Multivariate Difference-Differential Dimension Polynomials and New Invariants of Difference-Differential Field Extensions

Alexander Levin
The Catholic University of America
Washington, D. C. 20064
levin@cua.edu

ABSTRACT

We introduce a method of characteristic sets with respect to several term orderings for difference-differential polynomials. Using this technique, we obtain a method of computation of multivariate dimension polynomials of finitely generated difference-differential field extensions. Furthermore, we find new invariants of such extensions and show how the computation of multivariate difference-differential polynomials is applied to the equivalence problem for systems of algebraic difference-differential equations.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*algebraic algorithms*

General Terms

Theory, Algorithms

Keywords

Difference-differential field, dimension polynomial, reduction, characteristic set

1. INTRODUCTION

The role of Hilbert polynomials in commutative and homological algebra, as well as in algebraic geometry, is well known. A similar role in differential algebra is played by differential dimension polynomials, which describe in exact terms the freedom degree of a dynamic system, as well as the number of arbitrary constants in the general solution of a system of algebraic differential equations.

The notion of a differential dimension polynomial was introduced by E. Kolchin [4] for a finitely generated differential field extension $L = K\langle\eta_1, \dots, \eta_n\rangle$ ($\operatorname{Char} K = 0$). He proved that there is a polynomial $\omega_{\eta|K}(t)$ associated with the set of generators $\eta = \{\eta_1, \dots, \eta_n\}$ such that for all sufficiently large $r \in \mathbf{Z}$, $\omega_{\eta|K}(r)$ is the transcendence degree of the field extension of K generated by all derivations of η_i ($1 \le i \le n$) of order $\le r$. $\omega_{\eta|K}(t)$ is called the *differential dimension polynomial* of the extension L/K associated with the set of differential generators η.

If P is a prime differential ideal of a finitely generated differential algebra $R = K\{\zeta, \dots, \zeta_n\}$ over a differential field K, then the quotient field of R/P is a differential field extension of K generated by the images of ζ_i ($1 \le i \le n$) in R/P. The corresponding differential dimension polynomial, therefore, characterizes the ideal P; it is denoted by $\omega_P(t)$. Assigning such polynomials to prime differential ideals has led to a number of new results on the Krull-type dimension of differential algebras and dimension of differential varieties (see, for example, [2] and [3]).

Furthermore, as it was shown in [11], one can naturally assign a differential dimension polynomial to a system of algebraic differential equations and this polynomial expresses the A. Einstein's strength of the system (see [1]). Methods of computation of (univariate) differential dimension polynomials and the strength of systems of differential equations via the Ritt-Kolchin technique of characteristic sets can be found, for example, in [12] and [6, Chapters 5, 9]. Note also, that there are quite many works on computation of dimension polynomials of differential, difference and difference-differential modules with the use of various generalizations of the Gröbner basis method (see, for example, [6, Chapters V - XI], [7], [8], [9], [10, Chapter 3], and [13]). This method, however, does not work for non-linear difference-differential polynomial ideals, which, generally speaking, do not have finite Gröbner bases.

In this paper, we develop a method of characteristic sets with respect to several orderings for algebras of difference-differential polynomials over a difference-differential fields whose basic set of derivations is partied into several disjoint subsets. We apply this method to prove the existence, outline a method of computation, and determine invariants of a multivariate dimension polynomial associated with a finite system of generators of a difference-differential field extension (and a partition of the basic sets of derivations). We also show that most of these invariants are not carried by univariate dimension polynomials and show how the consideration of the new invariants can be applied to the isomorphism problem for difference-differential field extensions and equivalence problem for systems of algebraic difference-differential equations.

2. PRELIMINARIES

Throughout the paper, $\mathbf{N}, \mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ denote the sets of all non-negative integers, integers, rational numbers, and real numbers, respectively. $\mathbf{Q}[t_1, \dots, t_k]$ will denote the ring of polynomials in variables $t_1, \dots, t_k$ over $\mathbf{Q}$.

By a *difference-differential ring* we mean a commutative ring R together with finite sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivations and automorphisms of R, respectively, such that any two mappings of the set $\Delta \cup \sigma$ commute. The set $\Delta \cup \sigma$ is called the *basic set* of the difference-differential ring R, which is also called a Δ-σ-ring. If R is a field, it is called a *difference-differential field* or a Δ-σ-field. Furthermore, in what follows, we denote the set

$$\{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$$

by σ^*. If R is a Δ-σ-ring, then Λ will denote the free commutative semigroup of all power products of the form

$$\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$$

where $k_i \in \mathbf{N}$, $l_j \in \mathbf{Z}$ $(1 \le i \le m,\ 1 \le j \le n)$. For any such an element λ, we set

$$\lambda_\Delta = \delta_1^{k_1} \dots \delta_m^{k_m}, \quad \lambda_\sigma = \alpha_1^{l_1} \dots \alpha_n^{l_n},$$

and denote by Λ_Δ and Λ_σ the commutative semigroup of power products $\delta_1^{k_1} \dots \delta_m^{k_m}$ and the commutative group of elements of the form $\alpha_1^{l_1} \dots \alpha_n^{l_n}$, respectively. The *order* of λ is defined as

$$\operatorname{ord} \lambda = \sum_{i=1}^{m} k_i + \sum_{j=1}^{n} |l_j|,$$

and for every $r \in \mathbf{N}$, we set $\Lambda(r) = \{\lambda \in \Lambda \mid \operatorname{ord} \lambda \le r\}$ $(r \in \mathbf{N})$.

A subring (ideal) R_0 of a Δ-σ-ring R is called a difference-differential (or Δ-σ-) subring of R (respectively, difference-differential (or Δ-σ-) ideal of R) if R_0 is closed with respect to the action of any operator of $\Delta \cup \sigma^*$. If a prime ideal P of R is closed with respect to the action of $\Delta \cup \sigma^*$, it is called a *prime* difference-differential (or Δ-σ-) *ideal* of R.

If R is a Δ-σ-field and R_0 a subfield of R which is also a Δ-σ-subring of R, then R_0 is said to be a Δ-σ-subfield of R; R, in turn, is called a difference-differential (or Δ-σ-) field extension or a Δ-σ-overfield of R_0. In this case we also say that we have a Δ-σ-field extension R/R_0.

If R is a Δ-σ-ring and $\Sigma \subseteq R$, then the intersection of all Δ-ideals of R containing the set Σ is the smallest Δ-σ-ideal of R containing Σ; it is denoted by $[\Sigma]$. (Clearly, $[\Sigma]$ is generated, as an ideal, by the set $\{\lambda\xi \,|\, \xi \in \Sigma,\ \lambda \in \Lambda\}$). If the set Σ is finite, $\Sigma = \{\xi_1, \dots, \xi_q\}$, we say that the Δ-ideal I is finitely generated (we write this as $I = [\xi_1, \dots, \xi_q]$) and call $\xi_1, \dots, \xi_q$ difference-differential (or Δ-σ-)generators of I.

If K_0 is a Δ-σ-subfield of the Δ-σ-field K and $\Sigma \subseteq K$, then the smallest Δ-σ-subfield of K containing K_0 is denoted by $K_0\langle\Sigma\rangle$. If $K = K_0\langle\Sigma\rangle$ and the set Σ is finite, $\Sigma = \{\eta_1, \dots, \eta_s\}$, then K is said to be a finitely generated Δ-σ-extension of K_0 with the set of Δ-σ-generators $\{\eta_1, \dots, \eta_s\}$. In this case we write $K = K_0\langle\eta_1, \dots, \eta_s\rangle$. It is easy to see that the field $K_0\langle\eta_1, \dots, \eta_s\rangle$ coincides with the field $K_0(\{\lambda\eta_i \,|\, \lambda \in \Lambda, 1 \le i \le s\})$.

A ring homomorphism of Δ-σ-rings $\phi : R \longrightarrow S$ is called a *difference-differential* (or Δ-σ-) *homomorphism* if $\phi(\tau a) = \tau\phi(a)$ for any $\tau \in \Delta \cup \sigma$, $a \in R$.

