# Restricted Lattice walks in Three Dimensions

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joint work with
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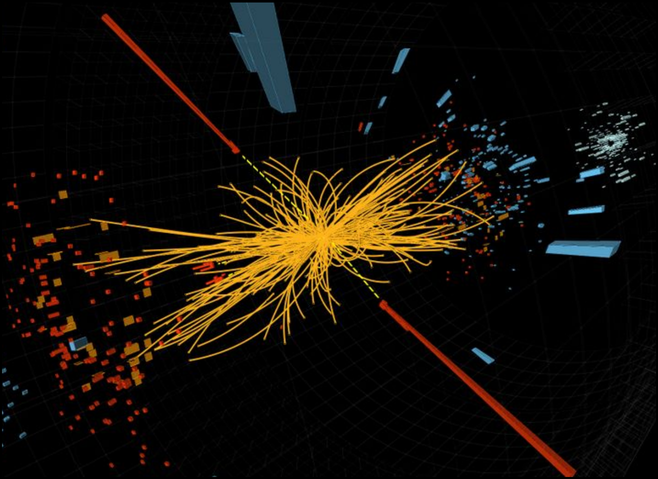

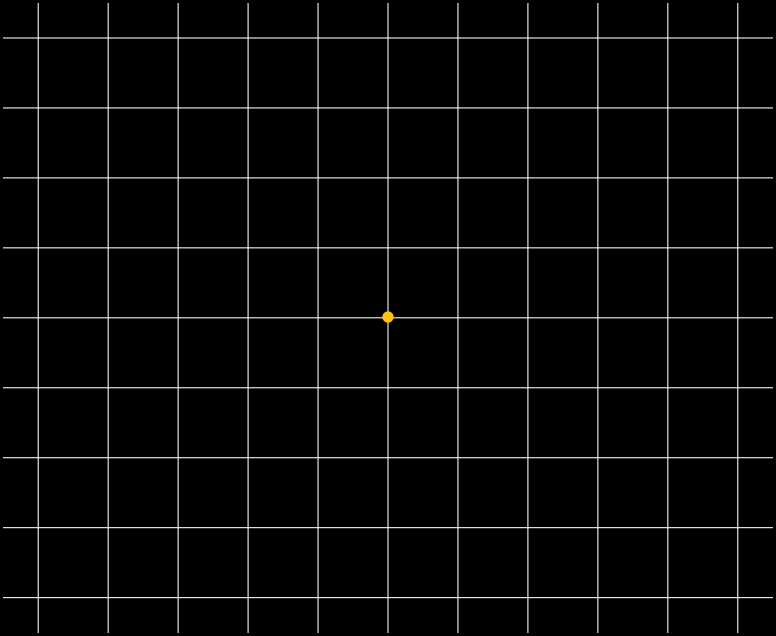

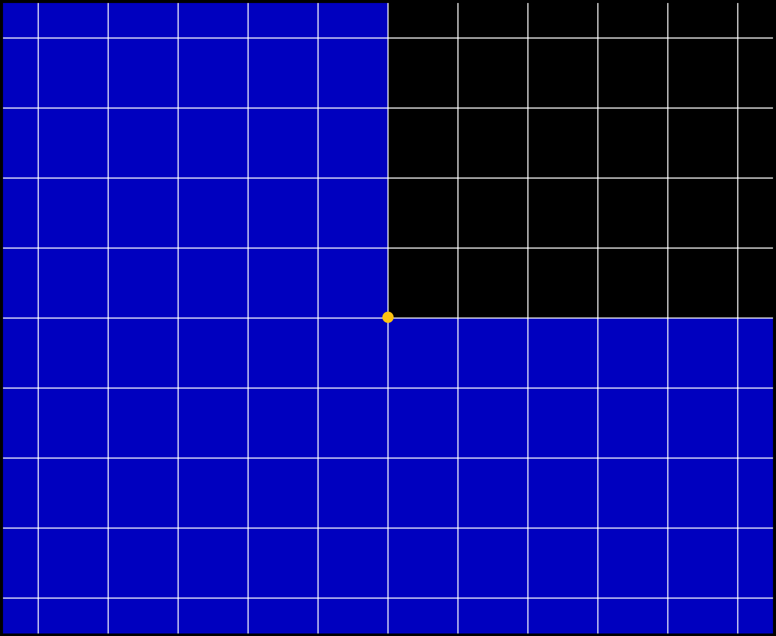

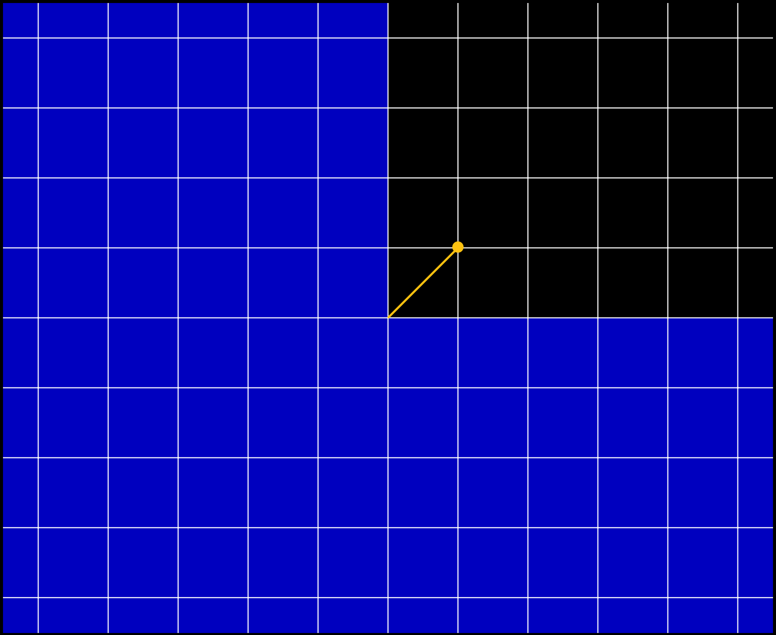

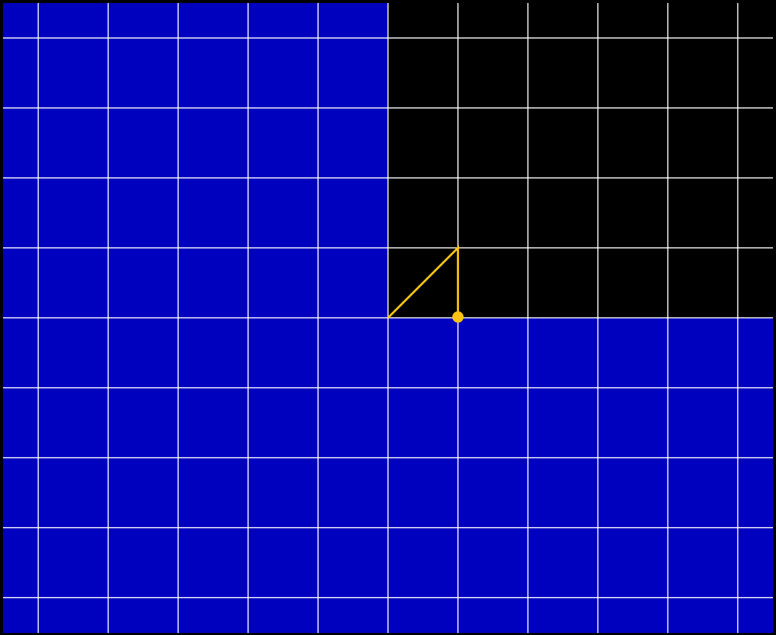

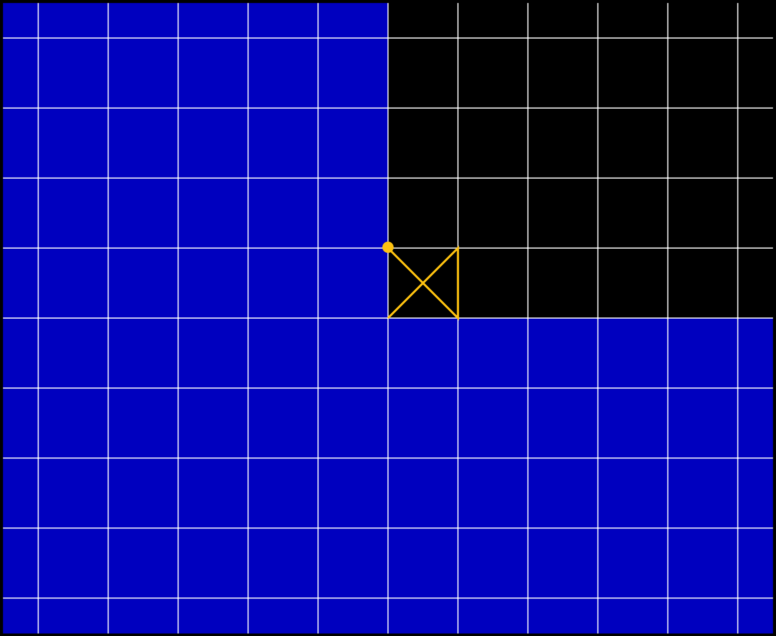

