) through (54). Let

$$\begin{aligned}L &:= (-1)^{\Lambda}. \qquad (63)\\ &= L_1L_2, \qquad (64)\end{aligned}$$

where

$$\begin{aligned}L_1 &= (-1)^{2\beta\left\lfloor\frac{1}{2}-\frac{\alpha\arg(z)}{2\pi}\right\rfloor}, \qquad (65)\\ L_2 &= (-1)^{\beta\Omega/(2\pi)}\begin{cases}(-1)^{2\beta} & \text{if } \arg(c(z))+\arg(z^{\alpha})\le-\pi,\\ (-1)^{-2\beta} & \text{if } \arg(c(z))+\arg(z^{\alpha})>\pi,\\ 1 & \text{otherwise,}\end{cases} \qquad (66)\\ &= \begin{cases}(-1)^{2\beta} & \text{if } \arg(c(z))+\arg(z^{\alpha})+\arg\left(1+\frac{g(z)}{c(z)z^{\alpha}}\right)\le-\pi,\\ (-1)^{-2\beta} & \text{if } \arg(c(z))+\arg(z^{\alpha})+\arg\left(1+\frac{g(z)}{c(z)z^{\alpha}}\right)>\pi,\\ 1 & \text{otherwise.}\end{cases} \qquad (67)\end{aligned}$$

Alternative (66) is preferable for L_2 when ω is constant. Techniques described in subsection 2.3 can further simplify the floor sub-expression in L_1. Also, the exponents of -1 in these formulas can be simplified and canonicalized by replacing them with their near-symmetric residue modulo 2.

Remark 11. In the common case where β is integer, then

$$(-1)^{2\beta} = (-1)^{-2\beta} = (-1)^{2\beta\lfloor 1/2-\alpha\arg(z)/(2\pi)\rfloor} = 1,$$

so L can be simplified to 1.

Remark 12. In the next most common case where β is a half-integer, L_1 can be expressed more candidly as

$$\hat{L}_1 = \begin{cases}1 & \text{if mods}\,(\pi-\alpha\arg(z),\,2\pi)\ge 0,\\ -1 & \text{otherwise,}\end{cases} \qquad (68)$$

and L_2 can be simplified to

$$\hat{L}_2 = (-1)^{\beta\Omega/(2\pi)} \begin{cases} 1 & \text{if } -\pi < \arg(c(z)) + \arg(z^\alpha) \le \pi \\ -1 & \text{otherwise.} \end{cases} \tag{69}$$

$$= \begin{cases} 1 & \text{if } -\pi < \arg(c(z)) + \arg(z^\alpha) + \arg\left(1+\dfrac{g(z)}{c(z)z^\alpha}\right) \le \pi, \\ -1 & \text{otherwise.} \end{cases} \tag{70}$$

It is easy and worthwhile to test for these cases. Alternative (69) is preferable when ω is a constant.

Techniques described in subsection 2.3 can further simplify local equivalents to L_2 and $\hat{L}_2$ – particularly for real z.

Also, if the local equivalent simplifies to a piecewise constant that is one value ℓ_0 at $z = 0$ for which $0\,\ell_0$ simplifies to 0, but another value ℓ_* everywhere else, then we can simply use ℓ_* for all of the terms having positive degree, because those powers of z are 0 at $z = 0$. This can dramatically simplify the result because often there are no non-positive degree terms or only one. For example, with real x there is only one non-positive degree term for

$$\text{series}\left(\left(-1+x-ix^2\right)^{7/2},\, x=0,\, o\left(x^2\right)\right) \;\to\; \left(\begin{cases} 1 & \text{if } x=0 \\ -1 & \text{otherwise} \end{cases}\right)(-i) - \frac{7}{2}ix + \frac{35}{8}ix^2 + o\left(x^2\right).$$

Otherwise if L doesn't simplify to a constant, then we have to include a perhaps complicated factor multiplying *every term* of the result, because extracting the dominant coefficient and exponent are mandatory for numeric powers of series. This is significantly more annoying than the possible one piecewise-constant term for a logarithm of a series. If we choose to distribute non-constant L over every term, then the result is significantly bulkier and less intelligible. If instead we choose to return an undistributed product of this factor with a sum of terms, then we have denied the user what they implicitly requested by using a function named series – a *sum* of terms.

If we are implementing the full generality of a hierarchical series, then such recursively-represented series have some appeal, because the jumps in the rotation angle make it have some properties of some essential singularities, which dominate any power of z. However, one can argue that such jumps belong in the coefficients because the magnitude of the rotation factor is always 1, whereas the logarithmic singularities that we already allow in the coefficients have infinite magnitude at $z = 0$.

It is easy for a human user to use an expand $(\ldots)$ function to distribute the factor over the terms after seeing an unexpanded result. However, the user might be other functions whose authors must be knowledgeable enough to realize that they should always apply expand $(\ldots, z)$ to the result of series $(\ldots)$ because it could be an expandable product.

If we are not implementing the full generality of hierarchical series, then a more serious disadvantage of not automatically distributing the factor is that it requires a special field for a multiplicative factor in the series data structure; and one such specialized exceptional field is inadequate for adding series containing such multiplicative factors if the series have different dominant exponents.

Expressions such as $(-1)^{2\beta}$ can alternatively be expressed as $L := e^{2\beta\pi i}$ or represented that way internally. However for presentation in results, $(-1)^{2\beta}$ avoids a perhaps-unnecessary i and more obviously indicates that the factor has magnitude 1.

Remark 13. Whenever z is real and L contains more than one piecewise factor, their product can often be combined into a single factor by separately simplifying their product under the alternative constraints $z < 0$, $z = 0$, and $z > 0$, then forming a single piecewise or constant factor accordingly if the three values are constants. This process often simplifies the product to the constant 1 when β is a half integer.

3.2 Real-branch series of fractional powers

For computing a fractional power of a series we must fully distribute the exponent over the dominant coefficient and dominant power, then combine the two exponents of the latter into a product. A way to accomplish this correctly for real-branch mode is to use in this mode the following rewrite rules for reduced exponents for a real expression t together with integers m and n:

$$\arg\left(t^{2m/(2n+1)}\right) \rightarrow 0, \tag{71}$$

$$\arg\left(t^{(2m+1)/(2n+1)}\right) \rightarrow \arg(t). \tag{72}$$

4 Branch bugs for inverse trig & hyperbolic series

Kahan [3] is often cited as a standard for defining the branch cuts of the inverse trigonometric and inverse hyperbolic functions, together with the principal values on those branch cuts and on the branch points that end them. In comparison to *Derive*, his definition of $\arctan(\ldots)$ has the disadvantage of violating the useful identity

$$\arctan\left(\frac{z}{\sqrt{1-z^2}}\right) \equiv \arcsin(z). \tag{73}$$

However, this section uses Kahan's definitions, which are used by TI-Nspire.