If K is a Δ-σ-field and $Y = \{y_1, \dots, y_s\}$ is a finite set of symbols, then one can consider the countable set of symbols

$$\Lambda Y = \{\lambda y_j \mid \lambda \in \Lambda, 1 \le j \le s\}$$

and the polynomial ring

$$R = K[\{\lambda y_j \mid \lambda \in \Lambda, 1 \le j \le s\}]$$

in the set of indeterminates ΛY over the field K. This polynomial ring is naturally viewed as a Δ-σ-ring where

$$\tau(\lambda y_j) = (\tau\lambda) y_j$$

for any $\tau \in \Delta \cup \sigma$, $\lambda \in \Lambda$, $1 \le j \le s$, and the elements of $\Delta \cup \sigma$ act on the coefficients of the polynomials of R as they act in the field K. The ring R is called a *ring of difference-differential* (or Δ-σ-) *polynomials* in the set of differential (Δ-σ-)indeterminates $y_1, \dots, y_s$ over K. This ring is denoted by $K\{y_1, \dots, y_s\}$ and its elements are called difference-differential (or Δ-σ-) polynomials.

Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a difference-differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. As a field, $L = K(\{\lambda\eta_j \mid \lambda \in \Lambda, 1 \le j \le s\})$.

The following is a unified version of E. Kolchin's theorem on differential dimension polynomial and the author's theorem on the dimension polynomial of a difference field extension (see [7] or [10, Theorem 4.2.5]).

THEOREM 2.1. *With the above notation, there exists a polynomial $\phi_{\eta|K}(t) \in \mathbf{Q}[t]$ such that*

(i) *$\phi_{\eta|K}(r) = \operatorname{tr.deg}_K K(\{\lambda\eta_j \mid \lambda \in \Lambda(r), 1 \le j \le s\})$ for all sufficiently large $r \in \mathbf{Z}$;*

(ii) *$\deg \phi_{\eta|K} \le m + n$ and $\phi_{\eta|K}(t)$ can be written as $\phi_{\eta|K}(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$, where $a_i \in \mathbf{Z}$ and $2^n | a_{m+n}$.*

(iii) *$d = \deg \phi_{\eta|K}$, a_{m+n} and a_d do not depend on the set of difference-differential generators η of L/K ($a_d \ne a_{m+n}$ if and only if $d < m + n$). Moreover, $\frac{a_{m+n}}{2^n}$ is equal to the difference-differential transcendence degree of L over K (denoted by Δ-σ-$\operatorname{tr.deg}_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the family $\{\lambda\xi_i \mid \lambda \in \Lambda, 1 \le i \le k\}$ is algebraically independent over K.*

The polynomial whose existence is established by this theorem is called a *univariate difference-differential* (or Δ-σ-) *dimension polynomial* of the extension L/K associated with the system of difference-differential generators η.

3. PARTITION OF THE SET Δ. FORMULATION OF THE MAIN THEOREM

Let K be a difference-differential field of zero characteristic with basic sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivations and automorphisms, respectively. Suppose that the set of derivations is represented as the union of p disjoint subsets $(p \ge 1)$:

$$\Delta = \Delta_1 \cup \dots \cup \Delta_p \tag{1}$$

where $\Delta_1 = \{\delta_1, \dots, \delta_{m_1}\}$, $\Delta_2 = \{\delta_{m_1+1}, \dots, \delta_{m_1+m_2}\}$, ...,

$$\Delta_p = \{\delta_{m_1+\dots+m_{p-1}+1}, \dots, \delta_m\} \quad (m_1 + \dots + m_p = m).$$

If

$$\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \in \Lambda$$

($k_i \in \mathbf{N}$, $l_j \in \mathbf{Z}$), then the orders of λ with respect to Δ_i ($1 \le i \le p$) and σ are defined as

$$\operatorname{ord}_i \lambda = \sum_{\nu=m_1+\cdots+m_{i-1}+1}^{m_1+\cdots+m_i} k_\nu \quad \text{and} \quad \operatorname{ord}_\sigma \lambda = \sum_{j=1}^{n} |l_j|,$$

respectively. (If $i = 1$, then ν changes from 1 to m_1 in the first sum.) For any $r_1, \dots, r_{p+1} \in \mathbf{N}$, the set

$$\{\lambda \in \Lambda \,|\, \operatorname{ord}_i \lambda \le r_i \ (i = 1, \dots, p), \operatorname{ord}_\sigma \lambda \le r_{p+1}\}$$

will be denoted by $\Lambda(r_1, \dots, r_{p+1})$.

In what follows, for any permutation $(j_1, \dots, j_{p+1})$ of the set $\{1, \dots, p+1\}$, $<_{j_1,\dots,j_{p+1}}$ will denote the lexicographic order on $\mathbf{N}^{p+1}$ such that

$$(r_1, \dots, r_{p+1}) <_{j_1,\dots,j_{p+1}} (s_1, \dots, s_{p+1})$$

if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbf{N}$, $1 \le k \le p$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$.

If $\Sigma \subseteq \mathbf{N}^{p+1}$, then Σ' will denote the set of all $e \in \Sigma$ that are maximal elements of this set with respect to one of the $(p+1)!$ orders $<_{j_1,\dots,j_{p+1}}$.

The following statement is the main result of this paper.

THEOREM 3.1. *Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-σ-field extension generated by a set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\Phi_\eta \in \mathbf{Q}[t_1, \dots, t_{p+1}]$ such that*

(i) $\Phi_\eta(r_1, \dots, r_{p+1}) = \operatorname{tr.deg}_K K(\bigcup_{j=1}^{s} \Lambda(r_1, \dots, r_{p+1})\eta_j)$

for all sufficiently large $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$ (it means that there exist $s_1, \dots, s_{p+1} \in \mathbf{N}$ such that the equality holds for all $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$ with $r_1 \ge s_1, \dots, r_{p+1} \ge s_{p+1}$);

(ii) *$\deg_{t_i} \Phi_\eta \le m_i$ ($1 \le i \le p$), $\deg_{t_{p+1}} \Phi_\eta \le n$ and $\Phi_\eta(t_1, \dots, t_{p+1})$ can be represented as*

$$\Phi_\eta = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n} a_{i_1\dots i_{p+1}} \binom{t_1 + i_1}{i_1} \cdots \binom{t_{p+1} + i_{p+1}}{i_{p+1}}$$

where $a_{i_1\dots i_{p+1}} \in \mathbf{Z}$ and $2^n \,|\, a_{m_1\dots m_p n}$.

(iii) *Let $E_\eta = \{(i_1, \dots, i_{p+1}) \in \mathbf{N}^{p+1} \,|\, 0 \le i_k \le m_k$ for $k = 1, \dots, p$, $0 \le i_{p+1} \le n$, and $a_{i_1\dots i_{p+1}} \ne 0\}$. Then $d = \deg \Phi_\eta$, $a_{m_1\dots m_{p+1}}$, elements $(k_1, \dots, k_{p+1}) \in E'_\eta$, the corresponding coefficients $a_{k_1\dots k_{p+1}}$ and the coefficients of the terms of total degree d do not depend on the choice of the system of Δ-σ-generators η.*

DEFINITION 3.2. *$\Phi_\eta(t_1, \dots, t_{p+1})$ is called the difference-differential* (or Δ-σ-) *dimension polynomial of the Δ-σ-field extension L/K associated with the set of Δ-σ-generators η and partition* (1) *of the basic set of derivations.*

The Δ-σ-dimension polynomial associated with partition (1) has the following interpretation as the strength of a system of difference-differential equations.

