Annihilating monomials with the integro-differential Weyl algebra

Johannes Middeke
Department of Applied Mathematics
University of Western Ontario
London (ON), Canada, N6A 5B7
jmiddeke@uwo.ca

Abstract

The integro-differential Weyl algebra provides an algebraic model for differential and integral operators with polynomial coefficients. It has a natural action on the ring of polynomials. We are interested in computing the annihilator of a given polynomial with respect to this action. This contribution contains a first step towards that goal—namely we give a description of the annihilator of a monomial.

1 The integro-differential Weyl algebra

The (univariate) integro-differential Weyl was introduced for the first time in [1]. It arose from the algebra of integro-differential operators first discussed in [2] and refined in [3, 4] with the goal of solving boundary value problems using purely algebraic methods. While the aforementioned papers construct integro-differential operators with arbitrary coefficients using a Gröbner basis approach, in [1] the authors chose to model operators with polynomial coefficients using Ore polynomials.

A way to construct the integro-differential Weyl algebra as a generalised Weyl algebra has been discussed in [5].

In this abstract we briefly recall the basic properties of the integro-differential Weyl algebra and refer to [1] for details. Let $\mathbb{K}$ be a field of characteristic 0. The *integro-differential Weyl algebra*—denoted $A_1(\partial, \int)$—is the $\mathbb{K}$-algebra generated by the symbols x, ∂ and $\int$ and defined by the equations

$$\partial x = x\partial + 1, \qquad \int\int = x\int - \int x, \qquad \text{and} \qquad \partial\int = 1. \tag{1}$$

One can prove that $A_1(D, \int)$ is neither simple nor left or right Noetherian; it even contains zero divisors. Also, the integro-differential operators from [2, 3, 4] are isomorphic to $A_1(\partial, \int)/(\mathrm{E}x - c\mathrm{E})$ where $\mathrm{E} = 1 - \int\partial$ and c is a constant depending on the integral operator—cf again [1].

Interpreting ∂ as derivation and $\int$ as an integral, the integro-differential Weyl algebra has a natural action on the polynomial ring $\mathbb{K}[x]$. More precisely, the action $*\colon A_1(\partial, \int) \times \mathbb{K}[x] \to \mathbb{K}[x]$ is defined by

$$x * x^n = x^{n+1}, \qquad \partial * x^n = \frac{dx^n}{dx} = nx^{n-1}, \qquad \text{and} \qquad \int * x^n = \int_0^x x^n dx = \frac{1}{n+1}x^{n+1}$$

where $n \geq 0$. With this action, the relations in (1) model the Leibniz rule, partial integration and the fundamental theorem of calculus, respectively. Moreover, E corresponds to the evaluation at 0.

2 Annihilators

We want to compute annihilators of polynomials, i. e., the set of all operators in $A_1(\partial, \int)$ whose action sends a given polynomial to zero. It is well-known that annihilators are left ideals in $A_1(\partial, \int)$. As a first step towards computing them, we give the following theorem:

Theorem 1 *For any $n \geq 1$, the annihilator of x^n within the integro-differential Weyl algebra $A_1(\partial, \int)$ is generated as a left ideal by*

$$\partial^{n+1}, \quad (x - \textstyle\int)\partial^n, \quad \mathrm{E}\partial^{n-1}, \quad \ldots, \quad \mathrm{E}\partial, \quad \mathrm{E}$$

for $n \geq 0$ and the annihilator of 1 is generated by ∂ and $x - \int$. In particular, all these annihilators are finitely generated.

As a next step, we intend to generalise this result to annihilate arbitrary polynomials. Also, we shall add more evaluation operators to our algebra in order to model boundary conditions instead of only initial value problems. Moreover, a theorem in [6] states that all finitely generated left ideals in $A_1(\partial, \int)$ can be generated by only two elements. Therefore, another path of research is to attempt to compute these two generators for the annihilators.

References

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