This section doesn't discuss the six secondary functions such as $\operatorname{arcsec}(z) := \arccos(1/z)$ and $\operatorname{arccot}(z) := \pi/2 - \arctan(z)$ because their series are easily computed from such definitions. Beware, however, that different mathematical software might use different definitions. For example, *Mathematica* uses

$$\operatorname{arccot}(z) := \begin{cases} \pi/2 & \text{if } z = 0, \\ \dfrac{i}{2}\left(\ln(1 - i/z) - \ln(1 + i/z)\right) & \text{otherwise,} \end{cases}$$

which is not equivalent everywhere to $\pi/2 - \arctan(z)$.

One way to compute series for the primary inverse trigonometric and inverse hyperbolic functions is by their integral definitions. However, the handling of non-constant coefficients that can include nested logarithms and piecewise constants is then problematic, and this method alone doesn't specify the degree-0 term of the result.

Consequently, this article instead uses Kahan's definitions in terms of logarithms and powers.

Many of the tests in this section entail comparing the imaginary part of a series V with 0. For both real and complex z these tests can often be simplified by determining critical angles and perhaps also computing $I_c(V, \theta_c)$, as described in Sub-subsections 2.1.1 and 2.1.2. Some of the following formulas instead entail comparing the real part of a series V with 0. They often can similarly be simplified by determining critical angles and perhaps also computing $I_c(iV, \theta_c)$, which maps the real part to an imaginary part.

4.1 Series for arctanh

The inverse hyperbolic tangent of a series U can be computed from the identity

$$\operatorname{arctanh}(U) \quad \equiv \quad \frac{\ln(1+U)-\ln(1-U)}{2}. \tag{74}$$

The logarithms in formula (74) can contribute expressions involving ω, ϕ or ψ as discussed in Section 2. The result is candid in most cases. However, if series $U = cz^{\alpha} + g(z)$ has a negative dominant exponent α, then the zero-degree term computed by (74) is

$$\ln(cz^{\alpha}) - \ln(-cz^{\alpha}) + (\Omega(U+1) + \Omega(-U+1)). \tag{75}$$

This expression can and should be replaced with the simpler local equivalent

$$\frac{\pi i}{2}\begin{cases} \pm 1 & \text{if } z = 0, \\ 1 & \text{if } \Im(U) > 0 \,\vee\, \Im(U) = 0 \,\wedge\, \Re(U) < 0\,, \\ -1 & \text{otherwise,} \end{cases} \tag{76}$$

which is also less prone to rounding errors.

4.2 Series for arctan

The inverse tangent of a series U can be computed from the identity

$$\arctan(U) \quad \equiv \quad -i \operatorname{arctanh}(iU). \tag{77}$$

The degree-0 term of $\arctan(U)$ is

$$\begin{cases} 0 & \text{if } \alpha > 0, \\ \pm\pi/2 & \text{if } \alpha < 0 \,\wedge\, z = 0, \\ \pi/2 & \text{if } \alpha < 0 \,\wedge\, (\Re(u) > 0 \,\vee\, \Re(u) = 0 \,\wedge\, \Im(u) > 0)\,, \\ -\pi/2 & \text{if } \alpha < 0 \,\wedge\, (\Re(u) < 0 \,\vee\, \Re(u) = 0 \,\wedge\, \Im(u) < 0)\,, \\ \frac{1}{2}\left(-i\ln\left(\frac{ibz^{\sigma}}{2}\right) + \Omega(ibz^{\sigma} + ih(z))\right) & \text{if } U = i + bz^{\sigma} + h(z), \\ \frac{1}{2}\left(i\ln\left(\frac{-ibz^{\sigma}}{2}\right) - \Omega(-ibz^{\sigma} - ih(z))\right) & \text{if } U = -i + bz^{\sigma} + h(z), \\ i\operatorname{arctanh}(t) + \frac{1}{2}\Omega_{\pi,0}(1 - t + ig(z)) & \text{if } U = ti + g(z) \,\wedge\, t > 1, \\ i\operatorname{arctanh}(t) - \frac{1}{2}\Omega_{\pi,0}(1 + t - ig(z)) & \text{if } U = ti + g(z) \,\wedge\, t < -1, \\ \arctan(c(z)) & \text{otherwise.} \end{cases} \tag{78}$$

Formula (77) should automatically achieve these simplified forms except probably for the three cases where $\alpha < 0$.

Table 3 lists some simplified correct 0-degree terms for $\arctan(\ldots)$. Corresponding examples for $\operatorname{arctanh}(\ldots)$ can be derived from (77).

4.3 Series for arcsinh, arcsin and arccos

The inverse arcsine, arccosine and hyperbolic arcsine of a series U can be computed from

$$\operatorname{arcsinh}(U) \equiv \ln\left(U+\sqrt{1+U^2}\right), \tag{79}$$

$$\arcsin(w) \equiv -i\operatorname{arcsinh}(iw), \tag{80}$$

$$\arccos(w) \equiv \frac{\pi}{2}-\arcsin(w). \tag{81}$$

The square root might contribute a factor containing expressions involving $\arg(\ldots)$ to all of its result terms. The subsequent logarithm in identity (79) might then contribute horrid nested instances of $\arg(\ldots)$ to a 0-degree term in the result.

Mathematica 7.0.1.0 avoids this by instead using correction terms and factors that are specific to inverse trigonometric and inverse hyperbolic functions: The resulting arguments of $\arg(\ldots)$ are unnested and entail the terms of U, which are often fewer and simpler than those of $1+U^2$ and $U+\sqrt{1+U^2}$. Extensive numeric experiments provide convincing evidence that they are correct in most instances. Most of the formulas in this subsection and the following one are simplified and corrected versions of those corrections. When using these corrections and identity (79), it is important to use the rewrite $\ln(pz^\alpha)\to\ln(p)+\alpha\ln(z)$, but suppress the branch corrections for logarithms and fractional powers discussed in previous sections. It is also important to compute intermediate series such as $\sqrt{1+U^2}$ to sufficient order so that the final order is as requested.