Consider a system of difference-differential equations

$$A_i(f_1, \dots, f_s) = 0 \qquad (i = 1, \dots, q) \qquad (2)$$

over a field of functions of m real variables $x_1, \dots, x_m$ (f_i are unknown functions of $x_1, \dots, x_m$). Suppose that $\Delta = \{\delta_1, \dots, \delta_m\}$ where δ_i is the partial differentiation $\partial/\partial x_i$ and the basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_m\}$ where

$$\alpha_i : f(x_1, \dots, x_m) \mapsto f(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_m)$$

($h_1, \dots, h_m \in \mathbf{R}$). Thus, we assume that the left-hand sides of the equations in (2) contain unknown functions f_i, their partial derivatives, their images under the shifts α_j, and various compositions of such shifts and partial derivations. Furthermore, we suppose that system (2) is algebraic, that is, all $A_i(y_1, \dots, y_s)$ are elements of a ring of Δ-σ-polynomials $K\{y_1, \dots, y_s\}$ over some functional Δ-σ-field K.

Let us consider a grid with equal cells of dimension $h_1 \times \cdots \times h_m$ that fills $\mathbf{R}^m$. We fix some node $\mathcal{P}$ and say that *a node $\mathcal{Q}$ has order i* if the shortest path from $\mathcal{P}$ to $\mathcal{Q}$ along the edges of the grid consists of i steps (by a step we mean a path from a node of the grid to a neighbor node along the edge between them). We also fix partition (1) of the set of basic derivations Δ (such a partition can be, for example, a natural separation of (all or some) derivations with respect to coordinates and the derivation with respect to time).

For any $r_1, \dots, r_{p+1} \in \mathbf{N}$, let us consider the values of the unknown functions $f_1, \dots, f_s$ and their partial derivatives, whose order with respect to Δ_i does not exceed r_i ($1 \le i \le p$), at the nodes whose order does not exceed r_{p+1}. If $f_1, \dots, f_s$ should not satisfy any system of equations (or any other condition), these values can be chosen arbitrarily. Because of the system (and equations obtained from the equations of the system by partial differentiations and transformations of the form

$$f_j(x_1, \dots, x_m) \mapsto f_j(x_1 + k_1h_1, \dots, x_m + k_mh_m)$$

with $k_1, \dots, k_m \in \mathbf{Z}$, $1 \le j \le s$), the number of independent values of the functions $f_1, \dots, f_s$ and their partial derivatives whose ith order does not exceed r_i ($1 \le i \le p$) at the nodes of order $\le r_{p+1}$ decreases. This number, which is a function of $p+1$ variables $r_1, \dots, r_{p+1}$, is the "measure of strength" of the system in the sense of A. Einstein. We denote it by $S_{r_1,\dots,r_{p+1}}$.

Suppose that the Δ-σ-ideal J of $K\{y_1, \dots, y_s\}$ generated by the Δ-σ-polynomials $A_1, \dots, A_q$ is prime (e. g., the polynomials are linear). Then the field of fractions L of the Δ-σ-integral domain $K\{y_1, \dots, y_s\}/J$ is a Δ-σ-field extension of K generated by the finite set $\eta = \{\eta_1, \dots, \eta_s\}$ where η_i is the canonical image of y_i in $K\{y_1, \dots, y_s\}/J$ ($1 \le i \le s$). It is easy to see that the Δ-σ-dimension polynomial $\Phi_\eta(t_1, \dots, t_{p+1})$ of the extension L/K associated with the system of Δ-σ-generators η has the property that

$$\Phi_\eta(r_1, \dots, r_{p+1}) = S_{r_1,\dots,r_{p+1}}$$

for all sufficiently large $(r_1, \dots, r_{p+q}) \in \mathbf{N}^{p+1}$, so this dimension polynomial is the measure of strength of the system of difference-differential equations (2) in the sense of A. Einstein.

4. NUMERICAL POLYNOMIALS

DEFINITION 4.1. *A polynomial $f(t_1, \dots, t_p) \in \mathbf{Q}[t_1, \dots, t_p]$ is called* numerical *if $f(r_1, \dots, r_p) \in \mathbf{Z}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{Z}^p$.*

The following theorem proved in [6, Chapter 2] gives the "canonical" representation of a numerical polynomial. (As usual, $\binom{t}{k} = \frac{t(t-1)\dots(t-k+1)}{k!}$.)

THEOREM 4.2. *Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables and let $\deg_{t_i} f = m_i$ ($m_1, \dots, m_p \in \mathbf{N}$).*

Then $f(t_1,\dots,t_p)$ can be represented as

$$f(t_1,\dots t_p)=\sum_{i_1=0}^{m_1}\dots\sum_{i_p=0}^{m_p}a_{i_1\dots i_p}\binom{t_1+i_1}{i_1}\dots\binom{t_p+i_p}{i_p}$$

with uniquely defined integer coefficients $a_{i_1\dots i_p}$.

In what follows, we deal with subsets of $\mathbf{N}^m\times\mathbf{Z}^n$ $(m,n\ge 1)$ and a fixed partition of the set $\mathbf{N}_m=\{1,\dots,m\}$ into p disjoint subsets ($p\ge 1$):

$$\mathbf{N}_m=N_1\cup\dots\cup N_p \tag{3}$$

where $N_1=\{1,\dots,m_1\},\dots,\ N_p=\{m_1+\dots+m_{p-1}+1,\dots,m\}\quad(m_1+\dots+m_p=m)$.

If $a=(a_1,\dots,a_{m+n})\in\mathbf{N}^m\times\mathbf{Z}^n$ we set

$$\mathrm{ord}_i\,a=\sum\nolimits_{j\in N_i}a_j,\quad 1\le i\le p$$

and

$$\mathrm{ord}_{p+1}\,a=\sum_{i=m+1}^{m+n}|a_i|.$$

Furthermore, we consider the set $\mathbf{Z}^n$ as a union

$$\mathbf{Z}^n=\bigcup\nolimits_{1\le j\le 2^n}\mathbf{Z}_j^{(n)} \tag{4}$$

where $\mathbf{Z}_1^{(n)},\dots,\mathbf{Z}_{2^n}^{(n)}$ are all different Cartesian products of n sets each of which is either $\mathbf{N}$ or $\mathbf{Z}_-=\{a\in\mathbf{Z}|a\le 0\}$. We assume that $\mathbf{Z}_1^{(n)}=\mathbf{N}^n$ and call $\mathbf{Z}_j^{(n)}$) the *jth orthant* of $\mathbf{Z}^n$. The set $\mathbf{N}^m\times\mathbf{Z}^n$ will be treated as a partially ordered set with the order $\trianglelefteq$ such that

$$(e_1,\dots,e_m,f_1,\dots,f_n)\trianglelefteq(e'_1,\dots,e'_m,f'_1,\dots,f'_n)$$

if and only if the n-tuples $(f_1,\dots,f_n)$ and $(f'_1,\dots,f'_n)$ lie in the same orthant $\mathbf{Z}_k^{(n)}$ and

$$(e_1,\dots,e_m,|f_1|,\dots,|f_n|)<_P(e'_1,\dots,e'_m,|f'_1|,\dots,|f'_n|)$$

where $<_P$ is the product order on $\mathbf{N}^{m+n}$. (Recall that the product order on $\mathbf{N}^k$ is a partial order $<_P$ such that

$$c=(c_1,\dots,c_k)<_P c'=(c'_1,\dots,c'_k)$$

if and only if $c_i<c'_i$ for $i=1,\dots,k$. We write $c\le_P c'$ if $c<_P c'$ or $c=c'$).

