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Analytic Combinatorics
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Interlude: An Automatic Zero Equivalence Test

1. Some classes of sequences:
   (a) $f(n+3) = f(n+1) - 3f(n)$: C-finite sequences (trivial)
   (b) $f(n+2) = (n+3)f(n+1) - 3nf(n)$: P-finite sequences (routine)
   (c) $f(n+2) = H(n+3)f(n+1) - 3^n f(n)$: $\Pi\Sigma$-sequences (Carsten Schneider)
   (d) $f(n+2) = H(n+3)f(n+1)^2 - 3^n f(n)$: nonlinear recurrences (MK)
2. The larger the class, the less can be done automatically with it.
3. Consider the class $\mathcal{C}$, defined by *structural induction:*
   (a) Constant sequences belong to $\mathcal{C}$
   (b) if p is a polynomial in $f(n)$, $f(n+1),\dots,$ $f(n+r-1)$ and $f_i(n+j)$ ($i = 1,\dots,s$, $j = 0,\dots,r$, $f_i \in \mathcal{C}$), then sequences defined by $f(n+r) = p$ or $f(n+r) = 1/p$ also belong to $\mathcal{C}$.
4. Examples:
   (a) all P-finite sequences
   (b) $\mathrm{Fib}(n), 2^{\mathrm{Fib}(n)}, \mathrm{Fib}(2^n) \in \mathcal{C}$ (why?)
   (c) $f(n) \in \mathcal{C} \implies \sum_{k=1}^{n} f(k), \prod_{k=1}^{n} f(k) \in \mathcal{C}$
   (d) continued fractions, orthogonal polynomials, Sylvester sequences, etc.
5. There is an algorithm which decides if a given element of $\mathcal{C}$ is equivalent to zero. (By $A = B \iff A - B = 0$ this allows to prove identities, note $A, B \in \mathcal{C} \implies A - B \in \mathcal{C}$)
6. Idea: Use polynomial ideal arithmetic.
   (a) Consider $f(n), f(n+1), \dots$ as variables in a polynomial ring.
   (b) Consider the ideal I generated by the recurrences.
   (c) Decide $f(n+r) \in I + \langle f(n), \dots, f(n+r-1)\rangle$. If yes: induction proof, otherwise increase r and try again.
7. Example: Prove $\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n$
   (a) Let $f_1(n) = n$, $f_2(n) = \sum_{k=1}^{n} f_1(k)$, $f_3(n) = f_2(n) - \frac{1}{2}f_1(n)^2 - \frac{1}{2}f_1(n)$.

(b) Consider

$$\begin{aligned} I = \big\langle & f_1(n+1) - f_1(n) - 1, \\ & f_2(n+1) - f_2(n) - f_1(n+1), \\ & f_3(n) - f_2(n) + \tfrac{1}{2} f_1(n)^2 - \tfrac{1}{2} f_1(n) \big\rangle \end{aligned}$$

(c) We have $f_3(n+1) \in I + \langle f_3(n) \rangle$, therefore the claim holds iff it holds for $n = 1$.

III Symbolic Summation

Hypergeometric Sequences

Let $\mathbb{K}$ be a field of characteristic zero.

Def. A sequence $(f_k)_{k\geq 0}$ in $\mathbb{K}$ is called *hypergeometric* if there exists an $r \in \mathbb{K}(x)$ such that $\dfrac{f_{k+1}}{f_k} = r(k)$ for all k.

Example The following sequences are easily seen to be hypergeometric.

- polynomials and rational functions
- $f_k = \dfrac{(-1)^k(2k+1)}{k(k+1)}$
- $f_k = q^k$ for $q \in \mathbb{K}$ fixed
- $f_k = (-1)^k \dbinom{a}{k}$ for $a \in \mathbb{K}$ fixed, because

$$\begin{aligned} \frac{f_{k+1}}{f_k} &= (-1)\frac{\binom{a}{k+1}}{\binom{a}{k}} = (-1)\frac{a(a-1)\cdots(a-k)}{(k+1)!}\frac{k!}{a(a-1)\cdots(a-k+1)} \\ &= (-1)\frac{a-k}{k+1} = \frac{k-a}{k+1} \end{aligned}$$

- etc.