A correction term or factor is unnecessary for $\operatorname{arcsinh}(U(z))$ if the dominant degree of $U(z)$ is positive or if the dominant degree is 0 and $U(0)$ isn't on either branch cut or either branch point. Otherwise, here are the different cases:

4.3.1 Case $\operatorname{arcsinh}(U(z)=\hat{c}i+g(z))$ with $\hat{c}<-1\vee\hat{c}>1$

If the dominant term of $U(z)$ is $\hat{c}i$ with $\hat{c}<-1\vee\hat{c}>1$, then

$$\operatorname{arcsinh}(\hat{c}i+g(z)) = \frac{i\pi\operatorname{sign}(\hat{c})}{2}+T\cdot\left(\operatorname{arccosh}(\hat{c})+o\left(z^0\right)\right),$$

where

$$T = \begin{cases}1 & \text{if } \operatorname{sign}(\hat{c})\,\Re\,(g(z))\geq 0,\\ -1 & \text{otherwise.}\end{cases} \tag{82}$$

4.3.2 Case $\operatorname{arcsinh}(U(z)=\bar{c}i+g(z))$ with $\bar{c}=1\vee\bar{c}=-1$

If the dominant term of $U(z)$ is $\bar{c}i$ with $\bar{c}=1\vee\bar{c}=-1$ and the next non-zero term is bz^σ, then

$$\operatorname{arcsinh}(\bar{c}i+bz^\sigma+h(z)) = \frac{i\pi\bar{c}}{2}+(-1)^P Q\cdot\left(i\bar{c}\sqrt{2i\bar{b}}z^{\sigma/2}+o\left(z^{\sigma/2}\right)\right)$$

where

$$P = \left\lfloor\frac{1}{2}-\frac{\arg\left(i\bar{c}b+\dfrac{i\bar{c}h(z)}{z^\sigma}\right)+\sigma\arg(z)}{2\pi}\right\rfloor, \tag{83}$$

$$Q = \begin{cases}-1 & \text{if } \arg(b)=\dfrac{\pi\bar{c}}{2}\wedge\Re\left(\dfrac{h(z)}{z^\sigma}\right)<0,\\ 1 & \text{otherwise.}\end{cases} \tag{84}$$

Expression $i\bar{c}b+\dfrac{i\bar{c}h(z)}{z^{\sigma}}\to i\bar{c}b$ as $z\to 0$, so the critical angles for P are

$$-\pi\;<\;\theta_c=\frac{(2n+1)\,\pi-\arg\left(i\bar{c}b\right)}{\sigma}\;\leq\;\pi \tag{85}$$

for all integer n satisfying

$$\left\lfloor\frac{\arg\left(i\bar{c}b\right)}{2\pi}-\frac{1}{2}-\frac{\sigma}{2}\right\rfloor<n\leq\left\lfloor\frac{\arg\left(i\bar{c}b\right)}{2\pi}-\frac{1}{2}+\frac{\sigma}{2}\right\rfloor. \tag{86}$$

There are no such angles if $0<\sigma<1$ and $\arg\left(i\bar{c}b\right)$ is sufficiently close to 0: If

$$(\sigma-1)\,\pi\;<\;\arg\left(i\bar{c}b\right)\;<\;(1-\sigma)\,\pi, \tag{87}$$

then $P\equiv 0$, which is easily tested.

If $0<\sigma\leq 2$, then a more candid and accurate representation of $(-1)^{P}$ is

$$\begin{cases}1 & \text{if }-\pi<\arg\left(i\bar{c}b+\dfrac{i\bar{c}h(z)}{z^{\sigma}}\right)+\sigma\arg(z)\leq\pi,\\ -1 & \text{otherwise.}\end{cases} \tag{88}$$

4.3.3 Case $\operatorname{arcsinh}(\ldots)$ with negative dominant exponent and imaginary dominant coefficient

Use formula (79) including the angle rotation factor for the square root and the correction term for the logarithm. Brace yourself for a truly ugly result.

4.3.4 Case $\operatorname{arcsinh}(\ldots)$ with negative dominant exponent and non-imaginary dominant coefficient

If the dominant exponent α of $U(z)$ is negative and the dominant coefficient c isn't pure imaginary, then

$$\operatorname{arcsinh}\left(cz^{\alpha}+g(z)\right)\;=\;(-1)^{W}N\cdot\left(i\pi W+\frac{\ln\left(4c^{2}\right)}{2}+\alpha\ln(z)+o\left(z^{0}\right)\right). \tag{89}$$

where

$$W\;=\;\left\lfloor\frac{1}{2}-\frac{\arg\left(\left(c+\dfrac{g(z)}{z^{\alpha}}\right)^{2}\right)+2\alpha\arg(z)}{2\pi}\right\rfloor, \tag{90}$$

$$N\;=\;\begin{cases}1 & \text{if }\Re(c)>0,\\ -1 & \text{otherwise.}\end{cases} \tag{91}$$

Expression $\left(c+\dfrac{g(z)}{z^{\alpha}}\right)^{2}\to c^{2}$ as $z\to 0$, so the critical angles for W are

$$-\pi\;<\;\theta_c=\frac{(2n+1)\,\pi-\arg\left(c^{2}\right)}{2\alpha}\;\leq\;\pi \tag{92}$$

for all integer n satisfying

$$\left\lfloor \frac{\arg\left(c^2\right)}{2\pi} - \frac{1}{2} - \frac{\alpha}{2} \right\rfloor < n \le \left\lfloor \frac{\arg\left(c^2\right)}{2\pi} - \frac{1}{2} + \frac{\alpha}{2} \right\rfloor . \tag{93}$$

There are no such angles if $-1/2 < \alpha < 0$ and $\arg\left(c^2\right)$ is sufficiently close to 0: If

$$(\alpha - 1)\,\pi \;<\; \arg\left(c^2\right) \;<\; (1-\alpha)\,\pi, \tag{94}$$

then $W \equiv 0$, which is easily tested.

If $-1 \le \alpha < 0$, then more candidly and accurately,

$$W \;=\; \begin{cases} 1 & \text{if } \arg\left(\left(c + \dfrac{g(z)}{z^\alpha}\right)^2\right) + 2\alpha \arg(z) > \pi, \\ -1 & \text{if } \arg\left(\left(c + \dfrac{g(z)}{z^\alpha}\right)^2\right) + 2\alpha \arg(z) \le -\pi, \\ 0 & \text{otherwise;} \end{cases} \tag{95}$$

$$(-1)^W \;=\; \begin{cases} 1 & \text{if } -\pi < \arg\left(\left(c + \dfrac{g(z)}{z^\alpha}\right)^2\right) + 2\alpha \arg(z) \le \pi, \\ -1 & \text{otherwise.} \end{cases} \tag{96}$$

Some incorrect series for arcsinh (...), arcsin (...) and arccos (...) in current implementations are attributable to incorrect or omitted correction terms and/or factors. Table 4 lists some examples for arcsinh (...). Corresponding examples for arcsin (...) and arccos (...) can be derived from (80) and (81).

4.4 Branch corrections for arccosh

Subsection 4.3 expressed the series for arcsin and arccos but not arccosh in terms of the series for arcsinh. The inverse hyperbolic arccosine of a series U can instead be computed from either of

$$\operatorname{arccosh}(U) \;\equiv\; 2\ln\left(\sqrt{\frac{U-1}{2}} + \sqrt{\frac{U+1}{2}}\right), \tag{97}$$

$$\operatorname{arccosh} \;\equiv\; \ln\left(U + \sqrt{U-1}\sqrt{U+1}\right). \tag{98}$$

The latter is slightly slower asymptotically because of the series multiplication. The alternative that gives a simpler result might depend on the general expression simplifier and the case, such as whether or not the dominant degree of U is negative.