If A is a subset of $\mathbf{N}^m\times\mathbf{Z}^n$, then W_A will denote the set of all elements $w\in\mathbf{N}^m\times\mathbf{Z}^n$ such that there is no $a\in A$ with $a\trianglelefteq w$. Furthermore, for any $r_1,\dots r_{p+1}\in\mathbf{N}$, $A(r_1,\dots r_{p+1})$ denotes the set

$$\{x=(x_1,\dots,x_m,x'_1,\dots,x'_n)\in A\mid \mathrm{ord}_i\,x\le r_i,1\le i\le p+1\}.$$

If $E\subseteq\mathbf{N}^m$ and $s_1,\dots,s_p\in\mathbf{N}$, then $E(s_1,\dots,s_p)$ will denote the set $\{(e_1,\dots,e_m)\in E\mid \mathrm{ord}_i(e_1,\dots,e_m,0,\dots,0)\le s_i$ for $i=1,\dots,p\}$ ($(e_1,\dots,e_m,0,\dots,0)$ ends with n zeros; it is treated as a point in $\mathbf{N}^m\times\mathbf{Z}^n$.) Furthermore V_E will denote the set of all m-tuples $v=(v_1,\dots,v_m)\in\mathbf{N}$ which are not greater than or equal to any m-tuple from E with respect to the product order on $\mathbf{N}^m$. Clearly, $v=(v_1,\dots,v_m)\in V_E$ if and only if for any element $(e_1,\dots,e_m)\in E$, there exists $i\in\mathbf{N},1\le i\le m$, such that $e_i>v_i$.

The following two theorems are proved in [6, Chapter 2].

THEOREM 4.3. *Let E be a subset of $\mathbf{N}^m$ where $m=m_1+\dots+m_p$ for some $m_1,\dots,m_p\in\mathbf{N}$ $(p\ge 1)$. Then there exists a numerical polynomial $\omega_E(t_1,\dots,t_p)$ such that*

(i) $\omega_E(r_1,\dots,r_p)=\mathrm{Card}\,V_E(r_1,\dots,r_p)$ *for all sufficiently large $(r_1,\dots,r_p)\in\mathbf{N}^p$.* (As usual, Card M denotes the number of elements of a finite set M).

(ii) $\deg_{t_i}\omega_E\le m_i$ *for all $i=1,\dots,p$.*

(iii) $\deg\omega_E=m$ *if and only if $E=\emptyset$. In this case*
$$\omega_E(t_1,\dots,t_p)=\prod_{i=1}^{p}\binom{t_i+m_i}{m_i}.$$

DEFINITION 4.4. *The polynomial $\omega_E(t_1,\dots,t_p)$ is called the dimension polynomial of the set $E\subseteq\mathbf{N}^m$ associated with the partition $(m_1,\dots,m_p)$ of m.*

THEOREM 4.5. *Let $E=\{e_1,\dots,e_q\}$ $(q\ge 1)$ be a finite subset of $\mathbf{N}^m$ and let partition* (3) *of $\mathbf{N}_m$ be fixed. Let $e_i=(e_{i1},\dots,e_{im})\quad(1\le i\le q)$ and for any $l\in\mathbf{N}$, $0\le l\le q$, let $\Gamma(l,q)$ denote the set of all l-element subsets of the set $\mathbf{N}_q=\{1,\dots,q\}$. Furthermore, for any $\sigma\in\Gamma(l,q)$, let $\bar e_{\emptyset j}=0$, $\bar e_{\sigma j}=\max\{e_{ij}\mid i\in\sigma\}$ if $\sigma\ne\emptyset$ $(1\le j\le m)$, and $b_{\sigma k}=\sum_{h\in N_k}\bar e_{\sigma h}$ $(k=1,\dots,p)$. Then*

$$\omega_E(t_1,\dots,t_p)=\sum_{l=0}^{q}(-1)^l\sum_{\sigma\in\Gamma(l,\,q)}\prod_{j=1}^{p}\binom{t_j+m_j-b_{\sigma j}}{m_j} \tag{5}$$

Remark. It is clear that if $E\subseteq\mathbf{N}^m$ and E^* is the set of all minimal elements of the set E with respect to the product order on $\mathbf{N}^m$, then the set E^* is finite and $\omega_E(t_1,\dots,t_p)=\omega_{E^*}(t_1,\dots,t_p)$. Thus, Theorem 4.5 gives an algorithm that allows one to find a numerical polynomial associated with any subset of $\mathbf{N}^m$ (and with a given partition of the set $\{1,\dots,m\}$): one should first find the set of all minimal points of the subset and then apply Theorem 4.5.

The following result can be obtained by mimicking the proof of [8, Theorem 3.4].

THEOREM 4.6. *Let $A\subseteq\mathbf{N}^m\times\mathbf{Z}^n$ and let partition* (3) *of $\mathbf{N}_m$ be fixed. Then there exists a numerical polynomial $\phi_A(t_1,\dots,t_{p+1})$ such that*

(i) *$\phi_A(r_1,\dots,r_{p+1})=\mathrm{Card}\,W_A(r_1,\dots,r_{p+1})$ for all sufficiently large $(r_1,\dots,r_{p+1})\in\mathbf{N}^{p+1}$.*

(ii) *$deg_{t_i}\phi_A\le m_i$ $(1\le i\le p)$, $deg_{t_{p+1}}\phi_A\le n$ and the coefficient of $t_1^{m_1}\dots t_p^{m_p}t_{p+1}^{n}$ in ϕ_A is of the form*

$$\frac{2^na}{m_1!\dots m_p!n!},$$

with $a\in\mathbf{Z}$.

(iii) *Let us consider a mapping*

$$\rho:\mathbf{N}^m\times\mathbf{Z}^n\longrightarrow\mathbf{N}^{m+2n}$$

such that $\rho((e_1,\dots,e_{m+n})=(e_1,\dots,e_m,\max\{e_{m+1},0\},\dots,\max\{e_{m+n},0\},\max\{-e_{m+1},0\},\dots,\max\{-e_{m+n},0\})$.

Let

$$B=\rho(A)\cup\{\bar e_1,\dots,\bar e_n\},$$

where $\bar e_i$ $(1\le i\le n)$ is a $(m+2n)$-tuple in $\mathbf{N}^{m+2n}$ whose $(m+i)$th and $(m+n+i)$th coordinates are equal to 1 and all other coordinates are equal to 0. Then

$$\phi_A(t_1,\dots,t_{p+1})=\omega_B(t_1,\dots,t_{p+1})$$

where $\omega_B(t_1,\dots,t_{p+1})$ is the dimension polynomial of the set B (see Definition 4.4) associated with the partition $\mathbf{N}_{m+2n}=\{1,\dots,m_1\}\cup\{m_1+1,\dots,m_1+m_2\}\cup\dots\cup\{m_1+\dots+m_{p-1}+1,\dots,m\}\cup\{m+1,\dots,m+2n\}$ of the set $\mathbf{N}_{m+2n}$.

(iv) *If $A = \emptyset$, then*

$$\phi_A = \prod_{j=1}^{p} \binom{t_j + m_j}{m_j} \sum_{i=0}^{n} (-1)^{n-i} 2^i \binom{n}{i} \binom{t_{p+1} + i}{i}. \quad (6)$$

The polynomial $\phi_A(t_1, \dots, t_{p+1})$ is called the *dimension polynomial* of the set $A \subseteq \mathbf{N}^m \times \mathbf{Z}^n$ associated with partition (3) of $\mathbf{N}_m$.

5. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 3.1 and give a method of computation of difference-differential dimension polynomials of Δ-σ-field extensions based on constructing a characteristic set of the defining prime Δ-σ-ideal of the extension.

In what follows we use the conventions of section 3. In particular, we assume that partition (1) of the set of basic derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ is fixed.

Let us consider total orderings $<_1, \dots, <_p, <_\sigma$ of the set of power products Λ such that

$$\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} <_i \lambda' = \delta_1^{k'_1} \dots \delta_m^{k'_m} \alpha_1^{l'_1} \dots \alpha_n^{l'_n}$$

$(1 \le i \le p)$ if and only if
$(\mathrm{ord}_i \lambda, \mathrm{ord}\, \lambda, \mathrm{ord}_1 \lambda, \dots, \mathrm{ord}_{i-1} \lambda, \mathrm{ord}_{i+1} \lambda, \dots, \mathrm{ord}_p \lambda,$ $\mathrm{ord}_\sigma \lambda, k_{m_1+\dots+m_{i-1}+1}, \dots, k_{m_1+\dots+m_i}, k_1, \dots, k_{m_1+\dots+m_{i-1}},$ $k_{m_1+\dots+m_i+1}, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n)$ is less than the corresponding $(m+2n+p+2)$-tuple for λ' with respect to the lexicographic order on $\mathbf{N}^{m+2n+p+2}$.

