Integrability Conditions for Parameterized Linear Difference Equations*

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ABSTRACT

We study integrability conditions for systems of parameterized linear difference equations and related properties of linear differential algebraic groups. We show that isomonodromicity of such a system is equivalent to isomonodromicity with respect to each parameter separately under a linearly differentially closed assumption on the field of differential parameters. Due to our result, it is no longer necessary to solve non-linear differential equations to verify isomonodromicity, which will improve efficiency of computation with these systems. Moreover, it is not possible to further strengthen this result by removing the requirement on the parameters, as we show by giving a counterexample. We also discuss the relation between isomonodromicity and the properties of the associated parameterized difference Galois group.

Categories and Subject Descriptors

I.1.2 [**Computing methodologies**]: Symbolic and algebraic manipulation—*Algebraic algorithms*

General Terms

Algorithms, Theory

Keywords

Differential algebra, difference algebra, integrability conditions, difference Galois theory, differential algebraic groups

1. INTRODUCTION

In this paper, we improve the algorithm that verifies if a system of linear difference equations with differential parameters is isomonodromic (Definition 1). Given a system of difference equations, it is natural to ask if its solutions satisfy extra differential equations. For example, it is a famous result of Hölder [19] that the gamma function Γ satisfying the linear difference equation

$$y(x+1) = xy(x)$$

satisfies no polynomial differential equations with rational coefficients. This was also recently shown in [13] using parameterized difference Galois theory (see also [12, 15, 17, 16, 18, 14, 38, 11, 31, 36]). To explain the results of our paper, consider a decision procedure whose input consists of a field $\mathbf{K}$ with commuting automorphisms $\phi_1, \ldots, \phi_q$ and derivations $\partial_1, \ldots, \partial_m$ and a system of difference equations

$$\phi_1(Y) = A_1 Y, \ldots, \phi_q(Y) = A_q Y, \tag{1}$$

where $A_i \in \mathbf{GL}_n(\mathbf{K})$. Its output consists of such additional linear differential equations

$$\partial_1 Y = B_1 Y, \ldots, \partial_m Y = B_m Y, \tag{2}$$

where $B_i \in \mathbf{M}_n(\mathbf{K})$, if they exist, for which there exists an invertible matrix solution of (1) that satisfies (2) as well. For solutions of (1) to possibly exist, the A_i's must satisfy the integrability conditions (4). Moreover, existence of the B_i's above is equivalent to the existence of B_i's satisfying another large collection of integrability conditions (6) and (7). Such systems (1) are called isomonodromic in analogy with differential equations [33, 34]. Isomonodromy problems for q-difference equations and their relations with the q-difference Painlevé equations were studied in [22, 23].

In our main result, Theorem 1, we show that (7), which are non-linear differential equations, do not have to be verified to check the existence of the B_i's. More precisely, we prove that the existence of B_i's satisfying (6) implies the existence of new matrices that satisfy both (6) and (7).

Since, due to our result, we only need to check the existence of solutions (that have entries in the ground field $\mathbf{K}$) of a system of linear difference equations, a complexity estimate for verifying whether a system of difference equations is isomonodromic becomes possible. For this, a full complexity analysis for finding a rational solution of an inhomogeneous linear difference equation is sufficient, which we expect to appear in the near future. For linear differential equations there are already such results [2].

Our main result, Theorem 1, has a restriction on the field of constants. Namely, we require that this field be linearly differentially closed with respect to all but, possibly, one of the derivations (Definition 2). However, if this restriction

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were not imposed, then the conclusion of Theorem 1 would not hold, as our Example 3 shows. Moreover, we also conjecture that this restriction is necessary and leave the discovery of a proof for future research.

A similar problem but for systems of differential equations was considered in [10], motivated by the classical results [20, 21]. Differential categories developed in [9] formed the main technical tool in [10]. On the contrary, our proofs are written in elementary terms, which makes them more accessible. Moreover, from our proof, one can produce an algorithm that, given a common solution of (6), computes a common solution of both (6) and (7).