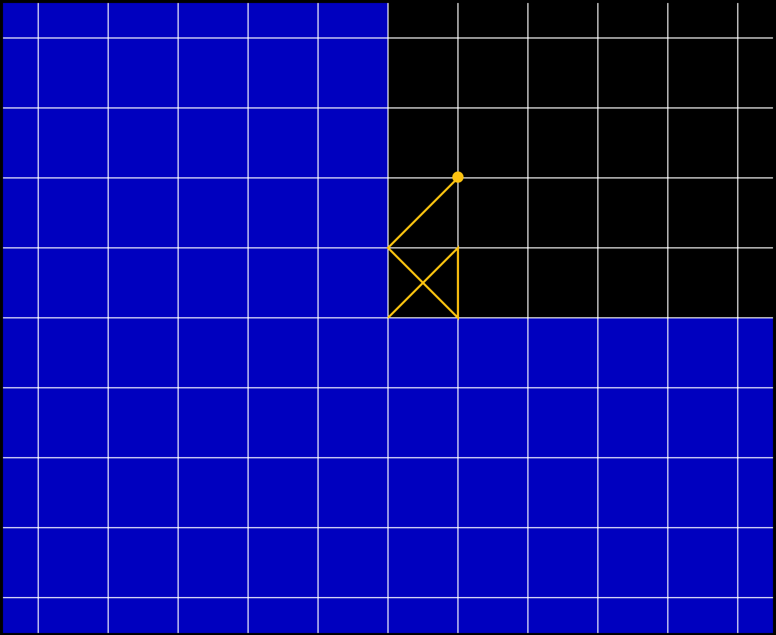

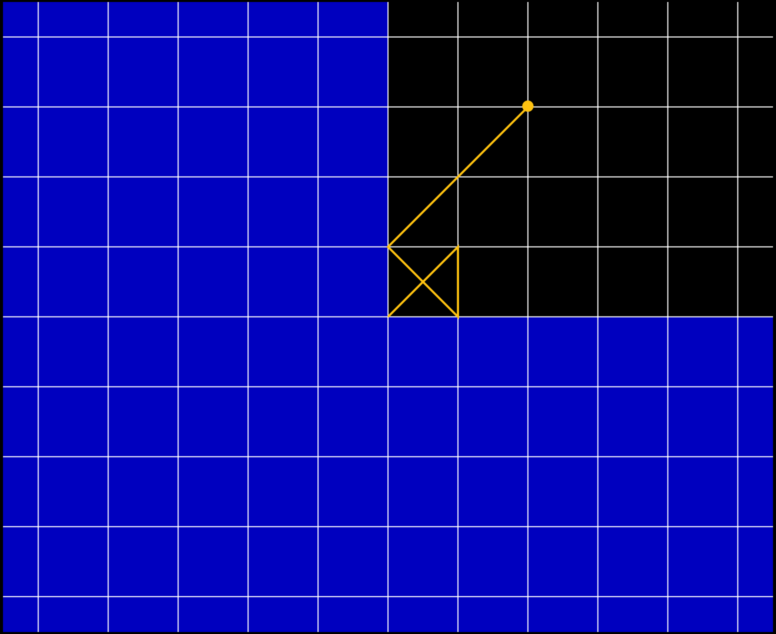

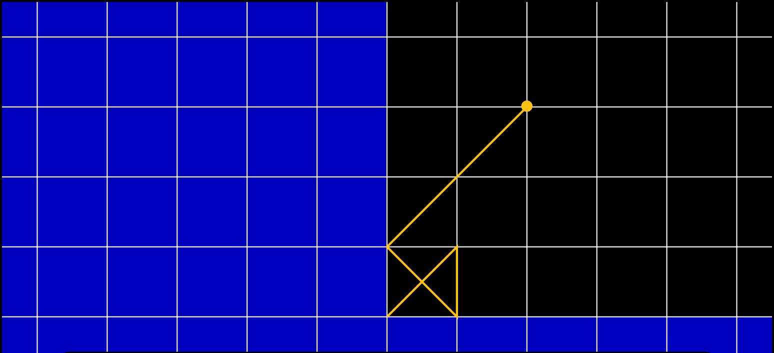

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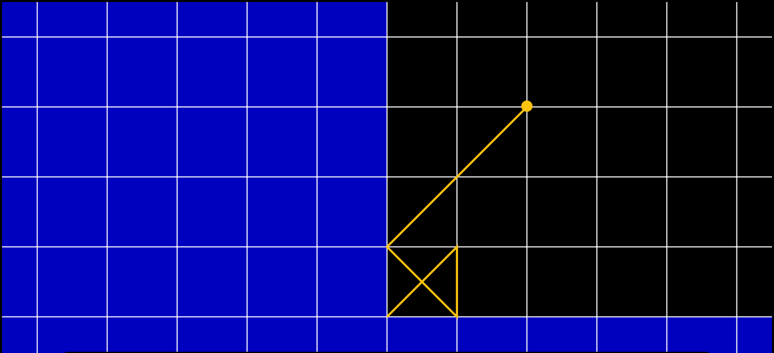

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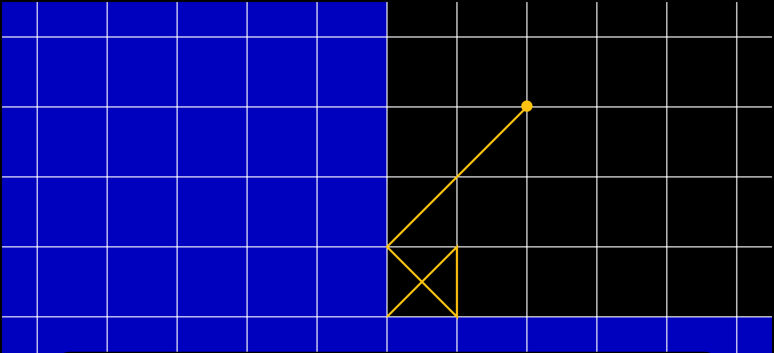

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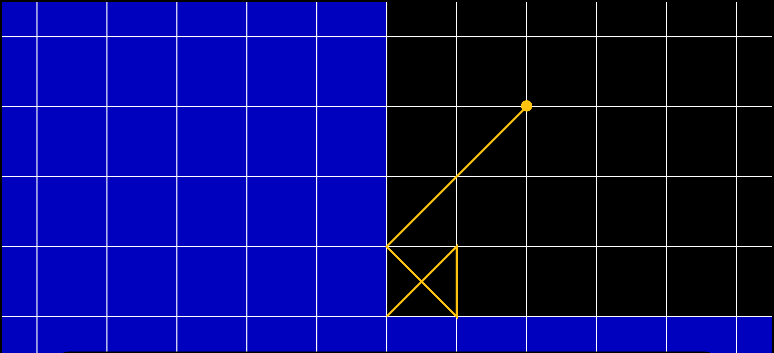

