

Integration of Unspecified Functions and Families of Iterated Integrals

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ABSTRACT

An algorithm for parametric elementary integration over differential fields constructed by a differentially transcendental extension is given. It extends current versions of Risch's algorithm to this setting and is based on some first ideas of Graham H. Campbell transferring his method to more formal grounds and making it parametric, which allows to compute relations among definite integrals. Apart from differentially transcendental functions, such as the gamma function or the zeta function, also unspecified functions and certain families of iterated integrals such as the polylogarithms can be modeled in such differential fields.

Categories and Subject Descriptors

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Symbolic Integration; Parametric Elementary Integration; Differentially Transcendental Functions; Differential Fields

1. INTRODUCTION

Among the many approaches to compute integrals in closed form we take the differential algebra approach using differential fields. The first complete algorithm for finding elementary integrals of a class of transcendental functions was published by Risch [10] and deals with integration of elementary functions. Later, this was generalized to other types of integrands, see [2, 9] and references therein. Given an element of a suitable differential field these algorithms look for an antiderivative in elementary extensions of that field. Often such algorithms consider an integrand depending linearly on parameters as in the following problem.

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PROBLEM 1 (PARAMETRIC ELEMENTARY INTEGRATION).
Given a differential field (F, D) and $f_0, \dots, f_m \in F$, compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$, where $C := \text{Const}(F)$, and corresponding g_1, \dots, g_n from some elementary extension of (F, D) such that

$$Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$$

for all j and $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ is a basis of the C -vector space of all $\mathbf{c} \in C^{m+1}$ for which $(f_0, \dots, f_m) \cdot \mathbf{c}$ has an elementary integral over (F, D) .

Parametric integration is important for definite integration as on an interval (a, b) a relation of the form

$$c_0 f_0(x) + \dots + c_m f_m(x) = g'(x),$$

where the c_i do not depend on x , yields a linear relation for the corresponding definite integrals

$$c_0 \int_a^b f_0(x) dx + \dots + c_m \int_a^b f_m(x) dx = g(b) - g(a).$$

For more on definite integration based on this principle see e.g. [1, 9] and references therein. Note that above formulation of the problem requires finding *all* choices for the parameters giving rise to an elementary integral.

Typically the field (F, D) is generated by functions satisfying first-order differential equations of certain forms, mostly linear but not exclusively. In strong contrast to this, in this paper we consider differential fields $(F, D) = (K\langle t \rangle, D)$ generated from some underlying differential field (K, D) by adjoining a generator t such that all derivatives t, Dt, D^2t, \dots are algebraically independent over K , i.e., t is *differentially transcendental* over (K, D) . As simple example we could choose $(K, D) = (\mathbb{Q}(x), \frac{d}{dx})$, then t differentially transcendental over (K, D) can represent, for example, the gamma function $\Gamma(x)$, the digamma function $\psi(x)$, or the Riemann zeta function $\zeta(x)$ as those functions do not satisfy any algebraic differential equation with rational function coefficients. For other functions with this property see [8] for example. A differentially transcendental t may also be regarded as differential indeterminate or unspecified function, which is what was informally considered by Campbell [3] for a single integrand, i.e. in the non-parametric case. We also note that certain families of iterated integrals, such as the polylogarithms $\text{Li}_2(x), \text{Li}_3(x), \dots$, are such that the generic element of the family is differentially transcendental. Hence an integral like

$$\int \frac{\text{Li}_{n-2}(x)\text{Li}_n(x)}{x\text{Li}_{n-1}(x)^2} dx = \ln(x) - \frac{\text{Li}_n(x)}{\text{Li}_{n-1}(x)}$$

with symbolic n can be obtained in the same framework, for example.

We carefully analyze the problem in the language of differential fields which leads to a new subproblem, see Problem 11, that we solve. Our algorithm, summarized as Algorithm 3, solves the parametric elementary integration problem over differential fields $(K\langle t \rangle, D)$, where t is differentially transcendental over (K, D) . Note that an algorithm for this type of field extensions can also be used to at least heuristically treat any given function to which no other algorithm applies, even if the function satisfies some algebraic differential equation with coefficients from K , it just may not find all solutions in that case.

An analogous problem for summation of unspecified sequences was treated by Kauers and Schneider [5, 6].

In Section 2 we recall the definitions used in this paper, give some properties of the notions used, and introduce a flexible way of representing elements of the fields dealt with. Then, in Section 3 we prove a refined version of Liouville's theorem, which will be crucial for the algorithm in Section 4. Section 5 deals with representing families of iterated integrals and a simple sufficient condition is given for algebraic independence of their members. Finally, Section 6 presents some examples of various applications of the algorithm.

Parts of the material presented in this paper were already included in the author's PhD thesis [9]. All fields are implicitly understood to be of characteristic 0.

2. DEFINITIONS AND BASIC PROPERTIES

We will write $\text{coeff}(p, x_1^{i_1} \cdots x_n^{i_n})$, explicitly keeping factors of the form x_j^0 in case $i_j = 0$, for the coefficient of $x_1^{i_1} \cdots x_n^{i_n}$ of p as a polynomial in x_1, \dots, x_n . Let a and b be univariate polynomials, then we write $a \div b$ and $a \bmod b$ for their quotient and remainder (the variable will be clear from the context). We sometimes write linear combinations as products of vectors $(f_0, \dots, f_m) \cdot \mathbf{c} = \sum_{i=0}^m c_i f_i$. By $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ we denote a matrix given by its column vectors. When extending a differential field (K, D) by adjoining new elements x_1, \dots, x_n we need to distinguish between the field $K(x_1, \dots, x_n)$ they generate and the differential field $K\langle x_1, \dots, x_n \rangle = K(x_1, \dots, x_n, Dx_1, \dots, Dx_n, D^2x_1, \dots)$ generated by those elements.

2.1 Elementary Extensions

We briefly recall the precise definition of the type of integrals we want to compute.

DEFINITION 2. *Let (F, D) be a differential field and let $(E, D) = (F(s_1, \dots, s_n), D)$ be a differential field extension. Then (E, D) is called an elementary extension of (F, D) , if each s_i is elementary over $(E_{i-1}, D) := (F(s_1, \dots, s_{i-1}), D)$, or more explicitly if each s_i*

1. *is a logarithm over (E_{i-1}, D) , i.e. there exists $a \in E_{i-1}$ such that $Ds_i = \frac{Da}{a}$, or*
2. *is an exponential over (E_{i-1}, D) , i.e. there exists $a \in E_{i-1}$ such that $\frac{Ds_i}{s_i} = Da$, or*
3. *is algebraic over E_{i-1} .*

DEFINITION 3. *Let (F, D) be a differential field and $f \in F$. Then we say that f has an elementary integral over (F, D) if there exist an elementary extension (E, D) of (F, D) and $g \in E$ such that $Dg = f$.*

2.2 Differentially Transcendental Extensions

DEFINITION 4. *Let (F, D) be a differential field, (K, D) a differential subfield, and $t \in F$. If all derivatives t, Dt, D^2t, \dots are algebraically independent over K , then t is differentially transcendental over (K, D) .*

We will, however, consider a more general way of representing the elements of the field $K\langle t \rangle = K(t, Dt, D^2t, \dots)$ by choosing $t_0, t_1, \dots \in K\langle t \rangle$ such that

$$t_0 = t \tag{1}$$

$$Dt_n = a_n t_{n+1} + b_n \tag{2}$$

with $a_n, b_n \in K(t_0, \dots, t_n)$ for $n \in \mathbb{N}$. Then, the following lemma shows that automatically $a_n \neq 0$ and the equality $K(t, Dt, \dots, D^n t) = K(t_0, \dots, t_n)$ are satisfied for all $n \in \mathbb{N}$. So we will mainly consider $K\langle t \rangle$ as $K(t_0, t_1, \dots)$. For instance, the flexibility in the representation introduced by (2) allows to represent the polylogarithms $\text{Li}_m(x)$ for symbolic m in a convenient way. If $(K, D) = (C(x), \frac{d}{dx})$ and $a_n = \frac{1}{x}$ and $b_n = 0$ in (2), then t_n corresponds to $\text{Li}_{m-n}(x)$ and functions involving $\text{Li}_m(x), \text{Li}_{m-1}(x), \dots$ are directly represented in terms of these functions instead of $\text{Li}_m(x), \text{Li}'_m(x), \dots$, see also Section 5.

