Syntax: Formulas of Prepositional Modal Logic

- Let Σ be a set of boolean variables
- The formula set MFor is defined by
  - $\Sigma \subseteq \mathrm{MFor}$,
  - $\mathrm{true}, \mathrm{false} \in \mathrm{MFor}$,
  - $A, B \in \mathrm{MFor} \implies (A \to B), (A \land B), (A \lor B) \in \mathrm{MFor}$,
  - $A \in \mathrm{MFor} \implies \neg A, \Box A, \Diamond A \in \mathrm{MFor}$.
- $\Box A$ “Box”: A is necessary
- $\Diamond A$ “Diamond”: A is possible
- $\Diamond A :\leftrightarrow \neg\Box\neg A$
- Examples: $\Box p \to \Box\Box p$, $\Diamond p \land \neg\Box q \lor (q \leftrightarrow p)$, etc.

Interpretations of Box and Diamond

- There are several possibilities

Name	$\Box F$	$\Diamond F$
classic modal	F is necessarily true	F is possibly true
temporal	F is true from now on	F will become true at some point
deontic	F is obligatory	F is permitted
doxastic	An agent believes F	F is consistent with the things an agent believes
epistemic	An agent knows F	An agent does not know that F is false
dynamic	F is true after each execution of a program	There is an execution of a program after which F is true
"automaton"	F is true in all reachable states	There is a reachable state in which F is true.
"Leibniz"	F is true "in all worlds"	There is "a world" where F is true

Axiom Candidates

- What tells us the intuition?

$\Box F$	$F \to \Box F$	$\Box F \to F$	$\Box F \to \Diamond F$	$\Box F \to \Box\Box F$	$\Box(F \to G) \to (\Box F \to \Box G)$
F is necessarily true,					
F is true from now on					
An agent believes F					
An agent knows F					
F is obligatory					
F is true after each execution of a program					
F is true in all reachable states					

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$\Box F$	$F \to \Box F$	$\Box F \to F$	$\Box F \to \Diamond F$	$\Box F \to \Box\Box F$	$\Box(F \to G) \to (\Box F \to \Box G)$
F is necessarily true,	yes	yes	yes	yes	yes
F is true from now on	yes	no	no	yes	yes
An agent believes F	yes	no	yes	yes	yes
An agent knows F	yes	yes	yes	yes	yes
F is obligatory	yes	no	yes	no	yes
F is true after each execution of a program	yes	no	no	no	yes
F is true in all reachable states	yes	no	no	no	yes

Axiom Systems

- Various axiom systems are defined
- The simples one is $\mathbf{K}$.
  - Axioms of $\mathbf{K}$:
    - All prepositional tautologies
    - $\Box(A \to B) \to (\Box A \to \Box B)$
  - Deduction rules of $\mathbf{K}$:
    - Modus Ponens (MP): $\dfrac{A,\ A \to B}{B}$
    - Necessitation (N): $\dfrac{A}{\Box A}$
- Other systems are obtained by adding further axioms to $\mathbf{K}$.
- For instance $\mathbf{T}$ is $\mathbf{K}$ plus $\Box A \to A$.
- Logic consequences can be proven from axioms and rules. Examples:
  - $\Box A \wedge \Box B \leftrightarrow \Box(A \wedge B)$ in $\mathbf{K}$
  - $\Box A \vee \Box B \to \Box(A \vee B)$ in $\mathbf{K}$

Three Philosophers With Hats I

- Three philosophers named 1, 2, 3, are sitting aroud.
- Each is wearing a hat which is black or white.
- There is at least one black hat.
- Everybody can see the color of the others' hats.
- No. 1 says he doesn't know the color of his hat.
- No. 2 says he deosn't know either.
- But no. 3 says that he must then have a black hat.
- How is this possible?
- Formalize and prove it in an appropriate modal logic!

Three Philosophers With Hats II

- Formalisation in multimodal logic.
- $\Box_i A$ means "No. i knows A."
- B_i be a boolean variable expressing that i's hat is black.
- One hat is black: $B_1 \vee B_2 \vee B_3$.
- Everybody can see the others' hats: $B_1 \to \Box_2 B_1$, $\neg B_1 \to \Box_2 \neg B_1$, etc.
- The first doesn't know: $\neg\Box_1 B_1$
- The second doesn't know: $\neg\Box_2 B_2$
- To prove: $\Box_3 B_3$.