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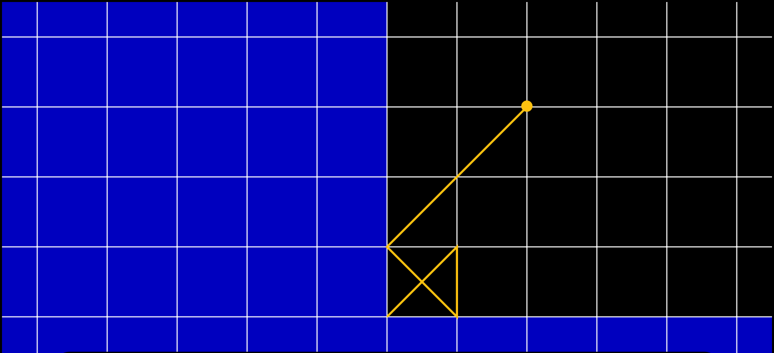

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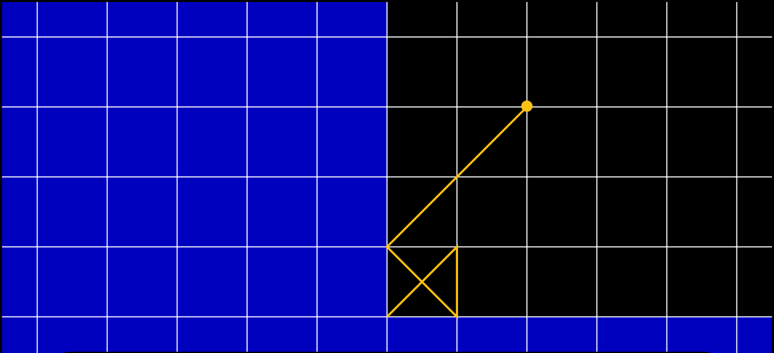

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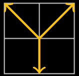

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$$a(x, y, t) = \sum_{n=0}^{\infty} \sum_{i,j} a_{n,i,j} \, x^i y^j t^n$$

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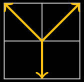
Example: For the step set  we have

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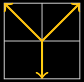
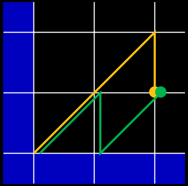
Example: For the step set  we have



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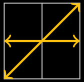
Example: For the step set  we have

$$\begin{aligned} a(x,y,t) = 1 &+ (x + xy)\, t \\ &+ (2 + x^2 + y + 2x^2y + x^2y^2)t^2 \\ &+ (5x + x^3 + 6xy + 3x^3y + 2xy^2 + 3x^3y^2 + x^3y^3)t^3 \\ &+ \cdots \end{aligned}$$

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Note:

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Question:

| How does the nature of $a(x, y, t)$ depend on the step set? |
| --- |

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Recall: $a$ is D-finite $:\Longleftrightarrow$

$$p_0\, a + p_1\, \frac{d}{dt} a + \cdots + p_r\, \frac{d^r}{dt^r} a = 0$$

for some polynomials $p_0, \ldots, p_r$ in $x, y, t$, not all zero.

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$|\{-1, 0, 1\}|$ → $3$

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What does it depend on?

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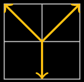
Example: For the step set  we obtain

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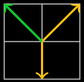
Example: For the step set  we obtain

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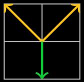
Example: For the step set  we obtain

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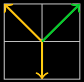
Example: For the step set  we obtain

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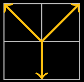
Example: For the step set  we obtain

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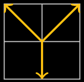
Example: For the step set  we obtain

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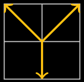
Example: For the step set  we obtain

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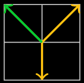
Example: For the step set  we obtain

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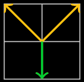
Example: For the step set  we obtain

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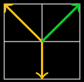
Example: For the step set  we obtain

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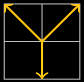
Example: For the step set  we obtain

$$a_{i,j,n+1} = a_{i+1,j-1,n} + a_{i,j+1,n} + a_{i-1,j-1,n}.$$

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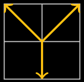
Example: For the step set  we obtain

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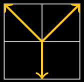
Example: For the step set  we obtain

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This functional equation uniquely describes $a(x, y, t)$.

All properties of $a(x, y, t)$ must somehow follow from it.

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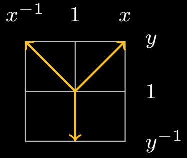

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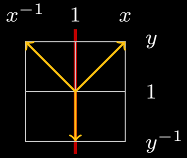


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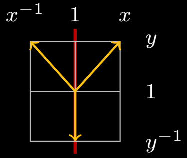


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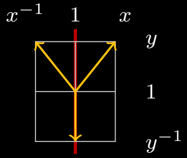


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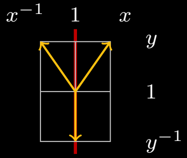


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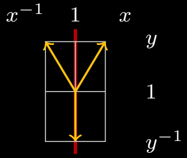


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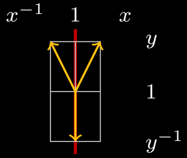


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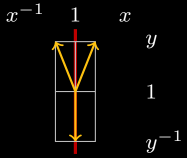


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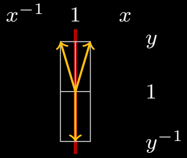


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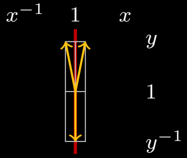


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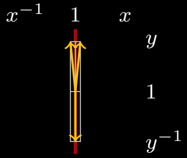


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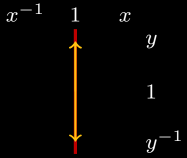


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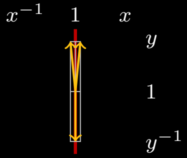


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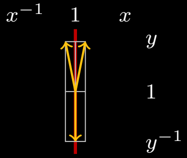


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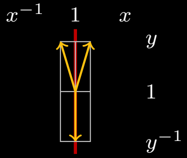


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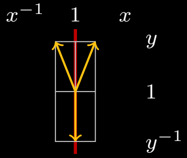


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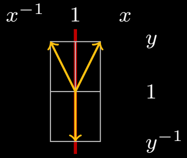


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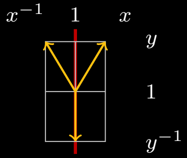


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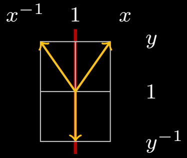


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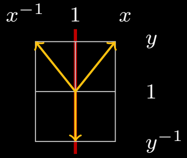


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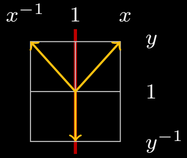


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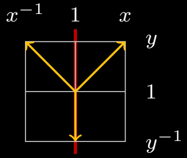


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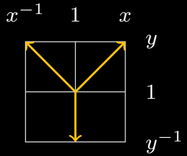


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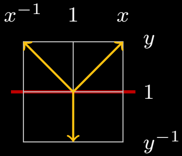


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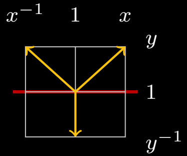


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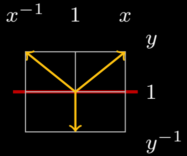


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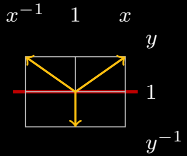


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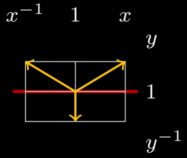


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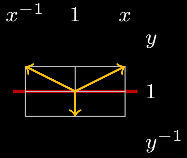


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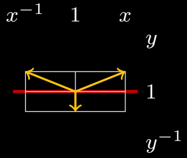


$$\phi:\ (x,\ y)\mapsto\left(\frac{1}{x},\ y\right)$$

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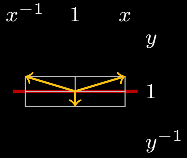


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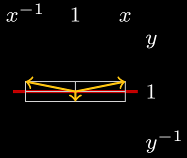


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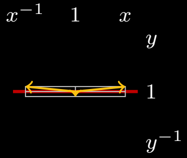


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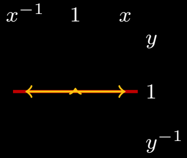


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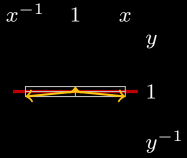


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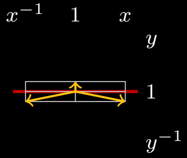