LEMMA 5. *Let t be differentially transcendental over (K, D) and let $t_0, t_1, \dots \in K\langle t \rangle$ such that (1) and (2). Then*

1. *$a_n \neq 0$ for all $n \in \mathbb{N}$ and*
2. *for all $n \in \mathbb{N}$ there are $\tilde{a}_n, \tilde{b}_n \in K(t, Dt, \dots, D^n t)$ such that $t_{n+1} = \tilde{a}_n D^{n+1} t + \tilde{b}_n$ and $\tilde{a}_n \neq 0$.*

PROOF. We prove both statements in parallel by induction, for which we artificially include the case $n = -1$. For $n = -1$ we define $a_{-1} := 1$, $\tilde{a}_{-1} := 1$, and $\tilde{b}_{-1} := 0$, then trivially $a_{-1}, \tilde{a}_{-1} \in K^*$ and $t_0 = \tilde{a}_{-1} t + \tilde{b}_{-1}$ by definition. For $n \in \mathbb{N}$ we assume that for all $i \in \{-1, 0, \dots, n-1\}$ there are $\tilde{a}_i, \tilde{b}_i \in K(t, Dt, \dots, D^i t)$ such that $t_{i+1} = \tilde{a}_i D^{i+1} t + \tilde{b}_i$ and $\tilde{a}_i \neq 0$. Then, from the assumptions above we obtain that $a_n t_{n+1} = Dt_n - b_n = D(\tilde{a}_{n-1} D^n t + \tilde{b}_{n-1}) - b_n = \tilde{a}_{n-1} D^{n+1} t + (D\tilde{a}_{n-1}) D^n t + D\tilde{b}_{n-1} - b_n$. If we had $a_n = 0$, then this and $\tilde{a}_{n-1} \neq 0$ would imply that $D^{n+1} t$ equals $\frac{(D\tilde{a}_{n-1}) D^n t + D\tilde{b}_{n-1} - b_n}{-\tilde{a}_{n-1}}$. The latter is in $K(t, \dots, D^n t)$ by induction hypothesis, which would be in contradiction to the algebraic independence of $t, \dots, D^{n+1} t$ over K . Hence, $a_n \neq 0$ and we set $\tilde{a}_n := \frac{\tilde{a}_{n-1}}{a_n}$ and $\tilde{b}_n := \frac{(D\tilde{a}_{n-1}) D^n t + D\tilde{b}_{n-1} - b_n}{a_n}$, which both are in $K(t, \dots, D^n t)$ by the induction hypothesis. \square

The second statement of the lemma above has some important immediate consequences, which we emphasize by stating the following corollary. The proof is trivial and so we omit it.

COROLLARY 6. *Let t be differentially transcendental over (K, D) and let $t_0, t_1, \dots \in K\langle t \rangle$ such that (1) and (2). Then*

1. *t_0, t_1, \dots are algebraically independent over K , and*
2. *$K(t, Dt, \dots, D^n t) = K(t_0, \dots, t_n)$ for all $n \in \mathbb{N}$.*

In the following we will formalize the ideas of Campbell [3] in this framework. In this context we define the

coefficient lifting $\kappa_D : K[t_0, t_1, \dots] \rightarrow K[t_0, t_1, \dots]$ of D by $\kappa_D(\sum_{\alpha} f_{\alpha} t^{\alpha}) := \sum_{\alpha} (Df_{\alpha}) t^{\alpha}$, where we use multiindex notation for brevity, and extend it to a derivation κ_D on $K(t_0, t_1, \dots)$ in the natural way by the quotient rule. It is easy to see that

$$Df = \kappa_D f + \sum_{k=0}^{\infty} \frac{\partial f}{\partial t_k} D t_k. \quad (3)$$

Note that the sum contains only finitely many nonzero summands since $\frac{\partial f}{\partial t_k} = 0$ from some point on. Generalizing the definition of κ_D above, for each $n \in \mathbb{N}$ we define the derivation $\kappa_{D,n}$ on $K(t_0, t_1, \dots)$ by

$$\kappa_{D,n} f := \kappa_D f + \sum_{k=0}^{n-1} \frac{\partial f}{\partial t_k} D t_k. \quad (4)$$

These derivations obviously obey $\kappa_{D,n} f + \frac{\partial f}{\partial t_n} D t_n = \kappa_{D,n+1} f$ for $f \in K(t_0, t_1, \dots)$ with $\kappa_{D,0} = \kappa_D$. For $f \in K(t_0, \dots, t_{n-1})$ we have in particular $\kappa_{D,n} f = Df$. An important measure on the elements of $K(t_0, t_1, \dots)$ is the highest index of any of the generators needed to represent the particular element of the field.

DEFINITION 7. Let t be differentially transcendental over (K, D) , then we define the differential degree of $f \in K\langle t \rangle$ by

$$\text{ddeg}_t(f) := \begin{cases} \min \{k \in \mathbb{N} \mid f \in K(t, \dots, D^k t)\} & \text{if } f \notin K \\ -\infty & \text{if } f \in K. \end{cases}$$

Note that for $t_0, t_1, \dots \in K\langle t \rangle$ with (1) and (2) the above corollary implies

$$\text{ddeg}_t(f) = \begin{cases} \min \{k \in \mathbb{N} \mid f \in K(t_0, \dots, t_k)\} & \text{if } f \notin K \\ -\infty & \text{if } f \in K. \end{cases}$$

So we can say that $\frac{\partial f}{\partial t_k} = 0$ for $k > \text{ddeg}_t(f)$ in (3). The differential degree obeys the following properties with respect to the operations of a differential field. For $f, g \in K(t_0, t_1, \dots)^*$ we have

$$\begin{aligned} \text{ddeg}_t(f + g) &\leq \max(\text{ddeg}_t(f), \text{ddeg}_t(g)), \\ \text{ddeg}_t(fg) &\leq \max(\text{ddeg}_t(f), \text{ddeg}_t(g)), \\ \text{ddeg}_t(1/f) &= \text{ddeg}_t(f), \\ \text{ddeg}_t(Df) &= \text{ddeg}_t(f) + 1 \end{aligned}$$

with equality in the first two relations if $\text{ddeg}_t(f) \neq \text{ddeg}_t(g)$. In particular, the last property implies that $\text{Const}_D(K\langle t \rangle) = \text{Const}_D(K)$. Furthermore, the following corollary highlights important properties implied by (2) and (3).

COROLLARY 8. Let t be differentially transcendental over (K, D) , let $t_0, t_1, \dots \in K\langle t \rangle$ such that (1) and (2), and let $F := K(t_0, t_1, \dots)$. Then $\text{Const}_D(F) = \text{Const}_D(K)$ and for all $f \in F$ and any $k \in \mathbb{N}$ with $k \geq \text{ddeg}_t(f)$ there exist $a, b \in K(t_0, \dots, t_k)$ with $a = a_k \frac{\partial f}{\partial t_k}$ and

$$Df = at_{k+1} + b.$$

3. LIOUVILLE'S THEOREM

Based on the properties stated in the previous section we are now ready to prove a refinement of Liouville's theorem for this situation. It was not made explicit in [3] which type

of expressions are dealt with exactly. Above we gave one interpretation in terms of differential fields and the following results present the corresponding precise details of the ideas from [3].