Gosper's Algorithm

Gosper proposed the following algorithm for indefinite hypergeometric summation in 1979.

Input $(f_k)_{k\geq 0}$ hypergeometric in $\mathbb{K}$.

Output $(g_k)_{k\geq 0}$ hypergeometric in $\mathbb{K}$ solving the *telescoping equation*

$$g_{k+1} - g_k = f_k \tag{T}$$

or IMPOSSIBLE if no such (g_k) exists.

0. Compute $f_{\mathrm{rat}} \in \mathbb{K}(x)$ such that $\dfrac{f_{k+1}}{f_k} = f_{\mathrm{rat}}(k)$.

1. Compute the *G-form* of f_{rat}:

$$f_{\text{rat}}(x) = \frac{p(x+1)}{p(x)} \cdot \frac{q(x)}{r(x+1)}$$

   with $p, q, r \in \mathbb{K}[x]$ such that

$$\gcd(q(x), r(x+j)) = \text{const} \neq 0 \qquad (j \in \mathbb{N})$$

2. Solve the *Gosper equation*

$$p(x) = q(x)y(x+1) - r(x)y(x) \tag{G}$$

   for a polynomial $y(x)$.

   If no solution exists, return IMPOSSIBLE.

3. Let $\quad g_{\text{rat}}(x) = \dfrac{y(x+1)}{y(x)} \dfrac{q(x)}{r(x)}$

4. Return $\quad g_k := \dfrac{1}{g_{\text{rat}}(k) - 1} f_k.$

Remarks

- Gosper's Algorithm solves the summation problem by $\displaystyle\sum_{k=0}^{n} f_k = g_{n+1} - g_0$.
- Correctness I: If there is no polynomial solution to (G), then there is no hypergeometric solution to (T).

  Proof: $\boxed{\textbf{HW}}$ (nontrivial)
- Correctness II: g_k really solves (T), because

$$\frac{g_{k+1}}{g_k} \overset{\boxed{\textbf{HW}}}{=} g_{\text{rat}}(k)$$

  and therefore

$$g_{k+1} - g_k = g_{\text{rat}} g_k - g_k = (g_{\text{rat}} - 1) \frac{1}{g_{\text{rat}} - 1} f_k = f_k.$$

- The G-form is not unique. Example: $\dfrac{p(x+1)}{p(x)} \dfrac{x}{1}$ is a G-form for the same rational function as $\dfrac{p(x+1)(x+3)}{p(x)(x+2)} \dfrac{x(x+2)}{x+3}$.
- The G-form can be made unique by imposing the additional conditions

$$\gcd(p(x), q(x)) = 1 \qquad \text{and} \qquad \gcd(p(x), r(x)) = 1.$$

- Step 2 can be done by means of the methods presented in Chapter I (recurrences).

Example

What is $\sum_{k=0}^{n} kk!$? We apply Gosper's algorithm.

0. $\dfrac{f_{k+1}}{f_k} = \dfrac{(k+1)(k+1)!}{kk!} \quad \Longrightarrow \quad f_{\text{rat}}(x) = \dfrac{(x+1)^2}{x}$
1. By inspection:
$$f_{\text{rat}}(x) = \frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)} = \frac{x+1}{x} \cdot \frac{x+1}{1}$$
i.e. $p(x) = x, \quad q(x) = x+1, \quad r(x) = 1.$
2. Solve the Gosper equation
$$x = (x+1)y(x+1) - y(x)$$
for a polynomial $y(x) \in \mathbb{K}[x]$. Solution: $y(x) = 1$.
3. $g_{\text{rat}}(x) = \dfrac{1}{1} \cdot \dfrac{x+1}{1} = x+1$
4. $g_k = \dfrac{1}{(k+1)-1} f_k = \dfrac{1}{k} kk! = k! \ (k \geq 0)$

(Check: $g_{k+1} - g_k = (k+1)! - k! = (k+1-1)k! = kk!$, OK)

It follows that $\sum_{k=0}^{n} kk! = (n+1)! - 0! = (n+1)! - 1$.

