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Algorithmic Combinatorics

Homework 7

Exercise 1 Let $(G, \circ)$ and $(H, *)$ be groups and define

$$\cdot\colon (G \times H) \times (G \times H) \to (G \times H), \qquad (g, h) \cdot (g', h') = (g \circ g', h * h').$$

Show that $(G \times H, \cdot)$ is a group.

Check the group laws:

$$\begin{aligned}
(g,h) \cdot (1_G, 1_H) &= (g \circ 1_G, h * 1_H) = (g, h)\\
\big((g,h) \cdot (g',h')\big) \cdot (g'', h'') &= (g \circ g', h * h') \cdot (g'', h'') = \big((g \circ g') \circ g'', (h * h') * h''\big)\\
&= \big(g \circ (g' \circ g''), h * (h' * h'')\big) = (g,h) \cdot (g' \circ g'', h' * h'')\\
&= (g,h) \cdot \big((g',h') \cdot (g'',h'')\big)\\
(g,h) \cdot (g^{-1}, h^{-1}) &= (g \circ g^{-1}, h * h^{-1}) = (1_G, 1_H).
\end{aligned}$$

Exercise 2 Let $N = [n]$, $X = \{a_1, \dots, a_n\}$ and define

$$(S_X \times S_N) \times X^N \to X^N \qquad ((\sigma, \pi), f) \mapsto (\sigma, \pi) f$$

by

$$((\sigma,\pi)f)(k) := \sigma(f(\pi^{-1}(k))) \qquad (k \in N).$$

Show that this is a group action.

Check the group action laws:

$$\begin{aligned}
(1_X, 1_N) f &= 1_X \circ f \circ 1_N = f\\
\big((\sigma,\pi)(\sigma',\pi')\big) f(k) &= (\sigma \circ \sigma', \pi \circ \pi') f(k) = \big((\sigma \circ \sigma') \circ f \circ ({\pi'}^{-1} \circ \pi^{-1})\big)(k)\\
&= \big(\sigma \circ (\sigma' \circ f \circ {\pi'}^{-1}) \circ \pi^{-1}\big)(k) = \big((\sigma,\pi)(\sigma' \circ f \circ \pi^{-1})\big)(k)\\
&= (\sigma,\pi)\big((\sigma',\pi')f\big)(k)
\end{aligned}$$

Exercise 2* In the notation of the lecture, does there exist a permutation $\sigma \in S_X$ with $f_2 = (\sigma, \pi) f_1$, when

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a_2 & a_4 & a_2 & a_1 \end{pmatrix},\ f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_2 & a_3 & a_3 \end{pmatrix},\ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}?$$

Yes:

$$\begin{aligned}
\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_2 & a_3 & a_3 \end{pmatrix}}_{=f_2} &= \sigma \circ \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ a_2 & a_4 & a_2 & a_1 \end{pmatrix}}_{=f_1} \circ \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}}_{=\pi=\pi^{-1}}\\
&= \sigma \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_4 & a_2 & a_2 \end{pmatrix}
\end{aligned}$$

gives the constraints

$$\sigma(a_1) \stackrel{!}{=} a_1, \qquad \sigma(a_4) \stackrel{!}{=} a_2, \qquad \sigma(a_2) \stackrel{!}{=} a_3, \qquad \sigma(a_2) \stackrel{!}{=} a_3$$

which are satisfied by

$$\sigma = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_3 & a_4 & a_2 \end{pmatrix}.$$

Exercise 3 Consider the group action $G \times G \to G$, $(g, g') \mapsto gg'$. Show that $\delta\colon G \to S_G$ is an injective group homomorphism.

First, δ is a homomorphism: For arbitrary $h \in G$ we have

$$\delta(g \circ g')(h) = g \circ g' \circ h = g \circ \delta(g')(h) = \delta(g)(\delta(g')(h)) = \big(\delta(g) \circ \delta(g')\big)(h).$$

Next, δ is injective: Let $g, g' \in G$ with $\delta(g) = \delta(g')$. Then

$$\forall\ h \in G : \delta(g)(h) = \delta(g')(h),$$

therefore $gh = g'h$ for all $h \in G$, and therefore $g = g'$.

Exercise 4 Let X be a G-set and $\delta\colon G \to S_X$ the induced permutation representation. Prove:

a) δ is a group homomorphism.

b) $\bar{G} := \delta(G) \leq S_X$ acts on X as a subgroup of S_X.

c) $\bar{G} \cong G/\ker\delta$

d) $G \backslash\backslash X = \bar{G} \backslash\backslash X$.