THEOREM 9. Let t be differentially transcendental over (K, D) and let $t_0, t_1, \dots \in K\langle t \rangle$ such that (1) and (2). Let $f \in F := K(t_0, t_1, \dots)$ such that f has an elementary integral over (F, D) and let $k := \text{ddeg}_t(f)$. Then there are $v \in K(t_0, \dots, t_{k-1})$, $c_1, \dots, c_n \in \overline{\text{Const}_D(K)}$, and $u_1, \dots, u_n \in K(c_1, \dots, c_n, t_0, \dots, t_{k-1})^*$ such that

$$f = Dv + \sum_{i=1}^n c_i \frac{D u_i}{u_i}. \quad (5)$$

If $k \geq 1$, we can also write this as

$$f = a_{k-1} \left(\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^n c_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i} \right) t_k + b$$

for some $b \in K(t_0, \dots, t_{k-1})$.

PROOF. By Liouville's theorem (e.g. [2, Thm 5.5.3]) we know that there are $v \in F$, $c_1, \dots, c_n \in \overline{\text{Const}_D(F)}$, and $u_1, \dots, u_n \in F(c_1, \dots, c_n)^*$ such that (5) and by Corollary 8 we deduce $c_1, \dots, c_n \in \overline{\text{Const}_D(K)}$. Define

$$m := \max(\text{ddeg}_t(v), \text{ddeg}_t(u_1), \dots, \text{ddeg}_t(u_n)).$$

If $m < 0$, then $f \in K$ and the statement is trivially fulfilled. So assume $m \geq 0$ now and assume without loss of generality that $v, c_1, \dots, c_n, u_1, \dots, u_n$ are chosen such that u_1, \dots, u_n are pairwise relatively prime polynomials from $K(c_1, \dots, c_n, t_0, \dots, t_{m-1})[t_m]$. Then, applying Corollary 8 to each summand in (5) implies that

$$f = a_m \left(\frac{\partial v}{\partial t_m} + \sum_{i=1}^n c_i \frac{\frac{\partial u_i}{\partial t_m}}{u_i} \right) t_{m+1} + b$$

for some $b \in K(c_1, \dots, c_n, t_0, \dots, t_m)$ and by comparing the differential degree of both sides we obtain $k \leq m + 1$. Since by Corollary 6 t_0, t_1, \dots are algebraically independent over K they are also algebraically independent over $K(c_1, \dots, c_n)$. So we even have $b \in K(t_0, \dots, t_m)$ by comparing the coefficient of t_{m+1}^0 . Next, Lemma 5 implies that $a_m \neq 0$. If we had $m > k - 1$, then by comparing the coefficient of t_{m+1} we could conclude $\tilde{f} := \frac{\partial v}{\partial t_m} + \sum_{i=1}^n c_i \frac{\frac{\partial u_i}{\partial t_m}}{u_i} = 0$.

From this we would obtain $\max(\text{ddeg}_t(u_1), \dots, \text{ddeg}_t(u_n)) < m$ by applying Lemma 5.6.1 from [2] in the differential field $(K(t_0, \dots, t_m), \frac{\partial}{\partial t_m})$ and noting that an irreducible polynomial $p \in K(c_1, \dots, c_n, t_0, \dots, t_{m-1})[t_m]$ can divide at most one of u_1, \dots, u_n as they are pairwise relatively prime. Therefore, the definitions of m and \tilde{f} would imply $\text{ddeg}_t(v) = m$ and $\tilde{f} = \frac{\partial v}{\partial t_m}$, respectively, which would give $\tilde{f} \neq 0$ altogether in contradiction to $\tilde{f} = 0$. Hence we have $m = k - 1$. \square

In particular, for the special case $k \leq 0$ this theorem contains analogs of Corollary 5.11.1 from [2] and Theorem 3.15 from [9], which we make explicit in the following corollary.

COROLLARY 10. Let t be differentially transcendental over (K, D) , let $t_0, t_1, \dots \in K\langle t \rangle$ with (1) and (2) and let $F := K(t_0, t_1, \dots)$. If $f \in K(t_0)$ has an elementary integral over (F, D) , then $f \in K$. If $f \in K$ has an elementary integral over (F, D) , then it has an elementary integral over (K, D) .

4. ALGORITHM

In the following we will assume that the differential field (K, D) is computable, i.e., that we can effectively compute the basic arithmetic operations as well as derivation and zero-testing. Furthermore, with $C := \text{Const}_D(K)$ we will need to compute C -vector space bases of the constant solutions $\{\mathbf{c} \in C^m \mid A \cdot \mathbf{c} = 0\}$ of linear systems with coefficients in K , i.e., $\ker(A) \cap C^m$ for $A \in K^{m \times n}$. This task can be reduced to the solution of linear systems with coefficients in C by an algorithm of Bronstein [2, Lemma 7.1.2], which computes $B \in C^{\tilde{m} \times n}$ such that $\ker(A) \cap C^m = \ker(B)$.

4.1 Computing the Logarithmic Part

Theorem 9 suggests that in order to compute elementary integrals over $(K(t_0, t_1, \dots), D)$ we should look at the following new subproblem, which was not part of [3]. The difficulty of this problem is related to the computation of the logarithmic part of the integral and will be taken care of by Algorithm 1. Based on this algorithm it is straightforward to solve the full subproblem, Algorithm 2 shows how this can be done.

PROBLEM 11. *Given: t differentially transcendental over (K, D) , $t_0, t_1, \dots \in K\langle t \rangle$ with (1) and (2), $k \in \mathbb{N}^+$, and $f_0, \dots, f_m \in K(t_0, \dots, t_{k-1})$.*

Find: a basis $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$, where $C := \text{Const}_D(K)$, of the C -vector space of all $\mathbf{c} \in C^{m+1}$ such that there exist $v \in K(t_0, \dots, t_{k-1})$, $d_1, \dots, d_l \in \overline{C}$, and $u_1, \dots, u_l \in K(d_1, \dots, d_l, t_0, \dots, t_{k-1})^$ with*

$$(f_0, \dots, f_m) \cdot \mathbf{c} = \frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^l d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i}$$

as well as corresponding $v_j \in K(t_0, \dots, t_{k-1})$, $d_{j,1}, \dots, d_{j,l_j} \in \overline{C}$, and $u_{j,1}, \dots, u_{j,l_j} \in K(d_{j,1}, \dots, d_{j,l_j}, t_0, \dots, t_{k-1})^$ for each $j \in \{1, \dots, n\}$.*

At first glance this problem may look like it was just parametric elementary integration over $(K(t_0, \dots, t_{k-1}), \frac{\partial}{\partial t_{k-1}})$ and we could solve it easily by well-known algorithms, but there is a subtle difference. Observe that the above formulation requires linear combinations with coefficients belonging to $C = \text{Const}_D(K)$ instead of $\text{Const}_{\frac{\partial}{\partial t_{k-1}}}(K(t_0, \dots, t_{k-1})) = K(t_0, \dots, t_{k-2})$. Furthermore, instead of allowing residues $d_i \in K(t_0, \dots, t_{k-2})$ they are restricted to $d_i \in \overline{\text{Const}_D(K)}$ above. Because of these requirements Problem 11 cannot be solved directly by standard algorithms. However, it can be solved algorithmically as follows by adapting the ideas of Theorem 3.9 from [9] in order to make use of both derivations D as well as $\frac{\partial}{\partial t_{k-1}}$. In the proof, we will use the following formula, implied by Lemma 3.2.2 from [2], where $p \in K(t_0, \dots, t_n)[z]$ and f is in some differential field extension of $(K(t_0, t_1, \dots), D)$.