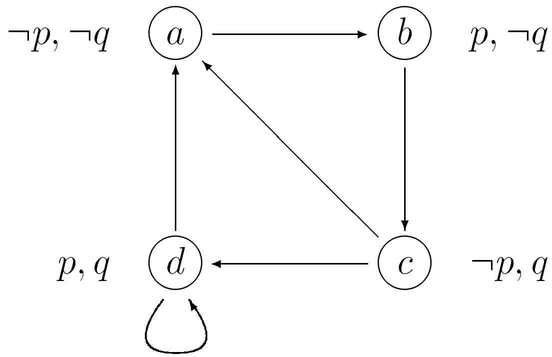
Three Philosophers With Hats III

- Proof:

1 (F)	$B_1 \vee B_2 \vee B_3$
2 (1)	$(\neg B_2 \wedge \neg B_3) \to B_1$
3 (2)	$\Box_1((\neg B_2 \wedge \neg B_3) \to B_1)$
4 (3)	$\Box_1(\neg B_2 \wedge \neg B_3) \to \Box_1 B_1$
5 (F)	$\neg\Box_1 B_1$
6 (4, 5)	$\neg\Box_1(\neg B_2 \wedge \neg B_3)$
7 (F)	$(\neg B_2 \wedge \neg B_3) \to \Box_1(\neg B_2 \wedge \neg B_3)$
8 (6, 7)	$\neg(\neg B_2 \wedge \neg B_3)$
9 (8)	$B_2 \vee B_3$
10 (9)	$\neg B_3 \to B_2$
11 (10)	$\Box_2(\neg B_3 \to B_2)$
12 (11)	$\Box_2\neg B_3 \to \Box_2 B_2$
13 (F)	$\neg\Box_2 B_2$
14 (12, 13)	$\neg\Box_2\neg B_3$
15 (F)	$\neg B_3 \to \Box_2\neg B_3$
16 (14, 15)	B_3
17 (16)	$\Box_3 B_3$ ■

Semantics: Kripke Structures

- Kripke Frame: pair (G, R) with a set G and a relation R on G. More or less a “graph.”
- Signature Σ: A set of boolean variables
- Kripke Structure: triple $\mathcal{K} = (G, R, v)$ where (G, R) is a Kripke frame, and $v\colon \Sigma \times G \to \{0, 1\}$ is a valuation.
- Example:



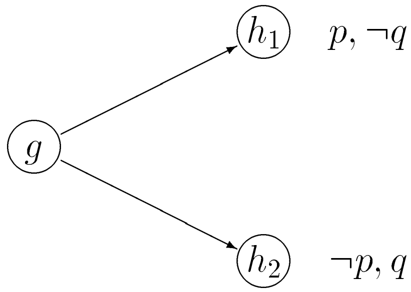
- G may or may not be finite
- There are no restrictions on R in general

Semantics: Interpretation

- Let $\mathcal{K} = (G, R, v)$ be a Kripke structure, $g \in G$, $F \in \mathrm{MFor}$.
- $g \models F$ ("F holds in g") is defined by structural induction
  - $g \models p$ iff $v(g, p) = 1$
  - $g \models \neg A$ iff not $g \models A$
  - $g \models A \vee B$ iff $g \models A$ or $g \models B$
  - $g \models A \wedge B$ iff $g \models A$ and $g \models B$
  - $g \models A \rightarrow B$ iff $g \models B$ or not $g \models A$
  - $g \models \Box A$ if $\forall\, h \in G : (R(g, h) \implies h \models A)$
  - $g \models \Diamond A$ if $\exists\, h \in G : (R(g, h) \wedge h \models A)$
- $\mathcal{K} \models F$ means that $g \models F$ for all $g \in G$.
- Semantic Consequences: Let $A, B \in \mathrm{MFor}$
  - $A \models_L B$ (local consequence) iff $\forall\, \mathcal{K} = (G, R, v)\ \forall\, g \in G : (g \models A \implies g \models B)$
  - $A \models_G B$ (global consequence) iff $\forall\, \mathcal{K} = (G, R, v) : (\forall g \in G : g \models A \implies \forall\, g \in G : g \models B)$

Syntax vs. Semantics

- Why is semantics defined like this?
- Because then $A \vdash_{\mathbf{K}} B \iff A \models_G B$
- "$\Leftarrow$" To show: Axioms and rules work.
  - $\mathcal{K} \models A$ if A is prepositional tautology
  - $\mathcal{K} \models \Box(A \to B) \to (\Box A \to \Box B)$
  - If $\mathcal{K} \models A$ and $\mathcal{K} \models A \to B$ then $\mathcal{K} \models B$
  - If $\mathcal{K} \models A$ then $\mathcal{K} \models \Box A$
- "$\Rightarrow$" nontrivial.
- Application: $\not\vdash_{\mathbf{K}} \Box(A \vee B) \to (\Box A \vee \Box B)$ can be shown by providing the following counterexample:



Semantics for other Axiom Systems

- Kripke Structures turned out to be adequate for $\mathbf{K}$
- What about other axiom systems?
- Other systems are obtained by extending $\mathbf{K}$ by further axioms.
- Adding axioms corresponds to restricting the class of admissible frames
- Example: $\mathbf{T}$ is $\mathbf{K}$ plus $\Box A \to A$.
- $\vdash_{\mathbf{T}} A$ iff $\mathcal{K} \models_G A$ for all $\mathcal{K}$ whose underlying frames are reflexive.
- More general: There is a "relationship" between the formula $\Box p \to p$ in MFor and the formula $\forall\, x : R(x, x)$ in predicate logic.
- Correspondence theory studies this kind of correspondence in depth and leads to interesting insights.

Other Aspects of Modal Logics

- Correspondence Theory
- Tense Logic
- Completness results
- Decidability and Automated Theorem Proving
- Modal Predicate Logic
- Specification Languages
- Dynamic Logic
- ...