On Isolated Singular Solutions in Polynomial System Solving

Abstract Nowadays, polynomial models are ubiquitous and widely applied across the engineering and sciences, such as in robotics, coding theory, optimization, mathematical biology, computer vision, game theory, statistics, machine learning, control theory, cryptography, and numerous others. A main challenge in algebra and geometry computing is to identify and tackle singular points, which naturally occur when computing the topology of implicit curves or surfaces, the intersection of parametric surfaces in geometric modeling.

A numerical approximation is usually computed to identify an isolated solution of a polynomial system. In practice, we often need to improve the quality of numerical approximations, but numerical methods such like Newton's method converge slowly at singular solutions (or not converge). On the other hand, it is well known that to certify whether a polynomial system has an isolated singular solution is an ill-posed problem, since arbitrary small perturbations of coefficients may transform the singular solution into a cluster of simple roots (or even make it disappear). Therefore, it is hardly possible to verify this problem, if not the entire computation is performed without any rounding error (exact arithmetics).

In this thesis, we first introduce the local dual space for characterizing an isolated singular solution of a polynomial system. By employing some regularization and reduction techniques, we present a novel algorithm for computing a reduced basis of such a space for the special case of breadth one. The algorithm also works for inputs only with limited accuracy, and is efficient both in time and memory use. Moreover, it leads to a parametric representation for a reduce basis (multiplicity structure) of the local dual space.

Based on such a parametric representation and presolving a regularized least squares problem, we propose a regularized Newton's method for refining an approximate singular solution of a given polynomial system. By a careful analysis, we prove the quadratic convergence of the algorithm if the numerical approximation is close to a breadth-one isolated singular solution.

By introducing some well-chosen smoothing parameters to the given system, we develop an improved deflation technique, which derives a square and regular augmented system from an isolated singular solution in a finite number of deflations. Based on this technique, we propose an algorithm for computing verified error bounds such that a slightly perturbed polynomial system is guaranteed to possess an isolated singular solution within the computed bounds.

Keywords solving polynomial systems, isolated singular solutions, local dual space, root refinement, verified error bounds