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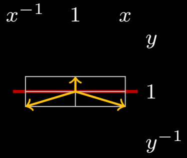


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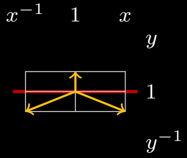


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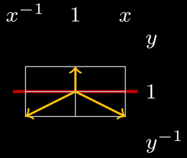


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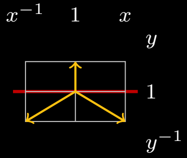


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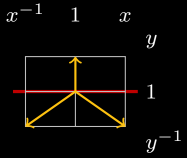


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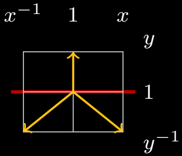


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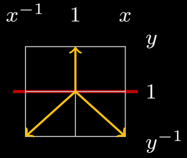


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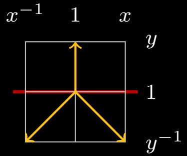


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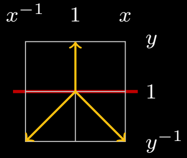


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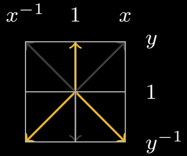


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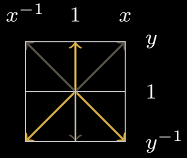


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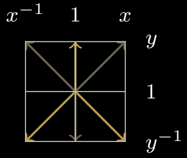


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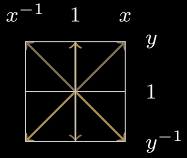


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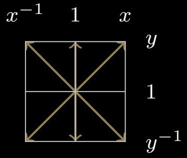


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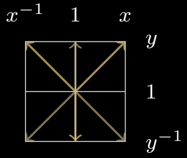


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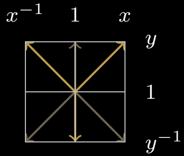


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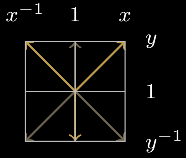


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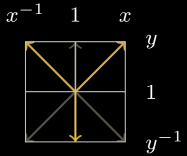


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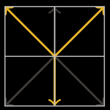


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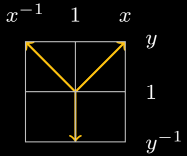


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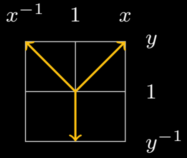


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These two maps together with composition generate a group, the so-called group of the model.

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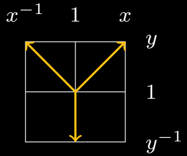


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These two maps together with composition generate a group, the so-called group of the model.

For some step sets this group is finite, for others it is infinite.

$$\Big(1 - \left(\tfrac{y}{x} + \tfrac{1}{y} + xy\right)t\Big)\, a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

Here, $G = \{1, \phi, \psi, \phi\psi\}$ is finite.

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Here, $G = \{1, \phi, \psi, \phi\psi\}$ is finite. Therefore we can do the following (writing $K = 1 - (\frac{y}{x} + \frac{1}{y} + xy)t$):

$$K\, xy\, a(x,y,t) = xy - xt\, a(x,0,t) - y^2 t\, a(0,y,t)$$

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only monomials $x^i y^j$ with $i < 0$ or $j < 0$

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$$

$$\Rightarrow xy\,a(x,y,t)=[x^>][y^>]\,\frac{xy-\frac{1}{x}y-x\frac{1}{1+\frac{1}{x}}\frac{1}{y}+\frac{1}{x}\frac{1}{1+\frac{1}{x}}\frac{1}{y}}{1-(\frac{y}{x}+\frac{1}{y}+xy)t}$$

$$\Big(1 - (\tfrac{y}{x} + \tfrac{1}{y} + xy)t\Big)\, a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

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$$\Rightarrow xy\,a(x,y,t) = \underbrace{[x^>][y^>]\,\underbrace{\frac{xy - \frac{1}{x}y - x\frac{1}{1+\frac{1}{x}}\frac{1}{y} + \frac{1}{x}\frac{1}{1+\frac{1}{x}}\frac{1}{y}}{1 - (\frac{y}{x} + \frac{1}{y} + xy)t}}_{\text{rational}}}_{\text{D-finite}}$$

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What to do then? Try using computer algebra, as follows.

$$\Big(1 - \big(\tfrac{y}{x} + \tfrac{1}{y} + xy\big)t\Big)\ a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

$$\underbrace{\left(1 - \left(\tfrac{y}{x} + \tfrac{1}{y} + xy\right)t\right)}_{=0 \text{ for } y=Y(x,t):=\frac{x-\sqrt{x(x-4t^2(1+x^2))}}{2t(1+x^2)}=t+\left(x+\frac{1}{x}\right)t^3+\cdots} a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

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For this choice of $Y(x,t)$ we find

$$0 = 1-\tfrac{t}{Y(x,t)}a(x,0,t)-\tfrac{Y(x,t)\,t}{x}\,a(0,Y(x,t),t)$$

$$\underbrace{\left(1 - \left(\tfrac{y}{x} + \tfrac{1}{y} + xy\right)t\right)}_{=0 \text{ for } y = Y(x,t) := \frac{x - \sqrt{x(x-4t^2(1+x^2))}}{2t(1+x^2)} = t + \left(x + \frac{1}{x}\right)t^3 + \cdots} a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

For this choice of $Y(x,t)$ we find

$$a(x,0,t) = \tfrac{Y(x,t)}{t} - x\, Y(x,t)^2\, a(0, Y(x,t), t)$$

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Setting $x \rightsquigarrow Y^{-1}(x,t)$ in this equation and rearranging terms gives

$$a(0,x,t) = \tfrac{1}{txY^{-1}(x,t)} - \tfrac{1}{Y^{-1}(x,t)\,x^2} a(Y^{-1}(x,t),0,t)$$

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Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

$$U(x,t) = \frac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t)$$
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Observe:

- This system has a unique solution.

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Observe:

- This system has a unique solution.
- By construction, the solution must be

$$U = a(x,0,t) \text{ and } V = a(0,x,t).$$

Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

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Now turn on the computer. . .

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Now turn on the computer...

- generate lots of coefficients of $a(x,0,t)$, and $a(0,x,t)$.

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- guess a system of D-finite differential equations possibly satisfied by these series.

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- generate lots of coefficients of $a(x,0,t)$, and $a(0,x,t)$.
- guess a system of D-finite differential equations possibly satisfied by these series.
- prove that the series solutions of the guessed D-finite system solve the functional equations.

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Conclude:

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Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.

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Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.
- Because of

$$\Big(1 - \big(\tfrac{y}{x} + \tfrac{1}{y} + xy\big)t\Big)\,a(x,y,t) = 1 - \tfrac{t}{y}\,a(x,0,t) - \tfrac{yt}{x}\,a(0,y,t)$$

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- Because of

$$a(x,y,t) = \frac{1 - \frac{t}{y}\,a(x,0,t) - \frac{yt}{x}\,a(0,y,t)}{1 - \left(\frac{y}{x} + \frac{1}{y} + xy\right)t}$$

Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

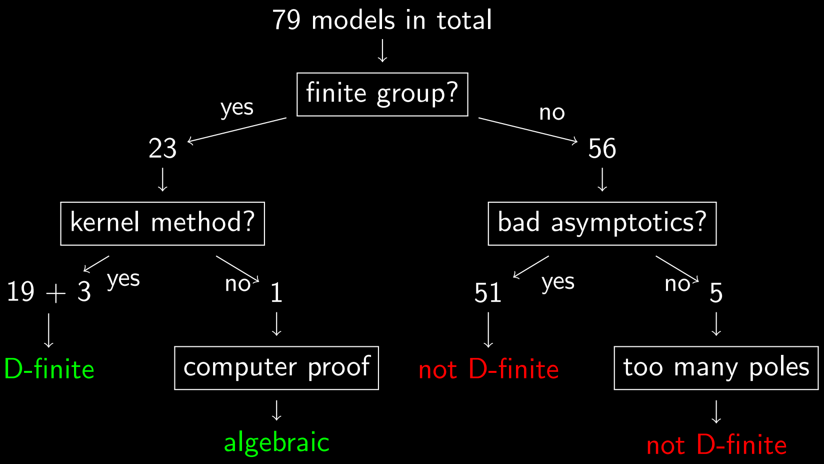
$$U(x,t) = \frac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t)$$
$$V(x,t) = \frac{1}{txY^{-1}(x,t)} - \frac{1}{Y^{-1}(x,t)\,x^2}U(Y^{-1}(x,t),t)$$