Similarly, $\lambda <_\sigma \lambda'$ if and only if $(\mathrm{ord}_\sigma \lambda, \mathrm{ord}\, \lambda, \mathrm{ord}_1 \lambda, \dots,$ $\mathrm{ord}_p \lambda, |l_1|, \dots, |l_n|, l_1, \dots, l_n, k_1, \dots, k_m)$ is less than the corresponding $(m+2n+p+2)$-tuple for λ' with respect to the lexicographic order on $\mathbf{N}^{m+2n+p+2}$.

Let $\lambda_1 = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$, $\lambda_2 = \delta_1^{r_1} \dots \delta_m^{r_m} \alpha_1^{s_1} \dots \alpha_n^{s_n}$ be elements of Λ. They are called *similar*, if $(l_1, \dots, l_n)$ and $(s_1, \dots, s_n)$ lie in the same orthant of $\mathbf{Z}^n$ (see (4)). In this case we write $\lambda_1 \sim \lambda_2$. We say that λ_1 *divides* λ_2 (or λ_2 is a *multiple* of λ_1) and write $\lambda_1 | \lambda_2$ if $\lambda_1 \sim \lambda_2$ and there exists $\lambda \in \Lambda$ such that $\lambda \sim \lambda_1$, $\lambda \sim \lambda_2$ and $\lambda_2 = \lambda\lambda_1$.

Let K be a Δ-σ-field (Char $K = 0$) and let partition (1) of the set Δ be fixed. Let $K\{y_1, \dots, y_s\}$ be the ring of Δ-σ-polynomials over K and let ΛY denote the set of all elements λy_i ($\lambda \in \Lambda$, $1 \le i \le s$) called *terms*. Note that as a ring,

$$K\{y_1, \dots, y_s\} = K[\Lambda Y].$$

Two terms $u = \lambda y_i$ and $v = \lambda' y_j$ are called *similar* if λ and λ' are similar; in this case we write $u \sim v$. If $u = \lambda y_i$ is a term and $\lambda' \in \Lambda$, we say that u is similar to λ' and write $u \sim \lambda'$ if $\lambda \sim \lambda'$. Furthermore, if $u, v \in \Lambda Y$, we say that u *divides* v or v *is a multiple of* u, if $u = \lambda' y_i$, $v = \lambda'' y_i$ for some y_i and $\lambda' | \lambda''$. (If $\lambda'' = \lambda\lambda'$ for some $\lambda \in \Lambda$, $\lambda \sim \lambda'$, we write $\frac{v}{u}$ for λ.)

Let us consider $p+1$ orders $<_1, \dots, <_p, <_\sigma$ on the set ΛY that correspond to the orders on Λ (we use the same symbols for the orders on Λ and ΛY). These orders are defined as follows: $\lambda y_j <_i$ (or $<_\sigma$) $\lambda' y_k$ if and only if $\lambda <_i$ (respectively, $<_\sigma$)λ' in Λ or $\lambda = \lambda'$ and $j < k$ ($1 \le i \le p$, $1 \le j, k \le s$).

The order of a term $u = \lambda y_k$ and its orders with respect to the sets Δ_i ($1 \le i \le p$) and σ are defined as the corresponding orders of λ (we use the same notation $\mathrm{ord}\, u$, $\mathrm{ord}_i\, u$, and $\mathrm{ord}_\sigma\, u$ for the corresponding orders).

If $A \in K\{y_1, \dots, y_s\} \setminus K$ and $1 \le k \le p$, then the highest with respect to $<_k$ term that appears in A is called the *k-leader* of A. It is denoted by $u_A^{(k)}$. The highest term of A with respect to $<_\sigma$ is called the *σ-leader* of A; it is denoted by v_A. If A is written as a polynomial in v_A,

$$A = I_d (v_A)^d + I_{d-1} (v_A)^{d-1} + \dots + I_0,$$

where all terms of $I_0, \dots, I_d$ are less than v_A with respect to $<_\sigma$, then I_d is called the *initial* of A. The partial derivative

$$\partial A / \partial v_A = d I_d (v_A)^{d-1} + (d-1) I_{d-1} (v_A)^{d-2} + \dots + I_1$$

is called the *separant* of A. The initial and the separant of a Δ-σ-polynomial A are denoted by I_A and S_A, respectively.

If $A, B \in K\{y_1, \dots, y_s\}$, then A is said to have lower rank than B (we write $\mathrm{rk}\, A < \mathrm{rk}\, B$) if either $A \in K$, $B \notin K$, or

$$\big(v_A, \deg_{v_A} A, \mathrm{ord}_1 u_A^{(1)}, \dots, \mathrm{ord}_p u_A^{(p)}\big)$$

is less than

$$\big(v_B, \deg_{v_B} B, \mathrm{ord}_1 u_B^{(1)}, \dots, \mathrm{ord}_p u_B^{(p)}\big)$$

with respect to the lexicographic order (v_A and v_B are compared with respect to $<_\sigma$). If the vectors are equal (or $A, B \in K$) we say that A and B are of the same rank and write $\mathrm{rk}\, A = \mathrm{rk}\, B$.

Definition 5.1. *If $A, B \in K\{y_1, \dots, y_s\}$, then B is said to be reduced with respect to A if*

(i) *B does not contain terms λv_A such that $\lambda \sim v_A$, $\lambda_\Delta \neq 1$, and*

$$\mathrm{ord}_i(\lambda u_A^{(i)}) \le \mathrm{ord}_i u_B^{(i)}, \quad i = 1, \dots, p.$$

(ii) *If B contains a term λv_A, where $\lambda \sim v_A$ and $\lambda_\Delta = 1$, then either there exists j, $1 \le j \le p$, such that*

$$\mathrm{ord}_j u_B^{(j)} < \mathrm{ord}_j \big(\lambda u_A^{(j)}\big) \quad \textit{or} \quad \mathrm{ord}_j \big(\lambda u_A^{(j)}\big) \le \mathrm{ord}_j u_B^{(j)}$$

for all $j = 1, \dots, p$ and $\deg_{\lambda v_A} B < \deg_{v_A} A$.

If $B \in K\{y_1, \dots, y_s\}$, then B is said to be *reduced with respect to a set $\Sigma \subseteq K\{y_1, \dots, y_s\}$* if B is reduced with respect to every element of Σ.

A set $\Sigma \subseteq K\{y_1, \dots, y_s\}$ is called *autoreduced* if $\Sigma \cap K = \emptyset$ and every element of Σ is reduced with respect to any other element of this set.

The proof of the following lemma can be found in [5, Chapter 0, Section 17].

Lemma 5.2. *Let A be any infinite subset of $\mathbf{N}^m \times \mathbf{N}_n$ ($n \ge 1$). Then there exists an infinite sequence of elements of A, strictly increasing relative to the product order, in which every element has the same projection on $\mathbf{N}_n$.*

As a consequence, we obtain the following statement.

Lemma 5.3. *Let S be any infinite set of terms λy_j ($\lambda \in \Lambda, 1 \le j \le s$) in $K\{y_1, \dots, y_s\}$. Then there exists an index j and an infinite sequence of terms $\lambda_1 y_j, \lambda_2 y_j, \dots, \lambda_k y_j, \dots$ such that $\lambda_k | \lambda_{k+1}$ for every $k = 1, 2, \dots$.*

Proposition 5.4. *Every autoreduced set is finite.*

Proof. Suppose that Σ is an infinite autoreduced subset of $K\{y_1, \dots, y_s\}$. Then Σ must contain an infinite set Σ' whose Δ-σ-polynomials have different σ-leaders similar to each other. Indeed, if it is not so, then Σ contains an infinite set Σ_1 whose Δ-σ-polynomials have the same σ-leader v. By Lemma 5.2, the infinite set

$$\big\{ \big(\mathrm{ord}_1 u_A^{(1)}, \dots, \mathrm{ord}_p u_A^{(p)} \big) \mid A \in \Sigma_1 \big\}$$

contains a nondecreasing infinite sequence

$$(\operatorname{ord}_1 u_{A_1}^{(1)}, \dots, \operatorname{ord}_p u_{A_1}^{(p)}) \leq_P (\operatorname{ord}_1 u_{A_2}^{(1)}, \dots, \operatorname{ord}_p u_{A_2}^{(p)}) \leq_P \dots$$

($A_1, A_2, \dots \in \Sigma_1$ and $\leq_P$ denotes the product order on $\mathbf{N}^p$). Since the sequence

$$\{\deg_{v_{A_i}} A_i | i = 1, 2, \dots\}$$

cannot be strictly decreasing, there are two indices i and j such that $i < j$ and $\deg_{v_{A_i}} A_i \leq \deg_{v_{A_j}} A_j$. We see that A_j is not reduced with respect to A_i that contradicts the fact that Σ is an autoreduced set.

