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A Linear Sparse Systems Solver (LSSS) applied to the Classification of Integrable non-abelian Laurent ODEs

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Sparse linear algebraic systems can be of different nature. A well known class are systems, we call them "numerical", that result in the discretization of partial differential equations (PDEs). A very different class of systems, we call them "selective", arises from integrability investigations of differential equations. Both types of systems behave very differently during the solution process and give results of different nature and therefore are also solved most efficiently with different methods.

The Linear Selective Systems Solver LSSS, described on the poster, was developed to solve "selective" systems that result when the aim is to find discrete mathematical objects of symbolic nature like Lie symmetries or first integrals if they exist. Table 1 compares both types of problems.

type	"numerical" systems	"selective" systems
examples	systems resulting from a discretization of PDEs	systems resulting from a symmetry investigation of PDEs
value of free parameters when applying the solution of the linear system	any floating point numbers (boundary values of PDE)	0 or 1 (to isolate the individual symmetries)
number of zero-valued variables in solution	essentially none	most variables
initial sparsity	yes	yes
sparsity throughout exact solution	yes	yes
overdetermination	no	yes
usability of iteration schemes for large problems of that type	useful	not useful

Table 1: Characterization of two different types of sparse linear systems

The special nature of selective systems allows a dedicated computer program LSSS (Linear Sparse Systems Solver) running in the computer algebra system REDUCE to be much more efficient than conventional computer programs for solving these systems [1]. Reasons are:

- Because of the existence of many variables that take the value of zero in the solution the systems involve 1-term equations which are utilized first to simplify the remaining system and generate more 1-term equations.
- The simplification of a system due to the vanishing of variables can be accomplished much faster than the simplification due to other substitutions.
- From the mathematical problem it is clear whether the type of a system is numerical or selective and thus to apply the most suitable technique from the start.
- Selective linear systems are typically formulated by separation of larger expressions. The complete separation and up-front formulation of the whole linear system can be avoided through a repeatedly selective splitting and thus formulation and solution of 1-term equations.
- Increasing the complexity of the mathematical problem (e.g. by a increased degree of the ansatz for symmetries or first integrals) the overdetermination and sparseness increases compensating partially the exploding size of the initial linear system if 1-term equations are used rigorously.

The package LSSS was developed in the course of investigating the integrability of the Kontsevich system ([3])

$$u_t = uv - uv^{-1} - v^{-1},\ v_t = -vu + vu^{-1} + u^{-1} \tag{1}$$

where u, v are non-commutative variables (in particular, square matrices of arbitrary size). This is the first non-abelian system with non-polynomial right hand sides for which integrability could be shown, in this case by computing a Lax pair with spectral parameter [2]. Essential ingredients were the computation of Lie-symmetries, first integrals, a pre-Hamiltonian operator and a recursion operator of (1), all of them requiring the solution of selective linear systems. With the program LSSS it was possible to compute Lie-symmtries of degree up to 16. The complete linear system that had been solved includes over 10^9 equations for 172 Mio variables. Despite of the majority of them being zero, the general solution is not trivial as it has 32 free parameters and its formulation requires already several mega byte.

Current applications of LSSS include the integrability investigation of generalizations of the form

$$u_t = uv + P(u, v, u^{-1}, v^{-1}),\ v_t = -vu + Q(u, v, u^{-1}, v^{-1}). \tag{2}$$

References

[1] Wolf, T., Schrüfer, E., Webster, K. Solving large linear algebraic systems in the context of integrable non-abelian Laurent ODEs, *Programming and Computer Software*, (2012) DOI:10.1134/S0361768812020065, also arXiv:1109.2785 (nlin.SI).

[2] Wolf, T., Efimovskaya, O. On integrability of the Kontsevich non-abelian ODE system, *Lett. in Math. Phys.*, vol 100, no 2 (2012), p 161-170 DOI:10.1007/s11005-011-0527-4, also arXiv:1108.4208v1 (nlin.SI).

[3] Kontsevich, M., private communication.