Each square root in these formulas might contribute a factor containing expressions involving arg (...) to all of its result terms. The subsequent logarithm in identity (97) or (98) might then contribute horrid nested instances of arg (...) to a 0-degree term in the result. Here are some simplified correction terms and factors that avoid this:

Let the dominant term of U be cz^α, and let the sum of the remaining terms be $g(z)$. Let bz^σ be the dominant term of $g(z)$, and let $h(z)$ be the remaining terms of $g(z)$. A correction term and/or factor is unnecessary if $\alpha = 0 \wedge (\Im(c) \neq 0 \vee c > 1)$. Otherwise we have the following cases:

4.4.1 Case $\operatorname{arccosh}(U(z))$ with dominant degree 0 & dominant coefficient < -1

If $\alpha = 0$ and $c < -1$, then

$$\operatorname{arccosh}(c(z) + g(z)) \;=\; 2i\pi J + \operatorname{arccosh}(c(z)) + o\left(z^0\right) \tag{99}$$

where

$$J \;=\; \begin{cases} -1 & \text{if } \Im(g(z)) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{100}$$

4.4.2 Case $\operatorname{arccosh}(U(z))$ with dominant degree 0 and dominant coefficient in (-1, 0) or (0,1)

If $\alpha = 0 \wedge (-1 < c < 0 \vee 0 < c < 1)$ then

$$\operatorname{arccosh}(c + g(z)) = K \cdot \left(\operatorname{arccosh}(c) + o\left(z^0\right)\right), \tag{101}$$

where

$$K \;=\; (-1)^J \;=\; \begin{cases} -1 & \text{if } \Im(g(z)) < 0, \\ 1 & \text{otherwise.} \end{cases} \tag{102}$$

4.4.3 Case $\operatorname{arccosh}(U(z))$ with dominant degree 0 and dominant coefficient 1

If instead $\alpha = 0$ and $c = 1$, then

$$\operatorname{arccosh}(1 + bz^\sigma + h(z)) \;=\; BE \cdot \left(\sqrt{2b}z^{\sigma/2} + o\left(z^{\sigma/2}\right)\right), \tag{103}$$

where

$$B \;=\; (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg\left(b + \frac{h(z)}{z^\sigma}\right) + \sigma \arg(z)}{2\pi} \right\rfloor}, \tag{104}$$

$$E \;=\; \begin{cases} -1 & \text{if } \arg(b) = \pi \wedge \Im\left(\dfrac{h(z)}{z^\sigma}\right) < 0, \\ 1 & \text{otherwise.} \end{cases} \tag{105}$$

Expression $b + \frac{h(z)}{z^\sigma} \to b$ as $z \to 0$, so the critical angles for B are

$$-\pi \;<\; \theta_c = \frac{(2n+1)\pi - \arg(b)}{\sigma} \;\leq\; \pi \tag{106}$$

for all integer n satisfying

$$\left\lfloor \frac{\arg(b)}{2\pi} - \frac{1}{2} - \frac{\sigma}{2} \right\rfloor < n \leq \left\lfloor \frac{\arg(b)}{2\pi} - \frac{1}{2} + \frac{\sigma}{2} \right\rfloor. \tag{107}$$

There are no such angles if $0 < \sigma < 1$ and $\arg(b)$ is sufficiently close to 0: If

$$(\sigma - 1)\pi \;<\; \arg(b) \;<\; (1 - \sigma)\pi, \tag{108}$$

then $B \equiv 0$, which is easily tested.

If $0 < \sigma \leq 2$, then a more candid and accurate representation of B is

$$\begin{cases} 1 & \text{if } -\pi < \arg\left(b + \frac{h(z)}{z^\sigma}\right) + \sigma \arg(z) \leq \pi, \\ -1 & \text{otherwise.} \end{cases} \tag{109}$$

4.4.4 Case $\operatorname{arccosh}(U(z))$ with dominant degree 0 and dominant coefficient -1

If $\alpha = 0$ and $c = -1$, then

$$\operatorname{arccosh}(-1 + bz^\sigma + h(z)) = i\pi C + CBE \cdot \left(i\sqrt{2b} z^{\sigma/2} + o\left(z^{\sigma/2}\right)\right)$$

where

$$C = \begin{cases} 1 & \text{if } \Im(bz^\sigma + h(z)) < 0, \\ -1 & \text{otherwise.} \end{cases} \tag{110}$$

4.4.5 Case $\operatorname{arccosh}(U(z))$ with negative dominant degree

If instead $\alpha < 0$ then

$$\operatorname{arccosh}(cz^\alpha + g(z)) = 2i\pi(D_1 + M) + \ln(2c) + \alpha \ln(z) + o\left(z^0\right) \tag{111}$$

where

$$D_1 = \left\lfloor \frac{1}{2} - \frac{\arg\left(c + \frac{g(z)}{z^\alpha}\right) + \alpha \arg(z)}{2\pi} \right\rfloor, \tag{112}$$

$$M = \begin{cases} 1 & \text{if } \arg(c) = \pi \wedge \Im\left(\frac{g(z)}{z^\alpha}\right) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{113}$$

Expression $c + \frac{g(z)}{z^\alpha} \to c$ as $z \to 0$, so the critical angles for D_1 are

$$-\pi < \theta_c = \frac{(2n+1)\pi - \arg(c)}{\alpha} \leq \pi \tag{114}$$

for all integer n satisfying

$$\left\lfloor \frac{\arg(c)}{2\pi} - \frac{1}{2} - \frac{\alpha}{2} \right\rfloor < n \leq \left\lfloor \frac{\arg(c)}{2\pi} - \frac{1}{2} + \frac{\alpha}{2} \right\rfloor. \tag{115}$$

There are no such angles if $-1 < \alpha < 0$ and $\arg(c)$ is sufficiently close to 0: If

$$-(1+\alpha)\pi < \arg(c) < (1+\alpha)\pi, \tag{116}$$

then $D_1 \equiv 0$, which is easily tested.