Thus, we can assume that all Δ-σ-polynomials in Σ have distinct σ-leaders similar to each other. Then (see Lemma 5.3) there exists an infinite sequence $B_1, B_2, \dots$ of elements of Σ such that $v_{B_i} | v_{B_{i+1}}$ and $\left(\frac{v_{B_{i+1}}}{v_{B_i}}\right)_\Delta \neq 1$ ($i = 1, 2, \dots$). Let

$$k_{ij} = \operatorname{ord}_j v_{B_i} \quad \text{and} \quad l_{ij} = \operatorname{ord}_j u_{B_i}^{(j)}, \quad 1 \leq j \leq p.$$

Then $l_{ij} \geq k_{ij}$, so that

$$\{(l_{i1} - k_{i1}, \dots, l_{ip} - k_{ip}) | i = 1, 2, \dots\} \subseteq \mathbf{N}^p.$$

By Lemma 5.2, there exists an infinite sequence of indices $i_1 < i_2 < \dots$ such that $(l_{i_1 1} - k_{i_1 1}, \dots, l_{i_1 p} - k_{i_1 p}) \leq_P (l_{i_2 1} - k_{i_2 1}, \dots, l_{i_2 p} - k_{i_2 p}) \leq_P \dots$. Then for any $j = 1, \dots, p$, we have

$$\operatorname{ord}_j \left(\frac{v_{B_{i_2}}}{v_{B_{i_1}}} u_{B_{i_1}}^{(j)} \right) = k_{i_2 j} - k_{i_1 j} + l_{i_1 j} \leq l_{i_2 j} = \operatorname{ord}_j u_{B_{i_2}}^{(j)},$$

so that B_{i_2} contains a term $\lambda v_{B_{i_1}} = v_{B_{i_2}}$ with $\lambda_\Delta \neq 1$ and

$$\operatorname{ord}_j \left(\lambda u_{B_{i_1}}^{(j)}\right) \leq \operatorname{ord}_j u_{B_{i_2}}^{(j)}, \quad j = 1, \dots, p.$$

Thus, B_{i_2} is reduced with respect to B_{i_1} that contradicts the fact that Σ is an autoreduced set. □

Throughout the rest of the paper, while considering and autoreduced set $\Sigma = \{A_1, \dots, A_d\}$ in $K\{y_1, \dots, y_s\}$, we always assume that $\operatorname{rk} A_1 < \dots < \operatorname{rk} A_d$.

The proof of the following statement is similar to the proof of Theorem 3.5.27 in [6].

PROPOSITION 5.5. *Let $\Sigma = \{A_1, \dots, A_d\}$ be an autoreduced set in $K\{y_1, \dots, y_s\}$ and let I_k and S_k denote the initial and separant of A_k, respectively. Let $I(\Sigma) = \{X \in K\{y_1, \dots, y_s\} \,|\, X = 1$ or X is a product of finitely many elements of the form $\gamma(I_k)$ and $\gamma'(S_k)$ where $\gamma, \gamma' \in \Lambda_\sigma\}$. Then for any Δ-σ-polynomial B, there exist $B_0 \in K\{y_1, \dots, y_s\}$ and $J \in I(\Sigma)$ such that B_0 is reduced with respect to Σ and $JB \equiv B_0 \mod [\Sigma]$ (that is, $JB - B_0 \in [\Sigma]$).*

With the notation of the last proposition, we say that the Δ-σ-polynomial B *reduces to* B_0 modulo Σ.

DEFINITION 5.6. *Let $\Sigma = \{A_1, \dots, A_d\}$ and $\Sigma' = \{B_1, \dots, B_e\}$ be two autoreduced sets in $K\{y_1, \dots, y_s\}$. Then Σ is said to have lower rank than Σ' if one of the following two cases holds:*

(i) *There exists $k \in \mathbf{N}$ such that $k \leq \min\{d, e\}$, $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, k-1$ and $\operatorname{rk} A_k < \operatorname{rk} B_k$.*

(ii) *$d > e$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, e$.*

If $d = e$ and $\operatorname{rk} A_i = \operatorname{rk} B_i$ for $i = 1, \dots, d$, then Σ is said to have the same rank as Σ'.

As in [5, Chapter I, Section 9], we obtain the following proposition.

PROPOSITION 5.7. *In every nonempty family of autoreduced sets of difference-differential polynomials there exists an autoreduced set of lowest rank.*

Let J be any ideal of $K\{y_1, \dots, y_s\}$. Since the set of all autoreduced subsets of J is not empty (if $A \in J$, then $\{A\}$ is an autoreduced subset of J), the last statement shows that J contains an autoreduced subset of lowest rank. Such an autoreduced set is called a *characteristic set* of the ideal J. The following statement can be obtained by mimicking the proof of Lemma 8 in [5, Chapter I].

PROPOSITION 5.8. *Let $\Sigma = \{A_1, \dots, A_d\}$ be a characteristic set of a Δ-σ-ideal J of the ring $R = K\{y_1, \dots, y_s\}$. Then an element $B \in R$ is reduced with respect to the set Σ if and only if $B = 0$.*

Since for any $A \in K\{y_1, \dots, y_s\}$ and $\gamma \in \Lambda_\sigma$, $\operatorname{ord}_i(\gamma A) = \operatorname{ord}_i A$ for $i = 1, \dots, p$, one can introduce the concept of a coherent autoreduced set of a linear Δ-σ-ideal of $K\{y_1, \dots, y_s\}$ (that is, a Δ-σ-ideal generated by a finite set of linear Δ-σ-polynomials) in the same way as it is defined in the case of difference polynomials (see [6, Section 6.5]): an autoreduced set $\Sigma = \{A_1, \dots, A_d\} \subseteq K\{y_1, \dots, y_s\}$ consisting of linear Δ-σ-polynomials is called *coherent* if it satisfies the following two conditions:

(i) λA_i reduces to zero modulo Σ for any $\lambda \in \Lambda$, $1 \leq i \leq d$.

(ii) If $v_{A_i} \sim v_{A_j}$ and $w = \lambda v_{A_i} = \lambda' v_{A_j}$, where $\lambda \sim \lambda' \sim v_{A_i} \sim v_{A_j}$, then the Δ-σ-polynomial

$$(\lambda' I_{A_j})(\lambda A_i) - (\lambda I_{A_i})(\lambda' A_j)$$

reduces to zero modulo Σ.

The following two propositions can be proved precisely in the same way as the corresponding statements for difference polynomials, see [6, Theorem 6.5.3 and Corollary 6.5.4]).

PROPOSITION 5.9. *Any characteristic set of a linear Δ-σ-ideal of $K\{y_1, \dots, y_s\}$ is a coherent autoreduced set. Conversely, if Σ is a coherent autoreduced set in $K\{y_1, \dots, y_s\}$ consisting of linear Δ-σ-polynomials, then Σ is a characteristic set of the linear Δ-σ-ideal $[\Sigma]$.*

PROPOSITION 5.10. *Let us consider a partial order $\preccurlyeq$ on $K\{y_1, \dots, y_s\}$ such that $A \preccurlyeq B$ if and only if $v_A | v_B$. Let A be a linear Δ-σ-polynomial in $K\{y_1, \dots, y_s\}$, $A \notin K$. Then the set of all minimal with respect to $\preccurlyeq$ elements of the set $\{\lambda A \,|\, \lambda \in \Lambda\}$ is a characteristic set of the Δ-σ-ideal $[A]$.*

Now we are ready to prove Theorem 3.1.