$$D(p(f)) = \sum_{i=0}^{\deg_z(p)} (\kappa_{D,n+1} \text{coeff}(p, z^i)) f^i + \frac{\partial p}{\partial z}(f) Df \quad (6)$$

THEOREM 12. *Let t be differentially transcendental over (K, D) , let $t_0, t_1, \dots \in K\langle t \rangle$ such that (1) and (2), and let $C := \text{Const}_D(K)$. Let $k \in \mathbb{N}^+$, let $\tilde{K} := K(t_0, \dots, t_{k-2})$, let $a_0, \dots, a_m, b \in \tilde{K}[t_{k-1}]$ with $b \neq 0$ and $\gcd(b, \frac{\partial b}{\partial t_{k-1}}) = 1$,*

and let z be an indeterminate. Then by Algorithm 1 we can compute linear independent $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ such that:

1. *If $\mathbf{c} \in C^{m+1}$ is such that there exist $v \in \tilde{K}(t_{k-1})$, $d_1, \dots, d_l \in \overline{C}$, and $u_1, \dots, u_l \in \tilde{K}(d_1, \dots, d_l, t_{k-1})^*$ with*

$$\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^l d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i} = \frac{(a_0, \dots, a_m) \cdot \mathbf{c}}{b},$$

then $\mathbf{c} \in \text{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.

2. *For all $j \in \{1, \dots, n\}$ there exists $r \in C[z]$ such that*

$$\frac{(a_0, \dots, a_m) \cdot \mathbf{c}_j}{b} - \sum_{r(\alpha)=0} \alpha \frac{\frac{\partial g_\alpha}{\partial t_{k-1}}}{g_\alpha} \in \tilde{K}[t_{k-1}],$$

where for all $\alpha \in \overline{C}$ with $r(\alpha) = 0$ we define $g_\alpha := \gcd((a_0, \dots, a_m) \cdot \mathbf{c}_j - \alpha \frac{\partial b}{\partial t_{k-1}}, b) \in \tilde{K}[t_{k-1}]$.

Algorithm 1 Restrict residues as specified in Theorem 12

1. Let $\tilde{a}_0, \dots, \tilde{a}_m, \tilde{b} \in \tilde{K}[z]$ such that $\tilde{a}_i(t_{k-1}) = a_i$ and $\tilde{b}(t_{k-1}) = b$
2. Let $\tilde{b}_j := \text{coeff}(\tilde{b}, z^j)$ for all $j \in \{0, \dots, \deg_z(\tilde{b})\}$
3. For all $i \in \{0, \dots, m\}$ compute $p_i \in \tilde{K}[z]$ with $\deg_z(p_i) < \deg_z(\tilde{b})$ such that

$$\tilde{a}_i \equiv p_i \frac{\partial \tilde{b}}{\partial z} \pmod{\tilde{b}}$$

4. For all $i \in \{0, \dots, m\}$ and $l \in \{0, 1\}$ compute $\tilde{p}_{i,l} \in \tilde{K}[z]$ with $\deg_z(\tilde{p}_{i,l}) < \deg_z(\tilde{b})$ such that

$$\frac{\partial p_i}{\partial z} \cdot \sum_{j=0}^{\deg_z(\tilde{b})} \text{coeff}(\kappa_{D,k-1} \tilde{b}_j, t_{k-1}^l z^j) z^j \equiv \tilde{p}_{i,l} \frac{\partial \tilde{b}}{\partial z} \pmod{\tilde{b}}$$

5. For all $i \in \{0, \dots, m\}$ compute

$$q_i := \sum_{j=0}^{\deg_z(p_i)} (\kappa_{D,k-1} \text{coeff}(p_i, z^j)) z^j - (\tilde{p}_{i,1} t_{k-1} + \tilde{p}_{i,0})$$

6. Construct a matrix $A \in \tilde{K}^{2 \deg_z(\tilde{b}) \times (m+1)}$ by

$$A := \begin{pmatrix} \text{coeff}(q_i, t_{k-1}^0 z^j)_{j,i} \\ \text{coeff}(q_i, t_{k-1}^1 z^j)_{j,i} \end{pmatrix},$$

where $j \in \{0, \dots, \deg(\tilde{b})-1\}$ and $i \in \{0, \dots, m\}$

7. Compute a C -vector space basis $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$
-

PROOF. First, we prove that the steps of Algorithm 1 can indeed be executed. Since $\gcd(b, \frac{\partial b}{\partial t_{k-1}}) = 1$ implies $\gcd(\tilde{b}, \frac{\partial \tilde{b}}{\partial z}) = 1$ the p_i and $\tilde{p}_{i,l}$ defined in Steps 3 and 4 exist and we apply the half-extended (i.e. computing one Bézout coefficient but not both) Euclidean algorithm in $\tilde{K}[z]$ for computing them. Step 7 can be reduced to computing a basis of the nullspace of a matrix with entries in C , either by exploiting the differential structure provided by D as done in Lemma 7.1.2 from [2], or by exploiting knowledge of the generators which generate the field \tilde{K} .

Next, we want prove that $q_0, \dots, q_m \in \tilde{K}[t_{k-1}][z]$ satisfy $\deg_z(q_i) < \deg_{t_{k-1}}(b)$ and

$$\forall \beta \in \overline{K}, \tilde{b}(\beta) = 0 : q_i(\beta) = D \left(\frac{\tilde{a}_i(\beta)}{\frac{\partial \tilde{b}}{\partial z}(\beta)} \right) \quad (7)$$

for all $i \in \{0, \dots, m\}$. By construction we have $\deg_z(q_i) < \deg_z(\tilde{b}) = \deg_{t_{k-1}}(b)$. For verifying (7) we take $\beta \in \overline{K}$ such that $\tilde{b}(\beta) = 0$ and obtain

$$p_i(\beta) = \frac{\tilde{a}_i(\beta)}{\frac{\partial \tilde{b}}{\partial z}(\beta)}$$

from the definition of p_i . From the definition of $\tilde{p}_{i,0}$ and $\tilde{p}_{i,1}$ by (4) and (6) we also get

$$\begin{aligned} \tilde{p}_{i,1}(\beta)t_{k-1} + \tilde{p}_{i,0}(\beta) &= \frac{\frac{\partial p_i}{\partial z}(\beta) \cdot \sum_{j=0}^{\deg_z(\tilde{b})} (\kappa_{D,k-1} \tilde{b}_j) \beta^j}{\frac{\partial \tilde{b}}{\partial z}(\beta)} \\ &= \frac{\partial p_i}{\partial z}(\beta) \frac{D(\tilde{b}(\beta)) - \frac{\partial \tilde{b}}{\partial z}(\beta) \cdot D\beta}{\frac{\partial \tilde{b}}{\partial z}(\beta)} = -\frac{\partial p_i}{\partial z}(\beta) \cdot D\beta. \end{aligned}$$

Therefore, using (4) and (6) again we obtain

$$\begin{aligned} q_i(\beta) &= \sum_{j=0}^{\deg_z(p_i)} (\kappa_{D,k-1} \text{coeff}(p_i, z^j)) \beta^j + \frac{\partial p_i}{\partial z}(\beta) \cdot D\beta \\ &= D(p_i(\beta)) = D \left(\frac{\tilde{a}_i(\beta)}{\frac{\partial \tilde{b}}{\partial z}(\beta)} \right). \end{aligned}$$