If $-2 < \alpha < 0$, then a more candid and accurate representation of D_1 is

$$\begin{cases} 1 & \text{if } -\pi < \arg\left(c + \frac{g(z)}{z^\alpha}\right) + \alpha \arg(z) \leq \pi, \\ -1 & \text{otherwise.} \end{cases} \tag{117}$$

4.4.6 Case $\operatorname{arccosh}(U(z))$ with positive dominant degree

If instead $\alpha > 0$, then

$$\operatorname{arccosh}(cz^{\alpha} + g(z)) = (-1)^{D_1} (-1)^{D_2} G \cdot \left(\frac{i\pi}{2} + o\left(z^0\right) \right) \tag{118}$$

where

$$D_2 = \left\lfloor \frac{1}{2} - \frac{1}{2\pi} \left(\arg\left(\frac{-1}{c + \frac{g(z)}{z^{\alpha}}} \right) - \alpha \arg(z) \right) \right\rfloor, \tag{119}$$

$$G = \begin{cases} -1 & \text{if } z \neq 0 \wedge \Im(c) < 0 \vee \\ & \quad \arg(c) = \pi \wedge \Im\left(\frac{g(z)}{z^{\alpha}}\right) < 0 \vee \\ & \quad \arg(c) = 0 \wedge \Im\left(\frac{g(z)}{cz^{\alpha} + g(z)}\right) < 0, \\ 1 & \text{otherwise.} \end{cases} \tag{120}$$

Expression $-1/\left(c + g(z)/z^{\alpha}\right) \to -1/c$ as $z \to 0$, so the critical angles for D_2 are

$$-\pi < \theta_c = \frac{(2n+1)\pi - \arg(-1/c)}{\alpha} \leq \pi \tag{121}$$

for all integer n satisfying

$$\left\lfloor \frac{\arg(-1/c)}{2\pi} - \frac{1}{2} - \frac{\alpha}{2} \right\rfloor < n \leq \left\lfloor \frac{\arg(-1/c)}{2\pi} - \frac{1}{2} + \frac{\alpha}{2} \right\rfloor. \tag{122}$$

There are no such angles if $0 < \alpha < 1$ and $\arg(c)$ is sufficiently close to 0: If

$$(\alpha - 1)\pi < \arg(-1/c) < (1-\alpha)\pi, \tag{123}$$

then $D_2 \equiv 0$, which is easily tested.

If $0 < \alpha \leq 2$, then a more candid and accurate representation of D_2 is

$$\begin{cases} 1 & \text{if } -\pi < \arg\left(-1/\left(c + \frac{g(z)}{z^{\alpha}}\right)\right) + \alpha \arg(z) \leq \pi, \\ -1 & \text{otherwise.} \end{cases} \tag{124}$$

When using these corrections and identity (97) or (98), it is important to use the rewrite $\ln(cz^{\alpha}) \to \ln(c) + \alpha \ln(z)$ but suppress in these formulas the logarithmic and fractional power adjustments discussed in Sections 2 and 3. Also, sub-expressions such as

$$\frac{-1}{c + \frac{g(z)}{z^{\alpha}}} \quad \text{and} \quad \frac{g(z)}{cz^{\alpha} + g(z)}$$

in (119) and (120) can and should be approximated by appropriate-order series. Note that a truncated series for these two sub-expressions usually can't be exact for non-zero $g(z)$. Therefore we should expect some narrow cusps in which the corrections are not piecewise constant. If $g(z)$ is known to $o(z^m)$, then we can compute the series for these two sub-expressions to $o(z^{m-\alpha})$.

For both $\operatorname{arcsinh}(U(z))$ and $\operatorname{arccosh}(U(z))$, if we are unable to narrow the choice to one of the above cases for reasons such as c containing insufficiently constrained indeterminates other than z, then we should construct a piecewise result from all of the cases that we can't preclude.

Table 5 contains some correct results for $\operatorname{arccosh}(\ldots)$ series.

Summary

The generalization from Taylor series to generalized Puiseux series introduces a surprising number of difficulties that haven't been fully addressed in previous literature and implementations. The most serious of these is incorrect results for expansion points that are on a branch cut or a branch point. Formulas are presented here that correct this for logarithms, fractional powers, inverse trigonometric functions, and inverse hyperbolic functions. These corrections typically entail an additive piecewise-constant multiple of $2\pi i$ and/or a unit-magnitude piecewise-constant multiplicative factor. Some of these corrections are applicable even to Taylor series. There are many alternative formulas for these corrections. The alternatives presented here are chosen:

- to reduce catastrophic cancellation when evaluated with approximate arithmetic,
- to candidly reveal the piecewise constancy,
- to reveal the boundaries of the pieces as explicitly as is practical,
- to simplify significantly where practical, such as for real expansion variables or numeric coefficients.
- to be as concise as possible subject to the above goals.

Tests, formulas and algorithms are given that compute simplified versions of these correction terms and factors near the expansion point for both real and complex expansion variables.

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Table 1: 0-degree term of series $(\ln u,\ \mathrm{var}{=}0,\ o(\mathrm{var}^n))$ with $n \geq 4$, $x, y \in \mathbb{R}$, $z = x{+}iy$:

#	u	Alternative 0-degree terms of $\ln u$ near variable $= 0$	why
1a	z^2+z^3	$\ln(z^2)+\begin{cases}2i\pi & \text{if } \Im(z)<0 \wedge \Re(z) \geq \Im(z)^2/2+\cdots+o(\Im(z)^n)\\ -2i\pi & \text{if } \Im(z)\geq 0 \wedge \Re(z) \geq \Im(z)^2/2+\cdots+o(\Im(z)^n)\\ 0 & \text{otherwise}\end{cases}$	(11) (23) rem. 6
1b		$2\ln z+2i\pi\left\lfloor\dfrac{\pi-2\arg z}{2\pi}\right\rfloor+\begin{cases}2i\pi, & \Im(z)<0 \wedge \Re(z) \geq \Im(z)^2/2+\cdots\\ -2i\pi, & \Im(z)\geq 0 \wedge \Re(z) \geq \Im(z)^2/2+\cdots\\ 0, & \text{otherwise}\end{cases}$	(52) (23) rem. 6
2a	$z^2+z^3e^z$	$\ln(z^2)+\begin{cases}2i\pi & \text{if } \arg(1{+}z{+}z^2{+}\cdots+o(z^n))+\arg(z^2) \leq -\pi\\ -2i\pi & \text{if } \arg(1{+}z{+}z^2{+}\cdots+o(z^n))+\arg(z^2) > \pi\\ 0 & \text{otherwise}\end{cases}$	(11)
2b		$2\ln z{+}2\pi i\left\lfloor\dfrac{\pi-2\arg z}{2\pi}\right\rfloor+\begin{cases}2i\pi, & \arg(1+\cdots+o(z^n))+\arg(z^2) \leq -\pi\\ -2i\pi, & \arg(1+\cdots+o(z^n))+\arg(z^2) > \pi\\ 0, & \text{otherwise}\end{cases}$	(54)
2c		$\ln(z^2)+(\arg(z^2{+}\cdots+o(z^n))-\arg(1{+}\cdots+o(z^n))-\arg(z^2))\,i$	(10)
2d		$2\ln(z)+(\arg(z^2{+}\cdots+o(z^n))-\arg(1{+}\cdots+o(z^n))-2\arg(z))\,i$	(53)
3	$-z^{-7/6}-z^{7/3}$	$\ln\left(-z^{-7/6}\right)$ or $-7\ln(z)/6+\begin{cases}i\pi & \text{if } \Im(z)\geq 0\\ -\pi i & \text{otherwise}\end{cases}$	prop 5 or (54)
4	$-1-z^2-z^3$	$\begin{cases}i\pi & \text{if } \left(\Im(z)\geq 0 \wedge \Re(z) \leq \Im(z)^2/2+\cdots+o(\Im(z)^n)\right) \vee \\ & \quad \left(\Im(z)>0 \wedge \Re(z) \geq \Im(z)^2/2+\cdots+o(\Im(z)^n)\right)\\ -i\pi & \text{otherwise}\end{cases}$	(29) rem. 6
5	$-1-z^2e^z$	$\begin{cases}i\pi & \text{if } \Im(z^2+2z^3+z^4+\cdots+o(z^n)) \leq 0\\ -i\pi & \text{otherwise}\end{cases}$	(29)
6	$-1{+}iz^{1/4}$	$i\pi$	(31)
7	$-1-iz^{\frac{1}{4}}+z$	$\begin{cases}i\pi & \text{if } z=0\\ -i\pi & \text{otherwise}\end{cases}$	(32)
8	$-1{+}iz^{\frac{1}{2}}{+}z$	$\begin{cases}i\pi & \text{if } z\leq 0\\ -i\pi & \text{otherwise}\end{cases}$	rem. 8
9	$c+z^2$	$\begin{cases}\ln(z^2) & \text{if } c=0\\ \ln(c)-2i\pi & \text{if } \arg(c)=\pi \wedge \Im(z^2)<0\\ \ln(c) & \text{otherwise}\end{cases}$	rem. 9
10	$-x^{-2}+e^x$	$\ln(x^{-2})$	prop 3
11	$x^2+x^3e^x$	$\ln(x^2)$, or $2\ln(\lvert x\rvert)$, or $2\ln(x)+\begin{cases}-2i\pi & \text{if } x<0\\ 0 & \text{otherwise}\end{cases}$	prop 3 or (54)
12a	$x^{4/3}+x^2$	$\ln\left(x^{4/3}\right)$	rem. 7
12b	real branch	$\ln\left(x^{4/3}\right)$, or $4\ln(\lvert x\rvert)/3$, or $4\ln(x)/3-\begin{cases}4i\pi/3 & \text{if } x<0\\ 0 & \text{otherwise}\end{cases}$	sec. 2.3.2

Table 2: For $n\geq 4,\ x,y\in\mathbb{R},\ z=x+iy$: $\text{series}\left(u^\beta,\ \text{var}=0,\ o(\text{var}^n)\right)\to(-1)^\lambda\,(S+\cdots)$

#	u^β	λ, or $L=(-1)^\lambda$	S	why
1a	$\left(z^2+z^3\right)^{3/2}$	$L=\begin{cases}1, & \left(\Im(z)\geq 0\wedge\Re(z)\geq\Im(z)^2/2+\ldots+o(\Im(z)^n)\right)\\ & \vee\left(\Im(z)<0\wedge\Re(z)>\Im(z)^2/2+\ldots+o(\Im(z)^n)\right)\\ -1, & \text{otherwise}\end{cases}$	z^3	(68) (70) rem. 6
1b		$L=\begin{cases}1 & \text{if } (\Re(z)>0\vee\Re(z)=0\wedge\Im(z)\geq 0)\\ & =(-\pi<\arg(z^2)+\arg(1+z)\leq\pi)\\ -1 & \text{otherwise}\end{cases}$	z^3	(68) (70)
1c		$\lambda=\frac{3}{2\pi}\left(\arg(z^2+z^3)-\arg(1+z)-2\arg(z)\right)$	z^3	(53,62)
2a	$\left(z^2-iz^3\right)^{3/2}$	$L=\begin{cases}1 & \text{if } \Re(z)>0\vee\Re(z)=0\wedge\Im(z)\geq 0\\ -1 & \text{otherwise}\end{cases}$	z^3	(50) (68)
2b		$\lambda=\lfloor 1/2-(\arg z)/2\rfloor$	z^3	(50)
3a	$\left(z^2+z^3\right)^{7/4}$	$L=\begin{cases}1, & \left(\Im(z)\geq 0\wedge\Re(z)\geq\Im(z)^2/2+\ldots+o(\Im(z)^n)\right)\\ & \vee\left(\Im(z)<0\wedge\Re(z)>\Im(z)^2/2+\ldots+o(\Im(z)^n)\right)\\ i, & \left(\Im(z)\geq 0\wedge\Re(z)<\Im(z)^2/2+\ldots+o(\Im(z)^n)\right)\\ -i, & \text{otherwise}\end{cases}$	$z^{7/2}$	(65) (67) rem. 6
3b		$\lambda=\frac{7}{4\pi}\left(\arg(z^2+z^3)-\arg(1+z)-2\arg(z)\right)$	$z^{7/2}$	(53,62)
4	$\left(-1+iz^{1/4}+z\right)^{3/2}$	$L=1$	$-i$	(31,52)
5	$\left(-1+iz^{1/2}+z\right)^{3/2}$	$L=1$	$-i$	rem. 8
6a	$\left(-1-z^2-z^3\right)^{3/2}$	$L=\begin{cases}-1 & \text{if } (y\geq 0\wedge x<y^2/2-3y^4/8+\ldots)\vee\\ & (y<0\wedge x\geq y^2/2-3y^4/8+\ldots)\\ 1 & \text{otherwise}\end{cases}$	$-i$	(68) (70) rem. 6
6b		$\lambda=\frac{3}{2\pi}\left(\arg(-1-z^2-z^3)-\arg(1+z^2+z^3)-\pi\right)$	$-i$	(29,62)
7	$\left(c+z^2\right)^{3/4}\mid c\neq 0$	$L=\begin{cases}i & \text{if } \arg c=\pi\wedge\Im(z^2/c)>0\\ 1 & \text{otherwise}\end{cases}$	$c^{3/4}$	rem. 9 (29)
8	$\left(-z^{-1/2}\right)^{1/2}$	$L=\begin{cases}-1 & \text{if } \arg(z)<0\\ 1 & \text{otherwise}\end{cases}$	$iz^{-1/4}$	rem. 2 (52)
9	$\left(\ln(z)+z\right)^{3/2}$	$L=1$	$\ln(z)^{3/2}$	(29)
10	$\left(iz+z^2\right)^{3/2}$	$L=\begin{cases}-1 & \text{if } \Re(z)<0\wedge\Im(z)\geq 0\\ 1 & \text{otherwise}\end{cases}$	$(-1)^{\frac{3}{4}}\,z^{\frac{3}{2}}$	(23) (69)
11	$\left(z^{-1}+1\right)^{3/2}$	$L=\begin{cases}-1 & \text{if } z<0\\ 1 & \text{otherwise}\end{cases}$	$z^{-3/2}$	prop. 5 (69)
12	$\left(-2+x\right)^{3/2}$	$L=1$	$-i$	prop. 3
13	$\left(-1-\sqrt{x}\right)^{7/2}$	$L=\begin{cases}1 & \text{if } x\geq 0\\ -1 & \text{otherwise}\end{cases}$	$-i$	(23)
14a	$\left(x^{4/3}+x^2\right)^{3/2}$	$L=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{otherwise}\end{cases}$	x^2	(23,69)
14b	real branch	$L=1$	x^2	rem. 4