PROOF. Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-σ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a natural Δ-σ-homomorphism Υ_η of the ring of Δ-σ-polynomials $K\{y_1, \dots, y_s\}$ onto $K\{\eta_1, \dots, \eta_s\}$ such that $\Upsilon_\eta(a) = a$ for any $a \in K$ and $\Upsilon_\eta(y_j) = \eta_j$ for $j = 1, \dots, s$. (If $A \in K\{y_1, \dots, y_s\}$, then $\Upsilon_\eta(A)$ is called the *value* of A at η; it is denoted by $A(\eta)$.) Obviously, the kernel P of the Δ-σ-homomorphism Υ_η is a prime Δ-σ-ideal of $K\{y_1, \dots, y_s\}$. This ideal is called the *defining* ideal of η over K or the defining ideal of the extension $L = K\langle \eta_1, \dots, \eta_s \rangle$.

It is easy to see that the quotient Δ-σ-field of the factor ring $\bar{R} = K\{y_1, \dots, y_s\}/P$ is naturally Δ-σ-isomorphic to the field L. The corresponding isomorphism is identity on K and maps the images of the Δ-σ-indeterminates $y_1, \dots, y_s$ in the factor ring $\bar{R}$ to the elements $\eta_1, \dots, \eta_s$, respectively.

Let $\Sigma = \{A_1, \dots, A_d\}$ be a characteristic set of P and for any $r_1, \dots, r_{p+1} \in \mathbf{N}$, let

$U_{r_1 \dots r_{p+1}} = \{u \in \Lambda Y \mid \mathrm{ord}_i\, u \le r_i$ for $i = 1, \dots, p$, $\mathrm{ord}_\sigma u \le r_{p+1}$, and either u is not a multiple of any v_{A_i} or for every $\lambda \in \Lambda, A \in \Sigma$ such that $u = \lambda v_A$ and $\lambda \sim v_A$, there exists $j \in \{1, \dots, p\}$ such that $\mathrm{ord}_j(\lambda u_A^{(j)}) > r_j\}$.

Applying the arguments of the proof of Theorem 6 in [5, Chapter II], we obtain that the set

$$\bar{U}_{r_1 \dots r_{p+1}} = \{u(\eta) \mid u \in U_{r_1 \dots r_{p+1}}\}$$

is a transcendence basis of $K\big(\bigcup_{j=1}^{n} \Lambda(r_1, \dots, r_{p+1})\eta_j\big)$ over K.

Let $U^{(1)}_{r_1 \dots r_{p+1}} = \{u \in \Lambda Y \mid \mathrm{ord}_i\, u \le r_i$ for $i = 1, \dots, p$, $\mathrm{ord}_\sigma u \le r_{p+1}$, and u is not a multiple of any v_{A_j}, $j = 1, \dots, d\}$ and let

$U^{(2)}_{r_1 \dots r_{p+1}} = \{u \in \Lambda Y \mid \mathrm{ord}_i\, u \le r_i$, $\mathrm{ord}_\sigma u \le r_{p+1}$ for $i = 1, \dots, p$ and there exists at least one pair i, j $(1 \le i \le p, 1 \le j \le d)$ such that $u = \lambda v_{A_j}$, $\lambda \sim v_{A_j}$, and $\mathrm{ord}_i(\lambda u^{(i)}_{A_j}) > r_i\}$. Clearly,

$$U_{r_1 \dots r_{p+1}} = U^{(1)}_{r_1 \dots r_{p+1}} \cup U^{(2)}_{r_1 \dots r_{p+1}}$$

and

$$U^{(1)}_{r_1 \dots r_{p+1}} \cap U^{(2)}_{r_1 \dots r_{p+1}} = \emptyset.$$

By Theorem 4.6, there exists a numerical polynomial $\phi \in \mathbf{Q}[t_1, \dots, t_{p+1}]$ such that

$$\phi(r_1, \dots, r_{p+1}) = \mathrm{Card}\, U^{(1)}_{r_1 \dots r_{p+1}}$$

for all sufficiently large $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$, $\deg_{t_i} \phi \le m_i$ $(1 \le i \le p)$, and $\deg_{t_{p+1}} \phi \le n$. Repeating the arguments of the proof of Theorem 4.1 of [9], we obtain that there is a linear combination $\psi(t_1, \dots, t_{p+1})$ of polynomials of the form (6) such that

$$\psi(r_1, \dots, r_{p+1}) = \mathrm{Card}\, U^{(2)}_{r_1 \dots r_{p+1}}$$

for all sufficiently large $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$. Then the polynomial

$$\Phi_\eta(t_1, \dots, t_{p+1}) = \phi(t_1, \dots, t_{p+1}) + \psi(t_1, \dots, t_{p+1})$$

satisfies conditions (i) and (ii) of Theorem 3.1.

In order to prove the last part of the theorem, suppose that $\zeta = \{\zeta_1, \dots, \zeta_q\}$ is another system of Δ-σ-generators of L/K, that is, $L = K\langle \eta_1, \dots, \eta_s \rangle = K\langle \zeta_1, \dots, \zeta_q \rangle$. Let

$$\Phi_\zeta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n} b_{i_1 \dots i_{p+1}} \binom{t_1 + i_1}{i_1} \dots \binom{t_{p+1} + i_{p+1}}{i_{p+1}}$$

be the dimension polynomial of L/K associated with the system of generators ζ. Then there exist positive integers $h_1, \dots, h_{p+1}$ such that

$$\eta_i \in K\left(\bigcup_{j=1}^{q} \Lambda(h_1, \dots, h_{p+1})\zeta_j\right)$$

and

$$\zeta_k \in K\left(\bigcup_{j=1}^{s} \Lambda(h_1, \dots, h_{p+1})\eta_j\right)$$

for any $i = 1, \dots, s$ and $k = 1, \dots, q$, whence

$$\Phi_\eta(r_1, \dots, r_{p+1}) \le \Phi_\zeta(r_1 + h_1, \dots, r_{p+1} + h_{p+1})$$

and

$$\Phi_\zeta(r_1, \dots, r_{p+1}) \le \Phi_\eta(r_1 + h_1, \dots, r_{p+1} + h_{p+1})$$

for all sufficiently large $(r_1, \dots, r_{p+1}) \in \mathbf{N}^{p+1}$. Now the statement of the third part of Theorem 3.1 follows from the fact that for any element $(k_1, \dots, k_{p+1}) \in E'_\eta$, the term $\binom{t_1+k_1}{k_1} \dots \binom{t_{p+1}+k_{p+1}}{k_{p+1}}$ appears in Φ_η and Φ_ζ with the same coefficient $a_{k_1 \dots k_{p+1}}$. The equality of the coefficients of the corresponding terms of total degree $d = \deg \Phi_\eta$ in Φ_η and Φ_ζ can be shown as in the proof of [10, Theorem 3.3.21]. $\square$

The result of Theorem 3.1 can be generalized to the case when both sets of basic operators Δ and σ are represented as unions of disjoint subsets. The proof is, however, essentially longer; the author will present the generalized version of Theorem 3.1 in his forthcoming paper.

Example 5.11. Let us find the Δ-σ-dimension polynomial that expresses the strength of the difference-differential equation

$$\frac{\partial^2 y(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2 y(x_1, x_2)}{\partial x_2^2} + y(x_1 + h, x_2) + a(x_1, x_2) = 0 \quad (7)$$

over some Δ-σ-field of functions of two real variables K, where the basic set of derivations $\Delta = \{\delta_1 = \frac{\partial}{\partial x_1}, \delta_2 = \frac{\partial}{\partial x_2}\}$ has the partition $\Delta = \{\delta_1\} \cup \{\delta_1\}$ and σ consists of one automorphisms $\alpha : f(x_1, x_2) \mapsto f(x_1 + h, x_2)$ $(h \in \mathbf{R})$.