Now let $\mathbf{c} \in C^{m+1}$ be fixed and define $q := (q_0, \dots, q_m) \cdot \mathbf{c} \in \tilde{K}[t_{k-1}][z]$. Then, by construction $\deg_z(q) < \deg_z(b)$. The roots of $r := \text{res}_{t_{k-1}}((a_0, \dots, a_m) \cdot \mathbf{c} - z \frac{\partial b}{\partial t_{k-1}}, b) \in \tilde{K}[z]$ are those $\alpha \in \overline{K}$ such that there exists a $\beta \in \overline{K}$ with $\tilde{b}(\beta) = 0$ and $(\tilde{a}_0(\beta), \dots, \tilde{a}_m(\beta)) \cdot \mathbf{c} - \alpha \cdot \frac{\partial \tilde{b}}{\partial z}(\beta) = 0$. Hence if $\beta \in \overline{K}$ ranges over the roots of b then $\alpha = \frac{(a_0(\beta), \dots, a_m(\beta)) \cdot \mathbf{c}}{\frac{\partial \tilde{b}}{\partial z}(\beta)}$ ranges over the roots of r . By (7) this implies

$$\{q(\beta) \mid \beta \in \overline{K}, b(\beta) = 0\} = \{D\alpha \mid \alpha \in \overline{K}, r(\alpha) = 0\}. \quad (8)$$

For verifying the first part of the statement of the theorem assume that there exist $v \in \tilde{K}(t_{k-1})$, $d_1, \dots, d_l \in \overline{C}$, and

$u_1, \dots, u_l \in \tilde{K}(d_1, \dots, d_l, t_{k-1})^*$ with $\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^l d_i \frac{\partial u_i}{u_i} = \frac{(a_0, \dots, a_m) \cdot \mathbf{c}}{b}$. Let $\alpha \in \overline{K}$ be such that $r(\alpha) = 0$. By Lemma 3.7

from [9] applied in $\tilde{K}(d_1, \dots, d_l)[t_{k-1}]$ there exists an irreducible $s \in \tilde{K}(d_1, \dots, d_l)[t_{k-1}]$ such that the residue satisfies $\text{res}_s(\frac{(a_0, \dots, a_m) \cdot \mathbf{c}}{b}) = \pi_s(\frac{(a_0, \dots, a_m) \cdot \mathbf{c}}{\frac{\partial \tilde{b}}{\partial t_{k-1}}}) = \alpha$. Here π_s

denotes the canonical projection onto the residue field of the valuation ring which is associated to s via the valuation $\nu_s(f) = \sup \{\nu \in \mathbb{Z} \mid \gcd(\text{den}(fs^{-\nu}), s) = 1\}$. Hence by Lemma 5.6.1 from [2] we obtain $\alpha = \text{res}_s(\frac{(a_0, \dots, a_m) \cdot \mathbf{c}}{b}) = \sum_i d_i \nu_s(u_i) \in \overline{C}$. Therefore, we have that $\alpha \in \overline{C}$ for all roots of r , i.e., $q(\beta) = 0$ for all roots $\beta \in \overline{K}$ of \tilde{b} by (8). Since \tilde{b} is squarefree it has $\deg_z(\tilde{b})$ distinct roots in \overline{K} and it follows that $q = 0$. Consequently, by definition we have $A \cdot \mathbf{c} = 0$, i.e., $\mathbf{c} \in \text{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ as required. For verifying the second part of the statement we fix some $j \in \{1, \dots, n\}$ and assume $\mathbf{c} = \mathbf{c}_j$. Then $q = (1, z, \dots, z^{\deg_z(\tilde{b})-1}, t_{k-1}, t_{k-1}z, \dots, t_{k-1}z^{\deg_z(\tilde{b})-1}) \cdot A \cdot \mathbf{c}_j =$

0. So by (8) all roots $\alpha \in \overline{K}$ of r lie in \overline{C} . Therefore $\frac{r}{\text{lcz}(r)} \in C[z]$ and it fulfils the statement by Theorem 3.8.1 from [9]. \square

COROLLARY 13. *We can solve Problem 11.*

Algorithm 2 Solve Problem 11

Abbreviate $\tilde{K} := K(t_0, \dots, t_{k-2})$

1. For all $i \in \{0, \dots, m\}$ compute $g_i \in \tilde{K}(t_{k-1})$ such that $h_i := f_i - \frac{\partial g_i}{\partial t_{k-1}} \in \tilde{K}(t_{k-1})$ has squarefree denominator (e.g. by Hermite reduction [2])
2. $b := \text{lcm}_{t_{k-1}}(\text{den}_{t_{k-1}}(h_0), \dots, \text{den}_{t_{k-1}}(h_m)) \in \tilde{K}[t_{k-1}]$
3. Apply Algorithm 1 to k and $h_0b, \dots, h_mb, b \in \tilde{K}[t_{k-1}]$ to obtain $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$
4. For all $j \in \{1, \dots, n\}$ compute $d_{j,1}, \dots, d_{j,l_j} \in \overline{C}$, $u_{j,1}, \dots, u_{j,l_j} \in \tilde{K}(d_{j,1}, \dots, d_{j,l_j}, t_{k-1})^*$, $p_j \in \tilde{K}[t_{k-1}]$ such that

$$(h_0, \dots, h_m) \cdot \mathbf{c}_j = \frac{\partial p_j}{\partial t_{k-1}} + \sum_{i=1}^{l_j} d_i \frac{\partial u_i}{u_i}$$

(see [7, 4] for example)

5. For all $j \in \{1, \dots, n\}$ compute $v_j := (g_0, \dots, g_m) \cdot \mathbf{c}_j + p_j$
-

4.2 Main Algorithm

The following theorem is the main result of this paper showing that we can do parametric elementary integration over $(K(t_0, t_1, \dots), D)$ provided we can do parametric elementary integration over (K, D) . The corresponding algorithm resembles the one stated in [3] and extends it to the parametric version of the integration problem. Moreover, by incorporating Algorithm 2, motivated by our Theorem 9, non-integrability is detected at an earlier stage in some situations.

THEOREM 14. *Let t be differentially transcendental over (K, D) , let $t_0, t_1, \dots \in K(t)$ with (1) and (2) and let $F := K(t_0, t_1, \dots)$ and $C := \text{Const}_D(F)$. Assume we can solve the parametric elementary integration problem over (K, D) . Then we can solve the parametric elementary integration problem over (F, D) by Algorithm 3.*

PROOF. Note that t_0, t_1, \dots are algebraically independent over K by Corollary 6 and $C = \text{Const}(K)$ by Corollary 8.