Given a system of linear difference equations with parameters, parameterized difference Galois theory [13] associates a linear differential algebraic group [3, 4, 32, 6, 35, 27, 28], which is called the parameterized difference Galois group and is a group of matrices whose entries satisfy a system of polynomial differential equations. In addition to our main result, we also show in Proposition 2 how isomonodromicity can be characterized using the Galois group. This extends the corresponding result of [13], as our version does not require the constants to be differentially closed.

Our main result has further applications. In parameterized differential Galois theory [5], there are several algorithms for computing the Galois groups. For 2×2 systems, they are given in [1, 7]. An algorithm, more general in terms of the order of the system, [30, 29] uses the differential analogues [10] of our results to make the computation more efficient. Our results may prove useful in the design of an algorithm for computing parameterized difference Galois groups, as it has happened in the differential case.

Also note that the algorithm in [30] performs simultaneous prolongations with respect to all derivations, which gives a linear growth of algebraic indeterminates to be dealt with. On the other hand, if one performs prolongations with respect to each derivation separately, there is no such growth. The results of our paper, as well as [10], describe a situation in which one can deal with each derivation separately. We hope that more such situations will be discovered in the future to speed up the algorithms that compute parameterized differential and difference Galois groups.

The paper is organized as follows. We introduce our notation and review the basic notions of differential and difference algebra in §2. The integrability conditions are introduced in §3.1. Our main result is described in §3.2. In §3.3, we demonstrate how isomonodromicity is reflected in the associated parameterized difference Galois group. An example showing that assumption 1 cannot be removed from our main result, Theorem 1, is given in §3.4.

2. BASIC DEFINITIONS

We will start with the basic definitions and notation of differential and difference algebra. A Δ-ring is a commutative associative ring with unit 1 together with a set $\Delta = \{\partial_1, \ldots, \partial_m\}$ of commuting derivations $\partial_i \colon R \to R$ such that

$$\partial_i(a+b) = \partial_i(a) + \partial_i(b), \quad \partial_i(ab) = \partial_i(a)b + a\partial_i(b)$$

for all $a, b \in R$. For example, $\mathbb{Q}$ is a $\{\partial\}$-field with the unique zero derivation. For every $f \in \mathbb{C}(x)$, there exists a unique derivation $\partial : \mathbb{C}(x) \to \mathbb{C}(x)$ with $\partial(x) = f$, turning $\mathbb{C}(x)$ into a $\{\partial\}$-field. Let

$$\Theta = \left\{\partial_1^{i_1} \cdot \ldots \cdot \partial_m^{i_m} \mid i_j \geqslant 0\right\}.$$

Since ∂_i acts on R, there is a natural action of Θ on R. Let R and B be Δ-rings. If $B \supset R$, then B is a Δ-R-algebra if, for all i, $1 \leqslant i \leqslant m$, the action of ∂_i on B extends the action of ∂_i on R. Let $Y = \{y_1, \ldots, y_n\}$ be a set of variables and

$$\Theta Y := \left\{\theta y_j \;\middle|\; \theta \in \Theta,\ 1 \leqslant j \leqslant n\right\}.$$

The ring of differential polynomials $R\{Y\}_\Delta$ in differential indeterminates Y over R is $R[\Theta Y]$ with the derivations ∂_i that extends the ∂_i-action on R as follows:

$$\partial_i\left(\theta y_j\right) := (\partial_i \cdot \theta) y_j, \quad 1 \leqslant j \leqslant n.$$

An ideal I in a Δ-ring R is called a differential ideal if $\partial_i(a) \in I$ for all $a \in I$, $1 \leqslant i \leqslant m$.

Let $\mathbf{k}$ be a Δ-field of characteristic zero. In what follows, $\mathbf{M}_n(\mathbf{k})$ denotes the set of $n \times n$ matrices with entries in $\mathbf{k}$ and $\mathbf{GL}_n(\mathbf{k})$ are the invertible matrices in $\mathbf{M}_n(\mathbf{k})$.

A Φ-ring R is a commutative associative ring with unit 1 and a set $\Phi = \{\phi_1, \ldots, \phi_q\}$ of commuting automorphisms $\phi_i : R \to R$. A $\{\Phi, \Delta\}$-ring R is a Φ-ring and a Δ-ring such that, for all $\phi \in \Phi$ and $\partial \in \Delta$, $\phi\partial = \partial\phi$.