Table 3: 0-degree term of series $(\arctan(u), z{=}0, o(z^n))$ with $z \in \mathbb{C}$, $x \in \mathbb{R}$, $n \geq 3$:

#	u	A correct 0-degree term near $z = 0$
1	$z^{-2} + z^{-1}$	$\frac{\pi}{2}\begin{cases} \pm 1 & \text{if } z = 0 \\ 1 & \text{if } -\pi/2 < \arg(z^{-2} + z^{-1}) \leq \pi/2 \\ -1 & \text{otherwise} \end{cases}$
2	$2i + ze^z$	$\frac{\ln 3}{2}i + \begin{cases} \frac{\pi}{2} & \text{if } \Re\left(z + z^2 + \frac{z^3}{6} + \cdots + o(z^n)\right) \leq 0 \\ -\frac{\pi}{2} & \text{otherwise} \end{cases}$
3	$2i + z^{1/4}e^z$	$\frac{\pi}{2} + \frac{\ln 3}{2}i$
4	$-2i + ze^z$	$\frac{\ln 3}{2}i + \begin{cases} \frac{\pi}{2} & \text{if } z \neq 0 \\ -\frac{\pi}{2} & \text{otherwise} \end{cases}$
5	$-2i - z^{1/4}e^z$	$-\frac{\pi}{2} - \frac{\ln 3}{2}i$
6	$i + ze^z$	$\frac{\pi - \ln(iz) + \ln 2}{2}i - \begin{cases} 0 & \text{if } \arg\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots+o(z^n)\right) + \arg(iz) \leq \pi \\ \pi & \text{otherwise} \end{cases}$
7	$i + ize^z$	$\ln\left(\frac{z}{2}\right) - \frac{i}{2}$
8	$-i + z^2e^z$	$\frac{i\ln\left(\frac{-iz^2}{2}\right)}{2} + \begin{cases} \pi & \text{if } \arg\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots+o(z^n)\right) + \arg(-iz^2) \leq -\pi \\ \pi & \text{if } \arg\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots+o(z^n)\right) + \arg(-iz^2) > \pi \\ 0 & \text{otherwise} \end{cases}$
9	$-i + iz^{1/2}e^z$	$\frac{i\ln(z/4)}{4}$
10	$x^{-2} + x^{-1}$	$\frac{\pi}{2}\begin{cases} \pm 1 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ -1 & \text{otherwise} \end{cases}$
11	$2i + x^{3/4}e^x$	$\frac{\ln 3}{2}i + \begin{cases} \frac{\pi}{2} & \text{if } x \geq 0 \\ -\frac{\pi}{2} & \text{otherwise} \end{cases}$
12	$2i + x^{1/2}e^x$	$\pi/2 + i(\ln 3)/2$
13	$-2i + x^{3/4}e^x$	$-\frac{\ln 3}{2}i + \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{otherwise} \end{cases}$
14	$-2i - x^{3/4}e^x$	$-\pi/2 - i(\ln 3)/2$
15	$i + x^{5/4}e^x$	$\frac{-\ln\left(ix^{5/4}/2\right)}{2}$
16	$-i + x^{5/4}e^x$	$\frac{\ln\left(-ix^{5/4}/2\right)}{2}$

Table 4: First two non-zero terms of series $\left(\operatorname{arcsinh}(u),\ z{=}0,\ o\left(z^{\infty}\right)\right)$ with $z \in \mathbb{C}$, $x \in \mathbb{R}$:

#	u	First two non-0 terms of $\operatorname{arcsinh} u$ near variable $= 0$	Sec.
1	$-2i + z^2 + z^3$	$-\frac{i\pi}{2} + \left(\begin{cases} 1 & \text{if } \Re\left(z^2+z^3\right) > 0 \\ -1 & \text{otherwise} \end{cases}\right) \left(\ln\left(2+\sqrt{3}\right) + \frac{iz^2}{\sqrt{3}} + \cdots\right)$	4.3.1
2	$2i + z^2 + z^3$	$\frac{i\pi}{2} + \left(\begin{cases} 1 & \text{if } \Re\left(z^2+z^3\right) \geq 0 \\ -1 & \text{otherwise} \end{cases}\right) \left(-\frac{\ln\left(2+\sqrt{3}\right)}{2} - \frac{iz^2}{\sqrt{3}} + \cdots\right)$	4.3.1
3	$2i + z^{1/4}$	$\frac{i\pi}{2} + \ln\left(2+\sqrt{3}\right) - \frac{iz^{1/4}}{\sqrt{3}} + \cdots$	4.3.1
4	$i + z^2 + z^3$	$\frac{i\pi}{2} + (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(-1+iz) + 2\arg z}{2\pi} \right\rfloor} (iz + \cdots)$	4.3.2
5	$i + iz^2 + iz^3$	$\frac{i\pi}{2} + (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(-1+iz)+2\arg z}{2\pi} \right\rfloor} \left(\begin{cases} 1 & \Re(z) \geq 0 \\ -1 & \text{otherwise} \end{cases}\right) \left(\sqrt{2}z + \cdots\right)$	4.3.2
6	$-i + iz^2 + iz^3$	$-\frac{i\pi}{2} + (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(-1+iz) + 2\arg z}{2\pi} \right\rfloor} \left(i\sqrt{2}z + \cdots\right)$	4.3.2
7	$-i - iz^2 + z^3$	$-\frac{i\pi}{2} + (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(-1+iz)+2\arg z}{2\pi} \right\rfloor} \left(\begin{cases} 1, & \Re(z) \leq 0 \\ -1 & \text{otherwise} \end{cases}\right) \left(-\sqrt{2}z + \cdots\right)$	4.3.2
8	$iz^{-1} + 1$	ugh!	4.3.3
9	$z^{-2} + z^{-1}$	$(-1)^w \left(\ln 2 - 2\ln z + \pi i w + z + \cdots\right) \mid w = \left\lfloor \frac{1}{2} - \frac{\arg\left(1+2z+z^2\right) + 4\arg z}{2\pi} \right\rfloor$	4.3.4
10	$-2i + x^{3/2}$	$\left(-\frac{\pi i}{2} - \ln\left(2+\sqrt{3}\right) \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{otherwise} \end{cases}\right) - \frac{ix^{3/2}}{\sqrt{3}} + \cdots$	4.3.1
11	$-2i - x^{3/2}$	$\left(-\frac{\pi i}{2} - \ln\left(2+\sqrt{3}\right)\right) + \frac{ix^{3/2}}{\sqrt{3}} + \cdots$	4.3.1
12	$2i + x^{3/4}$	$\left(\frac{\pi i}{2} + \ln\left(2+\sqrt{3}\right) \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases}\right) - \frac{ix^{3/4}}{\sqrt{3}} + \cdots$	4.3.1
13	$2i + x^{3/2}$	$\left(\frac{\pi i}{2} + \ln\left(2+\sqrt{3}\right)\right) - \frac{ix^{3/2}}{\sqrt{3}} + \cdots$	4.3.1
14	$i + x^{3/2}$	$\frac{\pi i}{2} + \left(\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases}\right) \left((1-i)\, x^{3/4} + \cdots\right)$	4.3.2
15	$i - x^{3/2}$	$\frac{\pi i}{2} - (1+i)\, x^{3/4} + \cdots$	4.3.2
16	$-i + x^{5/2}$	$-\frac{\pi i}{2} + \left(\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases}\right) \left((1+i)\, x^{5/4} + \cdots\right)$	4.3.2
17	$-i + x^{3/2}$	$-\frac{\pi i}{2} + (1+i)\, x^{3/4} + \cdots$	4.3.2
18	$x^{-2} + x^{-1}$	$\left(\ln 2 - 2\ln x + \begin{cases} 0 & \text{if } x \geq 0 \\ 2\pi i & \text{otherwise} \end{cases}\right) + x + \cdots$	4.3.4