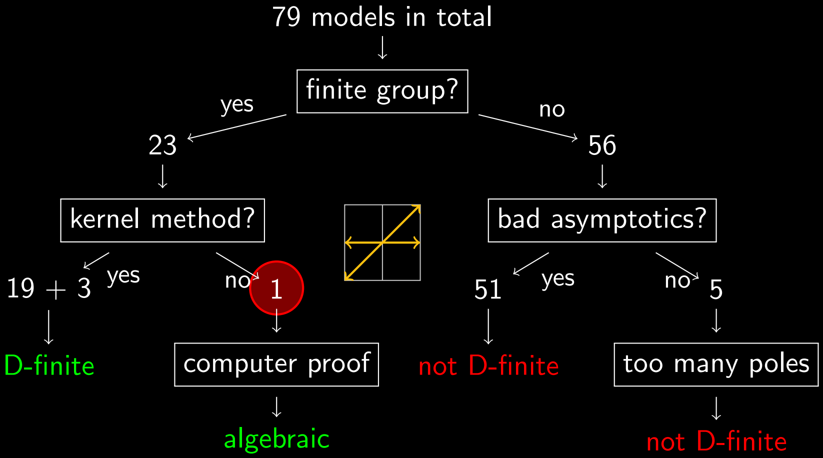
Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.
- Because of

$$a(x,y,t) = \frac{1 - \frac{t}{y}\,a(x,0,t) - \frac{yt}{x}\,a(0,y,t)}{1 - \left(\frac{y}{x} + \frac{1}{y} + xy\right)t}$$

  it follows that also $a(x,y,t)$ is D-finite.


79 models in total
finite group?
yes
no
23
56
kernel method?
bad asymptotics?
19 + 3
yes
no
1
51
yes
no
5
D-finite
computer proof
not D-finite
too many poles
algebraic
not D-finite



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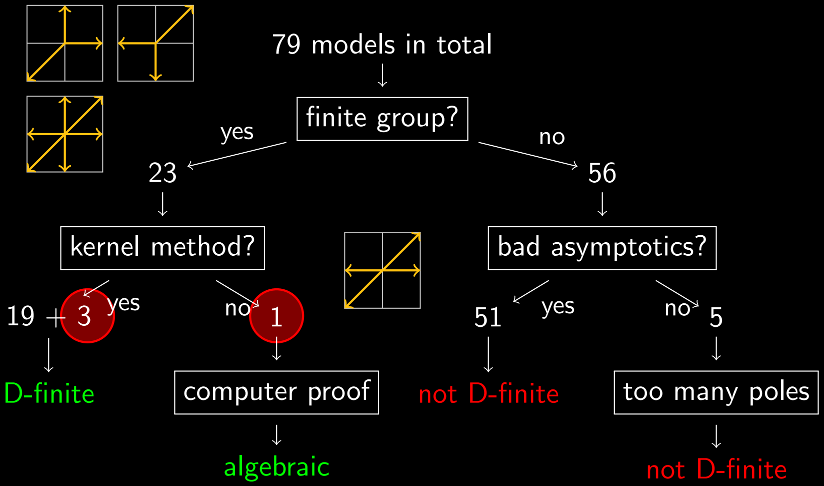
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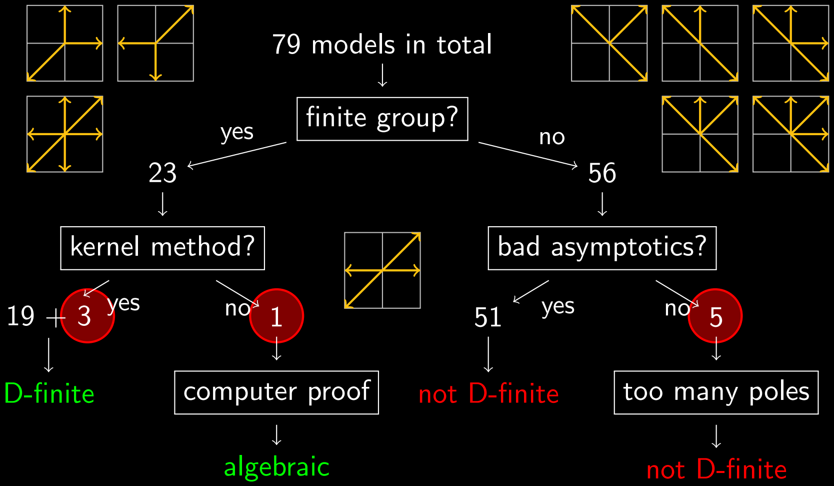
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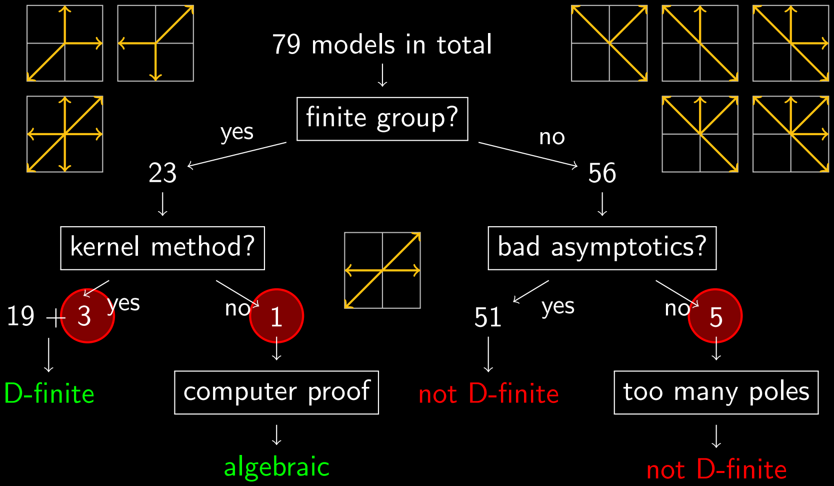
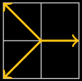
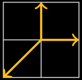
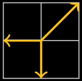
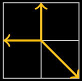
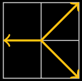
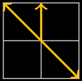
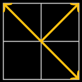
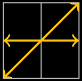
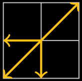
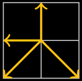
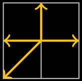
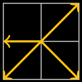
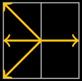
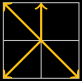
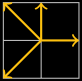


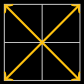
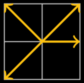
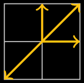
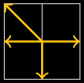
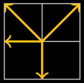

A posteriori observation:

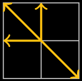
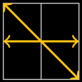
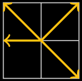
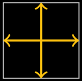
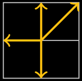
D-finite generating function $\iff$ finite group.

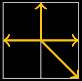
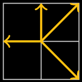
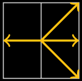
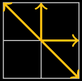
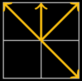
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | D-finite | D-finite | D-finite |
|  | algebraic | algebraic | algebraic |
|  | algebraic | algebraic | algebraic |
|  | D-finite | algebraic | D-finite |
|  | D-finite | D-finite | D-finite |

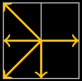
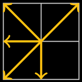
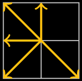
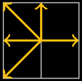
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | algebraic | not D-finite | not D-finite |
|  | algebraic | not D-finite | not D-finite |
|  | algebraic | algebraic | algebraic |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

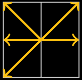
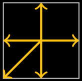
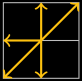
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

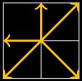
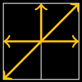
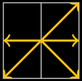
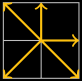
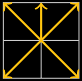
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

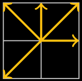
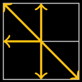
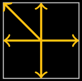
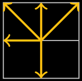
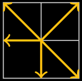
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |

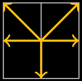
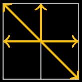
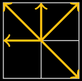
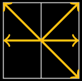
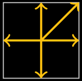
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | algebraic | not D-finite | not D-finite |
|  | algebraic | not D-finite | not D-finite |