In this case, the associated Δ-σ-extension $K\langle \eta \rangle / K$ is Δ-σ-isomorphic to the field of fractions of

$$K\{y\}/[\alpha y + \delta_1^2 y + \delta_2^2 y + a]$$

($a \in K$ corresponds to the function $a(x_1, x_2)$). Applying Proposition 5.10, we obtain that the characteristic set of the defining ideal of $K\langle \eta \rangle / K$ consists of the Δ-σ-polynomials

$$g_1 = \alpha y + \delta_1^2 y + \delta_2^2 y + a$$

and

$$g_2 = \alpha^{-1} g_1 = \alpha^{-1} \delta_1^2 y + \alpha^{-1} \delta_2^2 y + y + \alpha^{-1}(a).$$

With the notation of the proof of Theorem 3.1, the application of the procedure described in this proof, Theorem 4.6(iii), and formula (5) leads to the following expressions for the numbers of elements of the sets $U^{(1)}_{r_1 r_2 r_3}$ and $U^{(2)}_{r_1 r_2 r_3}$:

$$\mathrm{Card}\, U^{(1)}_{r_1 r_2 r_3} = r_1 r_2 + 2 r_2 r_3 + r_1 + r_2 + 2 r_3 + 1$$

and

$$\mathrm{Card}\, U^{(2)}_{r_1 r_2 r_3} = 4 r_1 r_3 + 2 r_2 r_3 - 2 r_3$$

for all sufficiently large $(r_1, r_2, r_3) \in \mathbf{N}^3$. Thus, the strength of equation (7) corresponding to the given partition of the basic set of derivations is expressed by the Δ-σ-polynomial

$$\Phi_\eta(t_1, t_2, t_3) = t_1 t_2 + 4 t_1 t_3 + 4 t_2 t_3 + t_1 + t_2 + 1.$$

Example 5.12. Let K be a Δ-σ-field where the basic set of derivations $\Delta = \{\delta_1, \delta_2\}$ is considered together with its partition $\Delta = \{\delta_1\} \cup \{\delta_2\}$ and $\sigma = \{\alpha\}$ for some automorphism α of K. Let $L = K\langle\eta\rangle$ be a Δ-σ-field extension with the defining equation

$$\delta_1^a\delta_2^b\alpha^c\eta + \delta_1^a\delta_2^b\alpha^{-c}\eta + \delta_1^a\delta_2^{b+c}\eta + \delta_1^{a+c}\delta_2^b\eta = 0 \qquad (8)$$

where a, b, and c are positive integers. Let $\Phi_\eta(t_1, t_2, t_3)$ denote the corresponding difference-differential dimension polynomial (which expresses the strength of equation (8) with respect to the given partition of Δ). In order to compute Φ_η, notice, first , that the defining Δ-σ-ideal P of the extension L/K is the linear Δ-σ-ideal of $K\{y\}$ generated by the Δ-σ-polynomial

$$f = \delta_1^a\delta_2^b\alpha^c y + \delta_1^a\delta_2^b\alpha^{-c}y + \delta_1^a\delta_2^{b+c}y + \delta_1^{a+c}\delta_2^b y.$$

By Proposition 5.10, the characteristic set of P consists of f and $\alpha^{-1}f = \alpha^{-(c+1)}\delta_1^a\delta_2^b y + \delta_1^a\delta_2^b\alpha^{c-1}y + \delta_1^a\delta_2^{b+c}\alpha^{-1}y + \delta_1^{a+c}\delta_2^b\alpha^{-1}y$. The procedure described in the proof of Theorem 3.1 shows that

$$\operatorname{Card} U^{(1)}_{r_1r_2r_3} = \phi_A(r_1, r_2, r_3)$$

for all sufficiently large $(r_1, r_2, r_3) \in \mathbf{N}^3$, where ϕ_A is the dimension polynomial of the set

$$A = \{(a, b, c), (a, b, -(c + 1))\} \subseteq \mathbf{N}^2 \times \mathbf{Z}.$$

Applying Theorem 4.6(iii), and formula (5) we obtain that $\phi_A(t_1, t_2, t_3) = 2ct_1t_2 + 2bt_1t_3 + 2at_2t_3 + (b + 2c - 2bc)t_1 + (a + 2c - 2ac)t_2 + (2a + 2b - 2ab)t_3 + a + b + 2c - ab - 2ac - 2bc + 2abc$. The computation of $\operatorname{Card} U^{(2)}_{r_1r_2r_3}$ with the use of the method of inclusion and exclusion described in the proof of Theorem 3.1 yields the following:

$$\operatorname{Card} U^{(2)}_{r_1r_2r_3} = (2r_3 - 2c + 1)[c(r_2 - b + 1) + c(r_1 - a + 1) - c^2]$$

for all sufficiently large $(r_1, r_2, r_3) \in \mathbf{N}^3$. Therefore, the Δ-σ-dimension polynomial that expresses the strength of equation (8), is as follows.

$$\Phi_\eta = 2ct_1t_2 + 2(b+c)t_1t_3 + 2(a+c)t_2t_3 + (b+3c-2bc-c^2)t_1$$

$$+(2a+2b+4c-2ab-2ac-2bc-2c^2)t_3 + a+b+4c-ab-3ac-3bc$$

$$+(a + 3c - 2ac - 2c^2)t_2 + +2abc + 2ac^2 + 2bc^2 + 2c^3 - 5c^2.$$

The computation of the univariate Δ-σ-dimension polynomial (see Theorem 2.1) via the method of Kähler differentials described in [6, Section 6.5] (by mimicking Example 6.5.6 of [6]) leads to the following result:

$$\phi_{\eta|K}(t) = \frac{D}{2}t^2 - \frac{D(D-2)}{2}t + \frac{D(D-1)(D-2)}{6} \qquad (9)$$

where $D = a + b + c$. In this case the polynomial $\phi_{\eta|K}(t)$ carries just one invariant $a+b+c$ of the extension L/K while $\Phi_\eta(t_1, t_2, t_3)$ determines three such invariants: c, $b + c$, and $a + c$ (see Theorem 3.1(iii)), that is, Φ_η determines all three parameters a, b, c of the defining equation while $\phi_\eta(t)$ gives just the sum of these parameters.

The extension $K\langle\zeta\rangle/K$ with a Δ-σ-generator ζ, the same basic set $\Delta \cup \sigma$ ($\Delta = \{\delta_1, \delta_2\}$, $\sigma = \{\alpha\}$), the same partition of Δ and defining equation

$$\delta_1^{a+b}\alpha^c\zeta + \delta_2^{a+b}\alpha^{-c}\zeta = 0 \qquad (10)$$

has the same univariate difference-dimension polynomial (9). However, its Δ-σ-dimension polynomial is not only different, but also has different invariants described in part (iii) of Theorem 3.1:

$$\Phi_\zeta = 2ct_1t_2 + 2(a+b)t_1t_3 + 2(a+b)t_2t_3 + At_1 + Bt_2 + Ct_3 + E$$

where $A = B = (a+b)(1-2c) + 2c$, $C = 2[1 - (a+b-1)^2]$, and $E = 1 + 2c(a + b - 1)^2$.

Two systems of algebraic difference-differential (Δ-σ-) equations with coefficients in a Δ-σ-field K are said to be *equivalent* if there is a Δ-σ-isomorphism between the Δ-σ-field extensions of K with these defining equations, which is identity on K. Our example shows that using a partition of the basic set of derivations and the computation of the corresponding multivariate Δ-σ-dimension polynomials, one can determine that two systems of Δ-σ-equations (see systems (8) and (10)) are not equivalent, even though they have the same univariate difference-dimension polynomial.

6. ACKNOWLEDGMENTS

This research was supported by the NSF Grant CCF-1016608

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