First, we prove that the steps of Algorithm 3 can indeed be executed. Step 3 can be computed by assumption. Steps 5 and 12 can be done by clearing the denominator

$$b := \text{lcm}_{t_k}(\text{den}_{t_k}(f_0), \dots, \text{den}_{t_k}(f_m))$$

and constructing the rows of A by coefficient extraction

$$(\text{coeff}(f_i b \div b, t_k^j))_{i=0, \dots, m}$$

for $j \in \{\min(k+1, 2), \dots, \max_i(\deg_{t_0}(f_i b)) - \deg_{t_0}(b)\}$ and

$$(\text{coeff}(f_i b \bmod b, t_k^j))_{i=0, \dots, m}$$

for $j \in \{0, \dots, \deg_{t_0}(b) - 1\}$. Alternatively, we can construct a matrix A based on partial fraction decomposition instead of computing b . In Step 12 the entries of a matrix generated that way are in \tilde{K} and a matrix with entries in K can be

Algorithm 3 Parametric elementary integration over differentially transcendental extensions

Require: t differentially transcendental over (K, D) , $t_0, t_1, \dots \in K\langle t \rangle$ with (1) and (2), $F := K\langle t_0, t_1, \dots \rangle$, $C := \text{Const}_D(F)$, and $f_0, \dots, f_m \in F$

Ensure: $\mathbf{c}_1, \dots, \mathbf{c}_n \in \text{Const}(K)^{m+1}$ and g_1, \dots, g_n from some elementary extension of (F, D) such that:

1. If $(f_0, \dots, f_m) \cdot \mathbf{c} \in F$ has an elementary integral over (F, D) for $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \text{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.
2. $\forall j \in \{1, \dots, n\} : Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$.
3. $k := \max_i(\text{ddeg}_t(f_i))$
4. **if** $k < 0$ **then**
5. Solve the parametric elementary integration problem over (K, D) with $f_0, \dots, f_m \in K$
6. **else if** $k = 0$ **then**
7. Compute a matrix $A \in K^{l \times (m+1)}$ such that $A \cdot \mathbf{c} = 0$ is equivalent to $(f_0, \dots, f_m) \cdot \mathbf{c} \in K$ for all $\mathbf{c} \in C^{m+1}$
8. Compute a C -vector space basis $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$
9. For all $j \in \{1, \dots, \tilde{n}\}$ set $\tilde{f}_j := (f_0, \dots, f_m) \cdot \tilde{\mathbf{c}}_j \in K$
10. Apply Algorithm 3 recursively to $\tilde{f}_1, \dots, \tilde{f}_{\tilde{n}} \in K$ to obtain some $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{\tilde{n}}$ and g_1, \dots, g_n from some elementary extension of (K, D)
11. Compute $(\mathbf{c}_1, \dots, \mathbf{c}_n) := (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}})$
12. **else** {i.e. $k > 0$ }
13. Abbreviate $\tilde{K} := K\langle t_0, \dots, t_{k-1} \rangle$
14. Compute a matrix $A \in K^{l \times (m+1)}$ such that $A \cdot \mathbf{c} = 0$ is equivalent to $(f_0, \dots, f_m) \cdot \mathbf{c} \in \tilde{K}[t_k]$ with $\text{deg}_{t_k} \leq 1$ for all $\mathbf{c} \in K^{m+1}$
15. Compute a C -vector space basis $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$
16. Set $\tilde{f}_{j,1}t_k + \tilde{f}_{j,0} := (f_0, \dots, f_m) \cdot \tilde{\mathbf{c}}_j$ with $\tilde{f}_{j,0}, \tilde{f}_{j,1} \in \tilde{K}$ for all $j \in \{1, \dots, \tilde{n}\}$
17. Apply Algorithm 2 to k and $\frac{\tilde{f}_{1,1}}{a_{k-1}}, \dots, \frac{\tilde{f}_{\tilde{n},1}}{a_{k-1}} \in \tilde{K}$ to obtain $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{\tilde{n}}$ and corresponding $v_j, d_{j,i}, u_{j,i}$ for $j \in \{1, \dots, \tilde{n}\}$
18. For $j \in \{1, \dots, \tilde{n}\}$ set $\tilde{g}_j := v_j + \sum_{i=1}^{l_j} d_{j,i} \log(u_{j,i})$ and $\tilde{f}_j := (\tilde{f}_{1,1}t_k + \tilde{f}_{1,0}, \dots, \tilde{f}_{\tilde{n},1}t_k + \tilde{f}_{\tilde{n},0}) \cdot \tilde{\mathbf{c}}_j - D\tilde{g}_j$
19. Apply Algorithm 3 recursively to $\tilde{f}_1, \dots, \tilde{f}_{\tilde{n}} \in \tilde{K}$ to obtain some $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_{\tilde{n}} \in C^{\tilde{n}}$ and $\hat{g}_1, \dots, \hat{g}_n$ from some elementary extension of (F, D)
20. For $j \in \{1, \dots, n\}$ set $g_j := (\hat{g}_1, \dots, \hat{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}}_j + \hat{g}_j$ and $(\mathbf{c}_1, \dots, \mathbf{c}_n) := (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_{\tilde{n}})$
21. **end if**

constructed from it by extracting appropriate coefficients. The definition in Step 5 implies $f_j \in K$ in Step 7, similarly we infer the existence of $\tilde{f}_{j,0}, \tilde{f}_{j,1} \in \tilde{K}$ in Step 14.

Showing the algorithm terminates requires to show $\tilde{f}_j \in \tilde{K}$ in Step 17. With $D\tilde{g}_j = Dv_j + \sum_i d_{j,i} \frac{D u_{j,i}}{u_{j,i}}$ in Step 16 and by Corollary 8 we can write \tilde{f}_j as

$$\left((\tilde{f}_{1,1}, \dots, \tilde{f}_{\tilde{n},1}) \cdot \tilde{\mathbf{c}}_j - a_{k-1} \left(\frac{\partial v_j}{\partial t_{k-1}} + \sum_i d_{j,i} \frac{\frac{\partial u_{j,i}}{\partial t_{k-1}}}{u_{j,i}} \right) \right) t_k + h_j$$

for some $h_j \in \tilde{K}$. Since we have $\frac{\partial v_j}{\partial t_{k-1}} + \sum_i d_{j,i} \frac{\frac{\partial u_{j,i}}{\partial t_{k-1}}}{u_{j,i}} = \frac{(\tilde{f}_{1,1}, \dots, \tilde{f}_{\tilde{n},1}) \cdot \tilde{\mathbf{c}}_j}{a_{k-1}}$ by Step 15, this implies $\tilde{f}_j = h_j \in \tilde{K}$.

We prove correctness by induction on $k = \max_i(\text{ddeg}_t(f_i))$. If $k < 0$, the $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ and g_1, \dots, g_n obtained satisfy the two properties by Corollary 10.

$k = 0$: For showing the first property we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c} \in K(t_0)$ has an elementary integral over (F, D) . By Corollary 10 we have $f \in K$ and hence by construction of $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}$ there exists a $\tilde{\mathbf{c}} \in C^{\tilde{n}}$ such that $\mathbf{c} = (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot \tilde{\mathbf{c}}$. Now, by invoking the case $k < 0$ we get $\tilde{\mathbf{c}} \in \text{span}_C\{\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}\}$ and therefore $\mathbf{c} \in \text{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. The second property is verified easily by just plugging in the definitions of g_j and \mathbf{c}_j .

$k > 0$: In order to prove the first property we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c} \in \tilde{K}(t_k)$ has an elementary integral over (F, D) . By Theorem 9 there are $v, b \in \tilde{K}$, $d_1, \dots, d_N \in \tilde{C}$, and $u_1, \dots, u_N \in \tilde{K}(d_1, \dots, d_N)$ such that $f = a_{k-1} \left(\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^N d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i} \right) t_k + b$. Hence by construction of $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}$ and $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}$ there is a $\hat{\mathbf{c}} \in C^{\tilde{n}}$ such that $\mathbf{c} = (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_{\tilde{n}}) \cdot \hat{\mathbf{c}}$. Now, by $\tilde{f}_j \in \tilde{K}$ we have $f - D((\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}}) = (\tilde{f}_1, \dots, \tilde{f}_{\tilde{n}}) \cdot \hat{\mathbf{c}} \in \tilde{K}$. So by construction of $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_{\tilde{n}}$ we obtain $\mathbf{c} \in \text{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. The second property is easily verified based on the construction. \square

5. FAMILIES OF ITERATED INTEGRALS

Now we will be concerned with representing functions depending on an additional (discrete) variable. Let (\tilde{F}, D) be a differential field and let $(s_k)_{k=1,2,\dots}$ be a sequence in \tilde{F} . For representing s_n for symbolic n we adapt the definitions from difference fields, see [11, Section 2.5], to our setting. Let (F, D) be a differential field containing the indeterminate n in its constant field. Then elements $t_0, t_1, \dots \in F$ represent s_n, s_{n-1}, \dots for symbolic n , if there exists a differential field monomorphism $\sigma : (F, D) \rightarrow (F, D)$ and maps $\varphi_1, \varphi_2, \dots$ from F to \tilde{F} with the following properties. One should think of φ_j as evaluation at $n = j$.