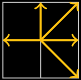
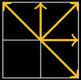
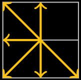
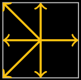
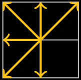
| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

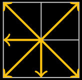
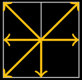
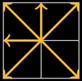
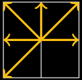
| step set | $a(0, 0, t)$ | $a(1, 1, t)$ | $a(x, y, t)$ |
|---|---|---|---|
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |

| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |

| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | algebraic | not D-finite | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |

| step set | $a(0,0,t)$ | $a(1,1,t)$ | $a(x,y,t)$ |
|---|---|---|---|
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |

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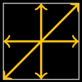
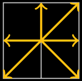
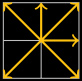
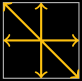
|---|---|---|---|
|  | D-finite | D-finite | D-finite |
|  | algebraic | algebraic | algebraic |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | algebraic | D-finite |

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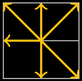
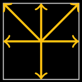
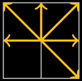
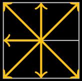
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |

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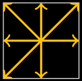
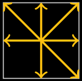
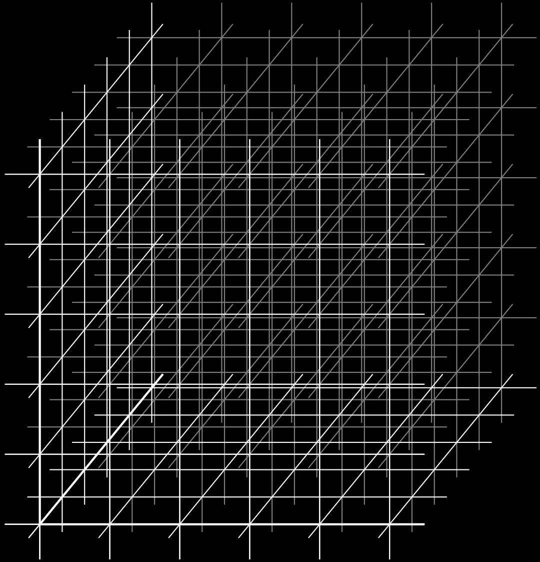
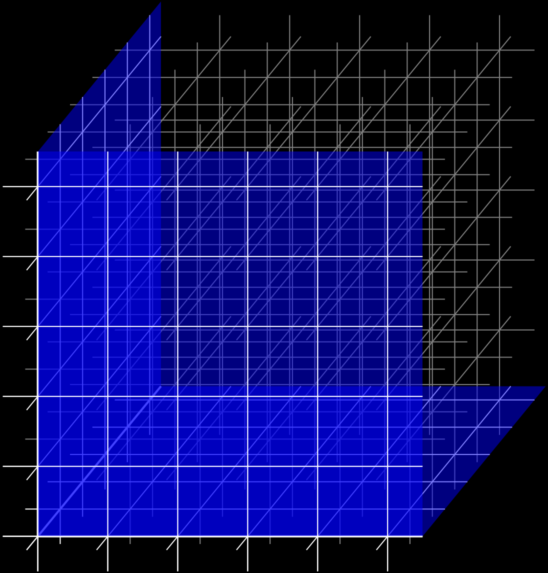
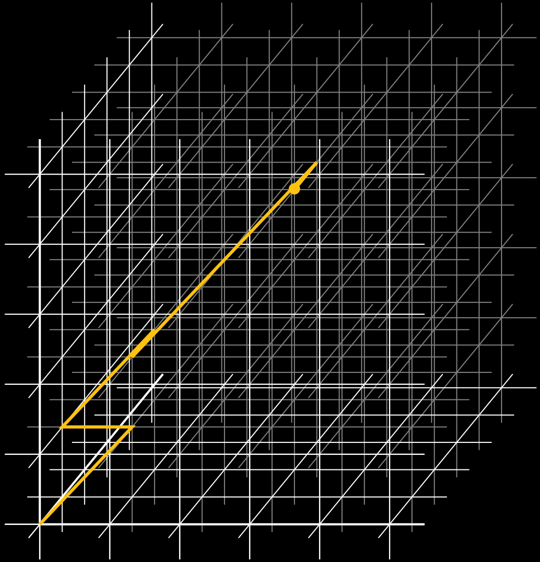
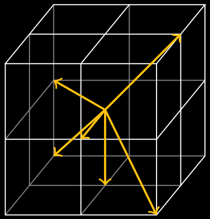
|  | not D-finite | not D-finite? | not D-finite |
|  | D-finite | D-finite | D-finite |
|  | not D-finite | not D-finite? | not D-finite |
|  | not D-finite | not D-finite? | not D-finite |







- start at $(0,0,0)$
- make $n$ steps (e.g., $n=7$)
- end at $(i,j,k)$ (e.g., $(i,j,k)=(3,4,2)$)
- never step out of the first octant
- use only steps taken from a prescribed step set, e.g.,



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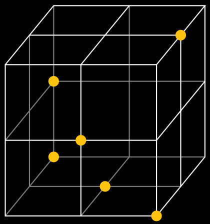

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$$2^{3^{3}-1} = 67108864$$

$|\{-1,0,1\}|$

$\dim = 3$

$\in$ or $\notin$

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$\in$ or $\notin$ except $(0, 0, 0)$ $-56034639$ models in bijection to others

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$|\{-1,0,1\}|$ ; $\dim = 3$ ; except $(0,0,0)$ ; $\in$ or $\notin$

$$2^{3^3-1} = 67108864$$

$- 56034639$ models in bijection to others

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$35548$ models to be investigated

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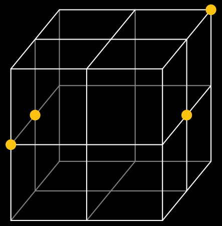
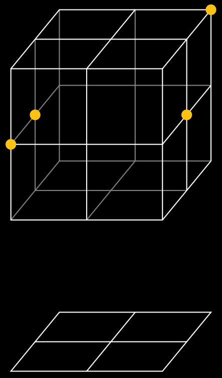
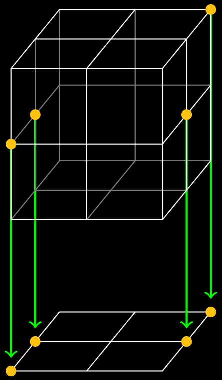
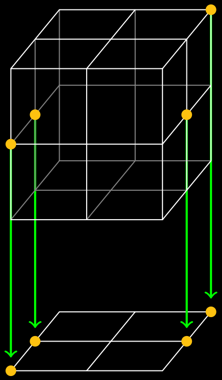
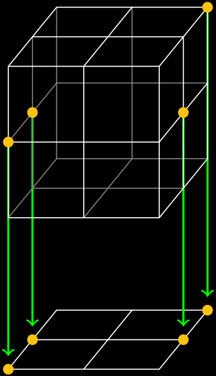


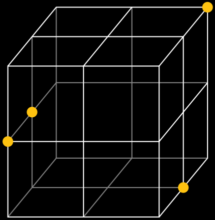






Models are in bijection!