Table 5: 1^{st} two non-0 terms of series $(\operatorname{arccosh}(u),\ z{=}0,\ o(z^{\infty}))$ with $z \in \mathbb{C}$, $x \in \mathbb{R}$:

#	u	1^{st} two non-0 terms of $\operatorname{arccosh} u$ near $z = x = 0$	Sec.
1	$-2 + z^2 + z^3$	$\left(\ln\left(2+\sqrt{3}\right) + \begin{cases} i\pi & \text{if } \Im\left(z^2+z^3\right) \geq 0 \\ -i\pi & \text{otherwise} \end{cases}\right) - \dfrac{z^2}{\sqrt{3}} + \cdots$	4.4.1
2	$-2 + iz^{1/4} + z$	$\left(\ln\left(2+\sqrt{3}\right) + i\pi\right) - iz^{1/4}/\sqrt{3} + \cdots$	4.4.1
3	$\dfrac{1}{2} + z^2 + z^3$	$\left(\begin{cases} 1 & \text{if } \Im\left(z^2+z^3\right) \geq 0 \\ -1 & \text{otherwise} \end{cases}\right)\left(\dfrac{i\pi}{3} - \dfrac{2iz^2}{\sqrt{3}} \cdots\right)$	4.4.2
4	$1 + z^2 + z^3$	$\left(\begin{cases} 1 & \text{if } -\pi < \arg(1+z) + 2\arg z \leq \pi \\ -1 & \text{otherwise} \end{cases}\right)\left(\sqrt{2}z + \dfrac{z^2}{\sqrt{2}} + \cdots\right)$	4.4.3
5	$-1 + z^2 + z^3$	$\left(\begin{cases} 1 & \Im\left(z^2+z^3\right) \geq 0 \\ -1 & \text{otherwise} \end{cases}\right)\left(i\pi + B \cdot \left(-i\sqrt{2}z + \cdots\right)\right)$ $\mid B = (-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(1+z) + 2\arg z}{2\pi} \right\rfloor}$	4.4.4
6	$z^{-2} + z^{-1}$	$\left(2i\pi\left\lfloor \dfrac{\pi - \arg(1+z) + 2\arg(z)}{2\pi} \right\rfloor + \ln 2 - 2\ln z\right) + z + \cdots$	4.4.5
7	$z + z^2$	$\left((-1)^{D_1+D_2}\begin{cases} 1 & \text{if } \Im(z+\cdots) \geq 0 \\ -1 & \text{otherwise} \end{cases}\right)\left(\dfrac{i\pi}{2} - iz + \cdots\right)$ $\mid D_1 = \left\lfloor \dfrac{1}{2} - \dfrac{\arg(1+z) + \arg z}{2\pi} \right\rfloor$ $\wedge\, D_2 = \left\lfloor \dfrac{1}{2} - \dfrac{\arg\left(-1+z+z^2+\cdots\right) - \arg z}{2\pi} \right\rfloor$	4.4.6
8	$-2 - x^{3/4}$	$\left(\ln\left(2+\sqrt{3}\right) + i\pi\begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}\right) + \dfrac{x^{3/4}}{\sqrt{3}} + \cdots$	4.4.1
9	$-2 + x$	$\left(\ln\left(2+\sqrt{3}\right) + i\pi\right) - ix/\sqrt{3} + \cdots$	4.4.1
10	$\dfrac{1}{2} - \sqrt{x}$	$\left(\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases}\right)\left(\dfrac{\pi i}{3} + \dfrac{2ix^{1/2}}{\sqrt{3}} + \cdots\right)$	4.4.2
11	${}^{1}\!/\!{}_{2} + x$	$\pi i/3 - 2ix/\sqrt{3}$	4.4.2
12	$1 - x^2 - x^3$	$\left(\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases}\right)\left(\sqrt{2}ix + \dfrac{i}{\sqrt{2}}x^2 + \cdots\right)$	4.4.3
13	$1 - x^2 - x^{3/2}$	$\sqrt{2}ix + ix^{3/2}/\sqrt{2} + \cdots$	4.4.3
14	$-1 + x^2$	$\pi i + \left(\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{otherwise} \end{cases}\right)\left(-\sqrt{2}ix + \cdots\right)$	4.4.4
15	$-x^{-2} - x^{-1}$	$\left(-2\ln x + \ln 2 + \begin{cases} \pi i & \text{if } x \geq 0 \\ 3\pi i & \text{otherwise} \end{cases}\right) - \dfrac{3ix}{2} + \cdots$	4.4.5
16	$-x^{-2} + x^{-1/2}$	$\left(-2\ln x + \ln 2 + \pi i\right) - x^{3/2} + \cdots$	4.4.5
17	$2x - x^{3/2}$	$\pi i/2 - 2ix + \cdots$	4.4.6