Ad a) For $g, g' \in G$ and $x \in X$, we have

$$\delta(g \cdot g')(x) = (g \cdot g')x = g(g'h) = \delta(g)\big(\delta(g')(x)\big) = \big(\delta(g) \circ \delta(g')\big)(x).$$

Ad b) By a) it is clear that $\bar{G} \leq S_X$. $\bar{G}$ acts on X: First,

$$1_{\bar{G}}x = \delta(1_G)(x) = \mathrm{id}(x) = x.$$

Secondly, for $\pi, \pi' \in \bar{G}$ we have certain $g, g' \in G$ with $\delta(g) = \pi, \delta(g') = \pi'$, and for all those choices of g, g' we have

$$\begin{aligned}(\pi \circ \pi')(x) &= \big(\delta(g) \circ \delta(g')\big)(x) = \delta(g \cdot g')(x) = (g \cdot g')(x) = g(g'x) \\ &= \delta(g)\big(\delta(g')(x)\big) = \pi(\pi'(x)).\end{aligned}$$

Ad c) Define $\bar{\delta}\colon G/\ker\delta \to \bar{G}$ by

$$\bar{\delta}(g\ker\delta) := \delta(g) \qquad (g \in G).$$

(1) $\bar{\delta}$ is well defined. For $h \in g\ker\delta$ we have $h = g \cdot \bar{h}$ with $\bar{h} \in \ker\delta$ and so

$$\bar{\delta}(h\ker\delta) = \delta(h) = \delta(g \cdot \bar{h}) = \delta(g) \circ \underbrace{\delta(\bar{h})}_{=1_G} = \delta(g) = \bar{\delta}(g\ker\delta).$$

(2) $\bar{\delta}$ is injective. Let $g, h \in G$ with $\bar{\delta}(g \ker \delta) = \bar{\delta}(h \ker \delta)$, i.e., $\delta(g) = \delta(h)$. We have to show $g \ker \delta = h \ker \delta$, or, w.l.o.g. $g \ker \delta \subseteq h \ker \delta$. Let $g' = g \cdot \bar{g} \in g \ker \delta$. Then $\bar{h} = h^{-1} \cdot g \cdot \bar{g} \in \ker \delta$ (note $\delta(h) = \delta(g)$) witnesses that $g' = h \cdot h' \in h \ker \delta$.

(3) $\bar{\delta}$ is surjective. Clear by $\bar{G} = \delta(G)$.

Ad d) It suffices to show $gX = \delta(g)X$ for all $g \in G$, which indeed is evident by the definition of $\delta(g)x$.

Exercise 5 Define $C_4 \times [4] \to [4], (g, k) \mapsto gk := g^2(k)$.

a) Show that this is a group action

b) Determine $\bar{C}_4$ and $\bar{C}_4 \setminus\!\!\setminus [4]$

Ad a) We have $1_{C_4}(k) = k$ and, by the commutativity of C_4,

$$(\pi \circ \sigma)k = (\pi \circ \sigma)^2(k) = (\pi^2 \circ \sigma^2)(k) = \pi^2(\sigma^2(k)) = \pi(\sigma k).$$

Ad b) Let $\pi = (1\,2\,3\,4)$. We have

$$\begin{aligned} \bar{C}_4 &= \delta(C_4) = \{\delta(\pi), \delta(\pi^2), \delta(\pi^3), \delta(\mathrm{id})\} = \{\pi^2, \pi^4, \pi^6, \pi^8\} = \{\pi^2, \mathrm{id}\} \\ &= \{(1\,3)(2\,4), \mathrm{id}\}, \\ \bar{C}_4 \setminus\!\!\setminus [4] &= \{\{k, (1\,3)(2\,4)k\} : k \in [4]\} = \{\{1,3\},\{2,4\}\}. \end{aligned}$$

Exercise 6 Define $C_5 \times [5] \to [5], (g, k) \mapsto gk := g^2(k)$.

a) Show that this is a group action

b) Determine $\bar{C}_5$ and $\bar{C}_5 \setminus\!\!\setminus [5]$

Ad a) *see Exercise 5 a).*

Ad b) Let $\pi = (1\,2\,3\,4\,5)$. We have

$$\begin{aligned} \bar{C}_5 &= \delta(C_5) = \{\delta(\pi), \delta(\pi^2), \delta(\pi^3), \delta(\pi^4), \delta(\mathrm{id})\} = \{\pi^2, \pi^4, \pi, \pi^3, \mathrm{id}\} = C_5, \\ \bar{C}_5 \setminus\!\!\setminus [5] &= \{\{\pi^i(k) : i \in [5]\} : k \in [5]\} = \{\{1,2,3,4,5\}\}. \end{aligned}$$