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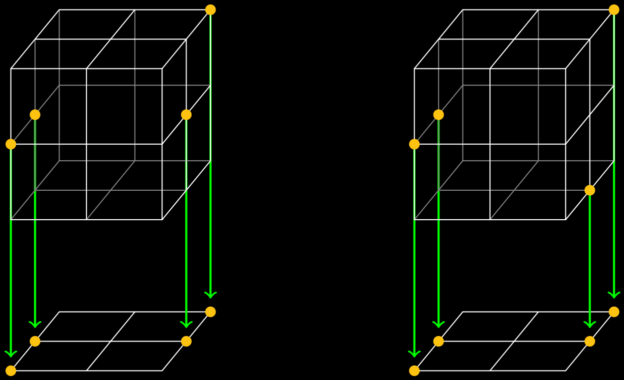

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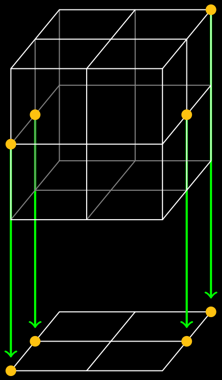

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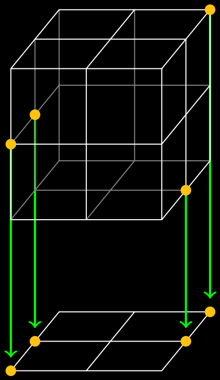
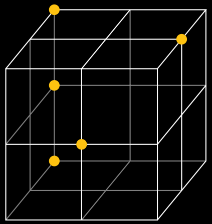
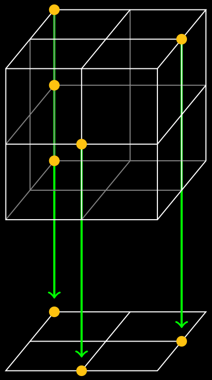
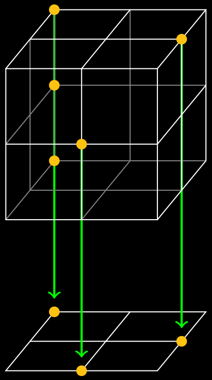


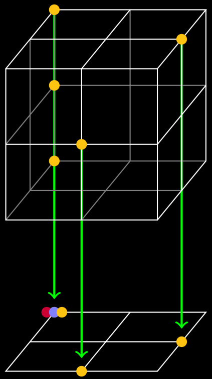
Not a valid bijection!







Bijection?



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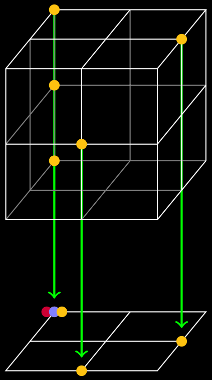
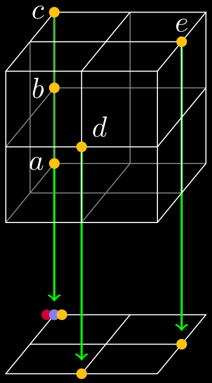


Bijection? YES!

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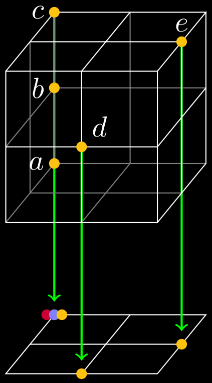


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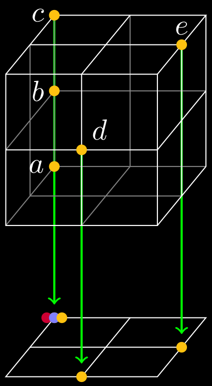
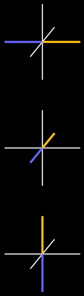
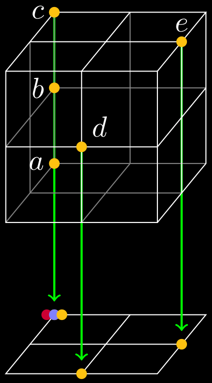


$$\#e \geq \#a + \#b + \#c$$

Bijection? YES!

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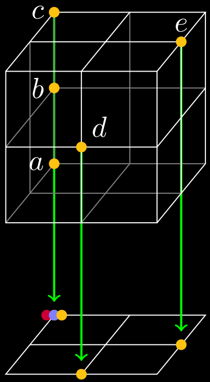
$\#a + \#b + \#c \geq \#d$

Bijection? YES!

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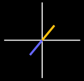

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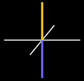
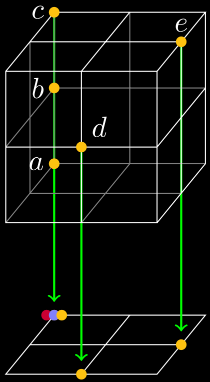


$$\#c + \#e \geq \#a$$




Bijection? YES!

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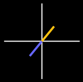

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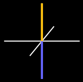

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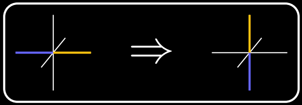

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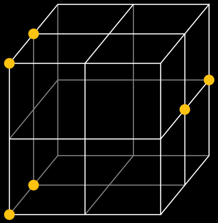

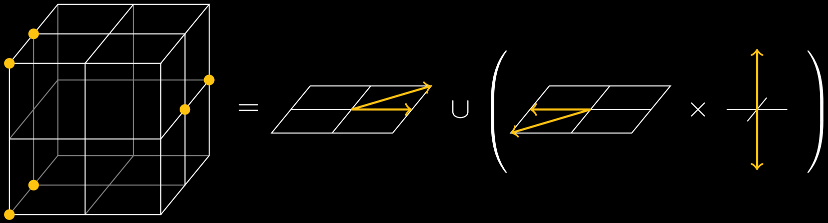

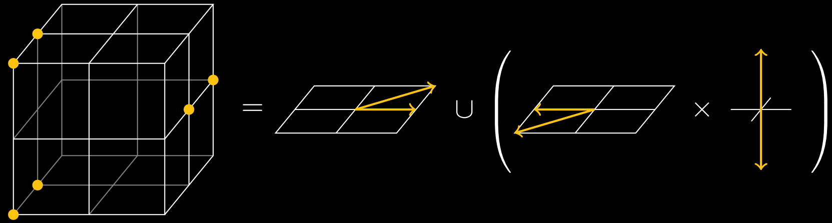

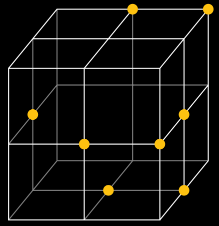

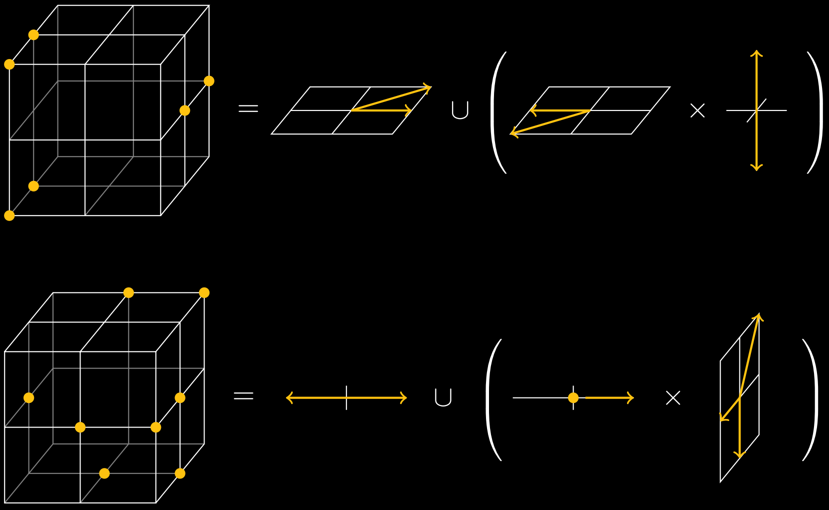

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| reducible to 2D | decomposable | finite group | 3 | 4 | 5 | 6 | Σ |
|---|---|---|---|---|---|---|---|
| T | T | T | | | | | |
| T | T | F | | | | | |
| T | F | T | | | | | |
| T | F | F | | | | | |
| F | T | T | | | | | |
| F | T | F | | | | | |
| F | F | T | | | | | |
| F | F | F | | | | | |
| | | Σ | | | | | |

| reducible to 2D | decomposable | finite group | 3 | 4 | 5 | 6 | $\Sigma$ |
|---|---|---|---|---|---|---|---|
| T | T | T | 8 | 47 | 110 | 175 | 340 |
| T | T | F | 46 | 437 | 1864 | 4821 | 7168 |
| T | F | T | 0 | 0 | 0 | 0 | 0 |
| T | F | F | 18 | 275 | 1599 | 5344 | 7236 |
| F | T | T | 0 | 18 | 47 | 82 | 147 |
| F | T | F | 0 | 9 | 125 | 411 | 545 |
| F | F | T | 0 | 8 | 0 | 15 | 23 |
| F | F | F | 1 | 185 | 2680 | 17223 | 20089 |
| | | $\Sigma$ | 73 | 979 | 6425 | 28071 | 35548 |