1. $\sigma(n) = n - 1$ and $\sigma(t_k) = t_{k+1}$ for all $k \in \mathbb{N}$.
2. $\varphi_j(n) = j$ and $\varphi_j(t_k) = s_{j-k}$ for all $j, k \in \mathbb{N}$ s.t. $k < j$.
3. For all $f, g \in F$ there exists $j_0 \in \mathbb{N}^+$ such that for all $j \geq j_0$ we have

$$\varphi_j(f + g) = \varphi_j(f) + \varphi_j(g), \quad \varphi_j(fg) = \varphi_j(f)\varphi_j(g),$$

$$\varphi_j(Df) = D\varphi_j(f), \quad \text{and} \quad \varphi_{j+1}(\sigma(f)) = \varphi_j(f).$$

These conditions imply that $f \mapsto (\varphi_1(f), \dots)$ is both a differential ring monomorphism and a difference ring monomorphism from F into the ring of sequences $\tilde{F}^{\mathbb{N}}/\sim$. The latter is defined by considering two sequences equivalent iff they differ only at finitely many entries and it is equipped with componentwise addition, multiplication, and derivation as well as with $\sigma(f_1, f_2, \dots) := (0, f_1, f_2, \dots)$. Note that the above properties in particular imply that $\varphi_j(0) = 0$ and $\varphi_j(1) = 1$ from some point on. Since F is a field they imply further that for all $f \in F \setminus \{0\}$ there exists j_0 such that $\varphi_j(f)\varphi_j(\frac{1}{f}) = 1$ and hence $\varphi_j(f) \neq 0$ for all $j \geq j_0$. In other

words, any relation among t_0, t_1, \dots corresponds to a relation among s_n, s_{n-1}, \dots that is valid for all n above some lower bound and vice versa.

Now, let (\tilde{K}, D) be a differential subfield of (\tilde{F}, D) and assume in particular that s_1, s_2, \dots are such that

$$Ds_1 = \tilde{a}_0 \tag{9}$$

$$Ds_k = \tilde{a}_{k-1}s_{k-1} \tag{10}$$

for $k \geq 2$ and some $\tilde{a}_0, \tilde{a}_1, \dots \in \tilde{K}^*$. We want to represent s_n for symbolic n by a differentially transcendental t over (K, D) . Therefore we set $(F, D) = (K\langle t \rangle, D)$ and require $\varphi_j(K) \subseteq \tilde{K}$. Following the considerations above, it is also necessary that there is no algebraic relation among s_n, s_{n-1}, \dots over K that is valid for all but possibly finitely many values of n . This needs to be verified for a particular sequence $(s_k)_{k=1,2,\dots}$ in some way. One way to do so is to check the stronger condition that there is no algebraic relation among s_1, s_2, \dots over \tilde{K} . This is by no means necessary. Nevertheless, in the following we will give a criterion to check the stronger condition provided the following additional assumption holds. Assume that for all $i \in \mathbb{N}^+$ there exist $b_{i,1}, \dots, b_{i,i} \in \tilde{K}$ such that

$$b_{i,i} = 1 \tag{11}$$

$$Db_{i,j} = -\tilde{a}_j b_{i,j+1}. \tag{12}$$

This assumption seems rather restrictive, but it covers several relevant cases as we will see below. First, it is important to note that we have

$$b_{i,1}\tilde{a}_0 = D \left(\sum_{j=1}^i b_{i,j}s_j \right). \tag{13}$$

Next, we make use of the following theorem from differential algebra in order to obtain a criterion on the algebraic independence of s_1, s_2, \dots over \tilde{K} .

THEOREM 15. *Let (K, D) be a differential field and define $C := \text{Const}(K)$. Let $w_1, \dots, w_n \in K$ such that no non-trivial C -linear combination of them has an integral in (K, D) . Then any t_1, \dots, t_n with $Dt_i = w_i$ are algebraically independent over K and $\text{Const}(K\langle t_1, \dots, t_n \rangle) = \text{Const}(K)$.*

Letting $w_i := b_{i,1}\tilde{a}_0$ and relying on (13) we obtain the following corollary on the sequence $(s_k)_{k=1,2,\dots}$.

COROLLARY 16. *Let (\tilde{K}, D) and $s_k, \tilde{a}_k, b_{i,j} \in \tilde{K}$ be as above in (9)-(12) and define $\tilde{C} := \text{Const}(\tilde{K})$. If no non-trivial \tilde{C} -linear combination of $b_{1,1}\tilde{a}_0, b_{2,1}\tilde{a}_0, \dots$ has an integral in (\tilde{K}, D) , then s_1, s_2, \dots are algebraically independent over \tilde{K} and $\text{Const}(\tilde{K}\langle s_1, s_2, \dots \rangle) = \text{Const}(\tilde{K})$.*

Once we verified for a particular choice of s_1, s_2, \dots that they are algebraically independent over \tilde{K} we can represent s_n for symbolic n in terms of a differentially transcendental extension t . If for some k the derivatives $t, Dt, \dots, D^k t$ (representing $s_n, Ds_n, \dots, D^k s_n$) were algebraically dependent over K , then as outlined above the evaluation maps $\varphi_1, \varphi_2, \dots$ would translate this to an algebraic dependence of $s_n, Ds_n, \dots, D^k s_n$, or equivalently of $s_n, s_{n-1}, \dots, s_{n-k}$, over \tilde{K} for all specific n sufficiently large.

Polylogarithms.

The polylogarithms are defined as $\text{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} dt$ and $\text{Li}_{k+1}(x) = \int_0^x \frac{1}{t} \text{Li}_k(t) dt$ for $k \geq 2$. Let $(\tilde{K}, D) = (\tilde{C}\langle x, \ln(x), \ln(1-x) \rangle, \frac{d}{dx})$ and $(\tilde{F}, D) = (K\langle s_1, s_2, \dots \rangle, D)$, then s_k represents $\text{Li}_{k+1}(x)$ if we choose $\tilde{a}_0 := -\frac{\ln(1-x)}{x}$ and $\tilde{a}_k := \frac{1}{x}$, $k \geq 1$. In addition, it is easily verified that $b_{i,j} := \frac{(-\ln(x))^{i-j}}{(i-j)!}$ satisfy (11) and (12). Now, $b_{k,1}\tilde{a}_0 = \frac{(-1)^k \ln(x)^{k-1} \ln(1-x)}{(k-1)!x}$ and no non-trivial \tilde{C} -linear combination of them has an integral in (K, D) as can be verified based on the fact that there are no $g \in \tilde{C}(x)$ and $c \in \tilde{C}$ such that $Dg = \frac{1}{1-x} + \frac{c}{x}$, for details on how the verification proceeds in general we refer to [2]. Hence by Corollary 16 we conclude that $\text{Li}_2(x), \text{Li}_3(x), \dots$ are algebraically independent over \tilde{K} . For symbolic n we deduce as above that $\text{Li}_n(x)$ is differentially transcendental over $(K, D) = C(n)\langle x, \ln(x), \ln(1-x) \rangle, \frac{d}{dx}$.