HW

Homework Use Gosper's Algorithm to evaluate the following sums:

a) $\sum_{k=0}^{n} q^k$ ($q \in \mathbb{C}$ fixed)

b) $\sum_{k=0}^{n} (-1)^k \binom{a}{k}$ ($a \in \mathbb{C}$ fixed)

c) $\sum_{k=1}^{n} (-1)^k \dfrac{k}{4k^2-1}$

d) $\sum_{k=1}^{n} \dfrac{1}{4k^2-1}$

HW

Homework Use Gosper's Algorithm to prove that the sequence H_n of harmonic numbers ($H_n = \sum_{k=1}^{n} 1/k$) is not hypergeometric (and hence, in particular, not rational).

Computation of a Gosper Form

Goal: Given a rational function $r(x) \in \mathbb{K}(x)$, find a representation
$$r(x) = \frac{p(x+1)}{p(x)} \cdot \frac{q(x)}{r(x+1)}$$

with $p, q, r \in \mathbb{K}[x]$ such that

$$\gcd(q(x), r(x+j)) = \text{const} \neq 0 \qquad (j \in \mathbb{N}) \tag{$*$}$$

Remark If we somehow only get to a representation $f_{\text{rat}}(x) = \dfrac{p(x+2)}{p(x)} \cdot \dfrac{q(x)}{r(x+1)}$ with q, r satisfying $(*)$, that's already fine, because then

$$f_{\text{rat}}(x) = \frac{p(x+2)p(x+1)}{p(x+1)p(x)} \cdot \frac{q(x)}{r(x+1)}$$

is of the required form, and similarly for other integers in place of 2.

In order to compute a Gosper Form of $f_{\text{rat}}(x) = \dfrac{s(x)}{t(x)}$, start with $p(x) = 1$ and $q(x) = s(x)$, $r(x+1) = t(x)$, and proceed as follows.

1. Find all $j \geq 2$ such that $\gcd(q(x), r(x+j)) \neq \text{const}$.
   Consider $\text{Res}_x(q(x), r(x+j))$. (This is a polynomial in j.) Recall that the resultant vanishes if and only if the gcd of the arguments is nontrivial. Ergo: Integer roots of the resultant give the critical j.
2. If no such j exists, we are done. Otherwise, take the smallest.
   Let $d(x) = \gcd(q(x), r(x+j))$, $q(x) = \bar{q}(x)d(x)$, $r(x+j) = \bar{r}(x+j)d(x)$.
   Then
   $$\frac{q(x)}{r(x+1)} = \frac{\bar{q}(x)}{\bar{r}(x+1)} \cdot \frac{d(x)}{d(x-j+1)}$$
   with $\gcd(\bar{q}(x), \bar{r}(x+l)) = 1$ for $l \leq j$.
   Update $q(x) = \bar{q}(x)$, $r(x+1) = \bar{r}(x+1)$, $p(x) = p(x)d(x-1)d(x)\cdots d(x-j+1)$.
3. Goto 1.

Example Let $q(x) = x^2 + x$ and $r(x) = 2x$. Then $r(x+j) = 2x + 2j$ and

$$\text{Res}_x(q(x), r(x+j)) = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2j & 2 \\ 0 & 0 & 2j \end{vmatrix} = 2j(2j-2)$$

vanishes iff $j = 0$ or $j = 1$.
$j = 0$ is not of interest because $0 \notin \mathbb{N}$.
Take $j = 1$. Then $\gcd(q(x), r(x+1)) = x + 1$, and so we continue with

$$q(x) = x, r(x) = 2, p(x) = x.$$

Done.