| reducible to 2D | decomposable | finite group | 3 | 4 | 5 | 6 | Σ |
|---|---|---|---|---|---|---|---|
| T | T | T | 8 | 47 | 110 | 175 | 340 |
| T | T | F | 46 | 437 | 1864 | 4821 | 7168 |
| T | F | T | 0 | 0 | 0 | 0 | 0 |
| T | F | F | 18 | 275 | 1599 | 5344 | 7236 |
| F | T | T | 0 | 18 | 47 | 82 | 147 |
| F | T | F | 0 | 9 | 125 | 411 | 545 |
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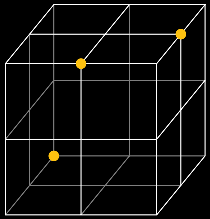

| reducible to 2D | decomposable | finite group | 3 | 4 | 5 | 6 | Σ | |
|---|---|---|---|---|---|---|---|---|
| T | T | T | 8 | 47 | 110 | 175 | 340 | |
| T | T | F | 46 | 437 | 1864 | 4821 | 7168 | |
| T | F | T | 0 | 0 | 0 | 0 | 0 | |
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| F | T | T | 0 | 18 | 47 | 82 | 147 | |
| F | T | F | 0 | 9 | 125 | 411 | 545 | not D-finite? |
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|---|---|---|---|---|---|---|---|---|
| T | T | T | 8 | 47 | 110 | 175 | 340 | look at the 527 resulting 2D models... |
| T | T | F | 46 | 437 | 1864 | 4821 | 7168 | |
| T | F | T | 0 | 0 | 0 | 0 | 0 | |
| T | F | F | 18 | 275 | 1599 | 5344 | 7236 | |
| F | T | T | 0 | 18 | 47 | 82 | 147 | D-finite! |
| F | T | F | 0 | 9 | 125 | 411 | 545 | not D-finite? |
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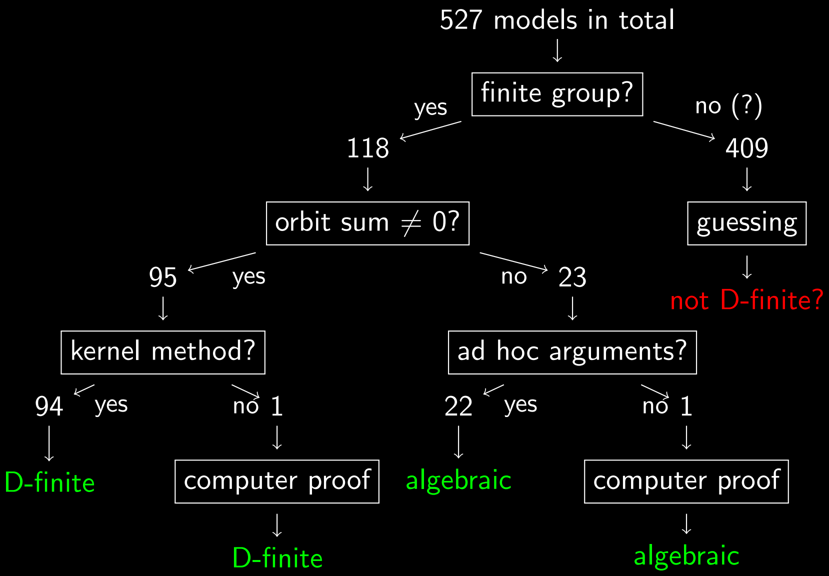
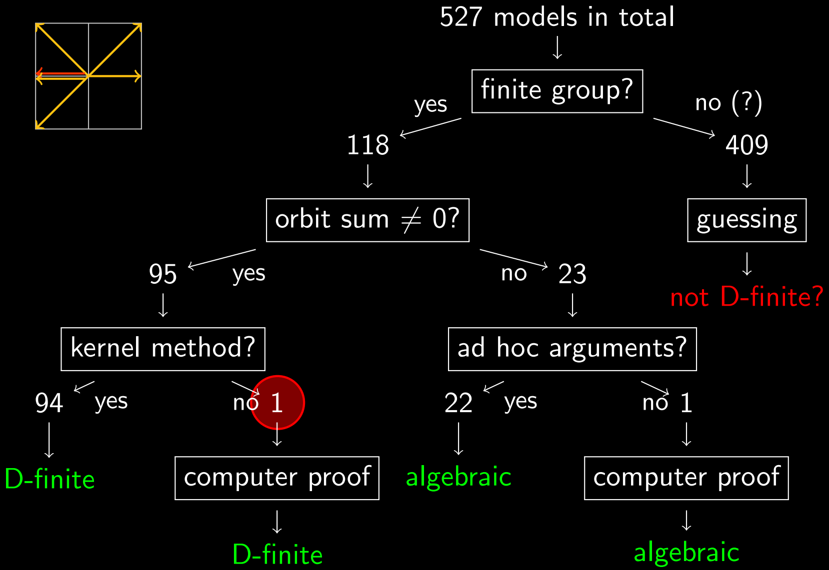

527 models in total
finite group?
yes
no (?)
118
409
orbit sum ≠ 0?
guessing
95
yes
no
23
not D-finite?
kernel method?
ad hoc arguments?
94
yes
no
1
22
yes
no
1
D-finite
computer proof
algebraic
computer proof
D-finite
algebraic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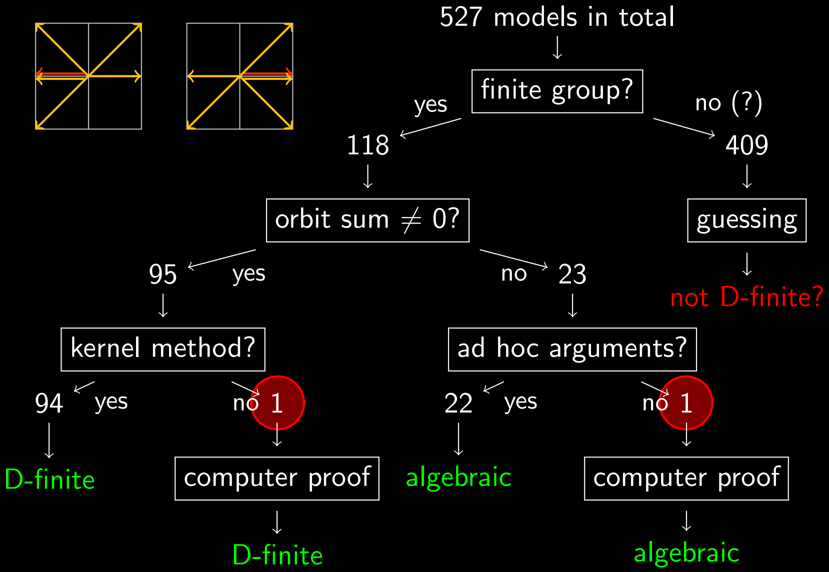
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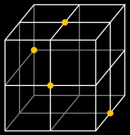
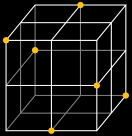
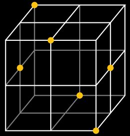
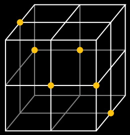
There are 23 models in 3D which are not reducible to 2D, which are not decomposable, and which have a finite group. For 4 of them, the orbit sum is nonzero and the kernel method implies that they are D-Finite.









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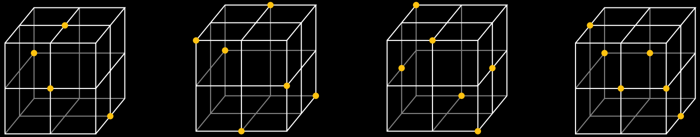

The remaining 19 models are mysterious. Even on a super-computer we were not able to find any evidence for possible differential equations. Can it be that they are not D-finite?

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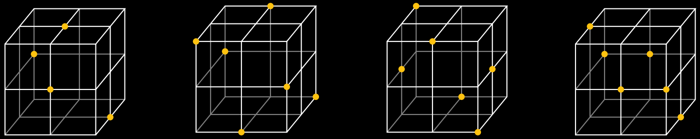

The remaining 19 models are mysterious. Even on a super-computer we were not able to find any evidence for possible differential equations. Can it be that they are not D-finite?

This would imply that the equivalence between D-finiteness and a finite group does not carry over to walks in three dimensions.