Repeated antidifferentiation.

For some function $f(x)$ we consider the iterated antiderivatives $\int f(x) dx, \int \int f(x) dx dx, \dots$, so let (\tilde{K}, D) be a differential field extension of $(\tilde{C}\langle x \rangle, \frac{d}{dx})$ such that $\tilde{a}_0 \in \tilde{K}$ represents $f(x)$. Then s_k represents the k -fold antiderivative if we choose $\tilde{a}_k := 1$ for $k \in \mathbb{N}^+$. We have that $b_{i,j} := \frac{(-x)^{i-j}}{(i-j)!}$ satisfy (11) and (12), so for applying Corollary 16 we need to check whether there exists a nonzero polynomial $p \in \tilde{C}[x]$ such that $p\tilde{a}_0$ has an integral in (\tilde{K}, D) . Depending on \tilde{a}_0 this may or may not be the case.

6. EXAMPLES

In this section we give some applications of the algorithm with various kinds of integrands. First, we look at some indefinite integrals. Then, we show how the parametric nature of the algorithm can be exploited in the context of definite integrals.

Example 1. We consider the polylogarithms $\text{Li}_n(x)$ for symbolic n and want to compute the integral

$$\int \frac{\text{Li}_{n-2}(x)\text{Li}_n(x)}{x\text{Li}_{n-1}(x)^2} dx.$$

For applying our algorithm we use $C := \mathbb{Q}(n)$ and $(K, D) := (C\langle x \rangle, \frac{d}{dx})$. We set $a_k = \frac{1}{x}$ and $b_k = 0$ in (2), so t_k from $(F, D) := (C\langle x, t_0, t_1, \dots \rangle, D)$ corresponds to $\text{Li}_{n-k}(x)$ as detailed in Section 5. The integrand is represented by

$$f := \frac{t_0 t_2}{x t_1^2},$$

which has $\text{ddeg}_t(f) = 2$ and even is of the form $\tilde{f}_1 t_2 + \tilde{f}_0$ with $\tilde{f}_1 = \frac{t_0}{x t_1} \in K(t_0, t_1)$ and $\tilde{f}_0 = 0$. So by applying Algorithm 3 we just need to solve Problem 11 for $x\tilde{f}_1$ and $k = 2$ (Step 15). We obtain $v = -\frac{t_0}{t_1}$ and $f - Dv = \frac{1}{x}$, which is easily dealt with in Step 17 resp. Step 3. Altogether, we obtain the following elementary integral of f over (F, D) .

$$-\frac{t_0}{t_1} + \log(x)$$

Translating back yields

$$\int \frac{\text{Li}_{n-2}(x)\text{Li}_n(x)}{x\text{Li}_{n-1}(x)^2} dx = \ln(x) - \frac{\text{Li}_n(x)}{\text{Li}_{n-1}(x)}.$$

Note that *Mathematica* and *Maple* in their current versions do not find the integral even for specific $n \in \{3, 4, \dots\}$. \square

With the following example we illustrate that the algorithm also works heuristically in cases where the assumption of t being differentially transcendental over (K, D) does not reflect the true situation. An integral computed that way will still be valid provided it does not represent a situation where division by zero occurs. More explicitly, the denominator of an expression in $K(t_0, t_1, \dots)$ should not reduce to zero after applying the true relations among t_0, t_1, \dots . This is completely analogous to specializing an unspecified function to a specific function.

Example 2. The Weierstraß elliptic function $\wp(x)$ fulfills the relation $\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3$ for some constants g_2 and g_3 , but in the following computation we ignore this relation and treat $\wp(x)$ and its derivatives as algebraically independent. Let the invariants g_2 and g_3 be fixed. In addition to $\wp(x)$ we also consider the Weierstraß zeta and sigma functions $\zeta(x)$ and $\sigma(x)$, where $\zeta'(x) = -\wp(x)$ and $\sigma'(x) = \zeta(x)\sigma(x)$. We will compute a closed form of

$$\int (x - \sigma(x))\wp(x) + \sigma(x)\zeta(x)^2 dx.$$

To this end, we set $C := \mathbb{Q}(g_2, g_3)$, $(K, D) := (C(x), \frac{d}{dx})$, and $F := K(t_0, \dots)$ where we choose $a_0 = t_0$, $a_1 = -1$, and $a_n = 1$ for $n \geq 2$ as well as $b_n = 0$ for all $n \in \mathbb{N}$ in (2). Hence we have

$$Dt_0 = t_0t_1 \quad Dt_1 = -t_2 \quad Dt_2 = t_3$$

and $\wp(x), \wp'(x), \zeta(x), \sigma(x)$ are represented by t_2, t_3, t_1, t_0 , respectively. Then the integrand is represented by

$$f := (x - t_0)t_2 + t_0t_1^2,$$

which has $\text{ddeg}_t(f) = 2$ and even is the form $\tilde{f}_1t_2 + \tilde{f}_0$ with $\tilde{f}_0, \tilde{f}_1 \in K(t_0, t_1)$. Following Algorithm 3 we first need to solve Problem 11 for $\frac{\tilde{f}_1}{a_1} = t_0 - x$ and $k = 2$, which gives $v = (t_0 - x)t_1$. Proceeding recursively with the remaining integrand $f - Dv = t_1$, which obviously has $\text{ddeg}_t(t_1) = 1$, we solve Problem 11 for $\frac{1}{a_0} = \frac{1}{t_0}$ and $k = 1$, giving $\frac{1}{t_0} = \frac{\partial}{\partial t_0} \log(t_0)$. Since $t_1 - D \log(t_0) = 0$ in Step 16 we are done and arrived at

$$(t_0 - x)t_1 + \log(t_0)$$

for the elementary integral of f over (F, D) . When translated back to

$$\int (x - \sigma(x))\wp(x) + \sigma(x)\zeta(x)^2 dx = (\sigma(x) - x)\zeta(x) + \ln(\sigma(x))$$

we verify that this remains valid and we successfully computed a closed form of the integral. \square

With the next example we illustrate the ability of our algorithm to find such relations of definite integrals.

Example 3. Consider the Laplace transform of an unspecified function $f(x)$. For the function $f(x)$ we merely assume that it is sufficiently regular, e.g. that $f(x)$ has a derivative on \mathbb{R}_0^+ that is continuous and bounded. Using our algorithm we want to relate the Laplace transform of $f'(x)$ to that of $f(x)$. For the differential fields we may choose $C := \mathbb{Q}(s)$,

$(K, D) := (C(e^{-sx}), \frac{d}{dx})$, as well as $F := K(t_0, \dots)$ with $a_n = 1$ and $b_n = 0$ in (2), so that t_n represents $f^{(n)}(x)$.

In order to relate $\int_0^\infty e^{-sx} f(x) dx$ and $\int_0^\infty e^{-sx} f'(x) dx$ we set $f_0 := e^{-sx}t_0$ and $f_1 := e^{-sx}t_1$. Then Algorithm 3 easily computes the relation

$$-sf_0 + f_1 = D(e^{-sx}t_0)$$

in (F, D) . Translating back and integrating gives

$$-s \int_0^\infty e^{-sx} f(x) dx + \int_0^\infty e^{-sx} f'(x) dx = e^{-sx} f(x) \Big|_{x=0}^\infty$$

where the right hand side evaluates to $0 - f(0)$ since $f(x)$ is assumed not to grow too fast. In other words, we automatically discovered the identity

$$\mathcal{L}_x(f'(x))(s) = s\mathcal{L}_x(f(x))(s) - f(0)$$

satisfied by the Laplace transform. \square

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