Example 1. We will list a few examples of $\{\Phi, \Delta\}$-rings:

1. $R = \mathbb{Q}(x_1, \ldots, x_m)$ with $\Phi = \{\phi_1, \ldots, \phi_m\}$ and $\Delta = \{\partial_1, \ldots, \partial_m\}$ defined by
$$\phi_i(f)(x_1, \ldots, x_m) = f(x_1, \ldots, x_i + 1, \ldots, x_m), f \in R,$$
$$\partial_i = \partial/\partial x_i, \quad 1 \leqslant i \leqslant m.$$
2. $R = \mathbb{Q}(x, y)$ with $\Phi = \{\phi\}$ and $\Delta = \{\partial_1, \partial_2\}$ defined by
$$\phi(f)(x, y) = f(xy, y),\ \partial_1 = x\partial/\partial_x,\ \partial_2 = \partial/\partial_y.$$

3. MAIN RESULT

We will start by introducing integrability conditions in §3.1 and then show our main result in §3.2. Applications of the parameterized difference Picard–Vessiot theory to this will be discussed in §3.3. §3.4 contains an example showing that the conclusion of Theorem 1 does not hold if one drops the linearly differentially closed assumption 1.

3.1 Integrability conditions

Let $\mathbf{K}$ be a $\{\Phi, \Delta\}$-field of characteristic zero and

$$A_1, \ldots, A_q \in \mathbf{GL}_n(\mathbf{K}).$$

Consider the system of difference equations

$$\phi_1(Y) = A_1 Y, \ldots, \phi_q(Y) = A_q Y. \tag{3}$$

If $L \supset \mathbf{K}$ is a Φ-field extension and $Z \in \mathbf{GL}_n(L)$ satisfies (3) then, for all i, j, $1 \leqslant i, j \leqslant q$, we have

$$\phi_j(A_i) A_j Z = \phi_j(\phi_i(Z)) = \phi_i(\phi_j(Z)) = \phi_i(A_j) A_i Z.$$

Therefore, we obtain

$$\phi_j(A_i) A_j = \phi_i(A_j) A_i \tag{4}$$

Moreover, if, in addition, L is a $\{\Phi, \Delta\}$-field extension of $\mathbf{K}$ and there exist $B_1, \ldots, B_m \in \mathbf{M}_n(\mathbf{K})$ such that

$$\partial_1(Z) = B_1 Z, \ldots, \partial_m(Z) = B_m Z, \tag{5}$$

then

$$\phi_j(B_i)A_jZ = \phi_j(\partial_i(Z)) = \\ = \partial_i(\phi_j(Z)) = \partial_i(A_j)Z + A_jB_iZ$$

and

$$\partial_j(B_i)Z + B_iB_jZ = \partial_j(\partial_i(Z)) = \\ = \partial_i(\partial_j(Z)) = \partial_i(B_j)Z + B_jB_iZ,$$

which imply that

$$\phi_j(B_i) = \partial_i(A_j)A_j^{-1} + A_jB_iA_j^{-1}, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant q, \quad (6)$$

and

$$\partial_j(B_i) - \partial_i(B_j) = [B_j, B_i], \quad 1 \leqslant i, j \leqslant m. \quad (7)$$

Therefore, (4), (6), and (7) are necessary conditions for the existence of a common invertible matrix solution of (3) and (5) with entries in a field L (see Definition 6.6(d) in [13]). They are also sufficient. Indeed, let

$$L = \mathbf{K}(x_{11}, \ldots, x_{nn}),$$

$$\partial_i((x_{rs})) = B_i(x_{rs}), \quad 1 \leqslant i \leqslant m,$$

and

$$\phi_j((x_{rs})) = A_j(x_{rs}), \quad 1 \leqslant j \leqslant q.$$

Then (4), (6), and (7) imply that L is a $\{\Phi, \Delta\}$-field.

Definition 1. The system of linear difference equations

$$\phi_1(Y) = A_1Y, \ldots, \phi_q(Y) = A_qY$$

with $A_1, \ldots, A_q \in \mathbf{GL}_n(\mathbf{K})$ satisfying (4) is called *isomonodromic* if there exist $B_1, \ldots, B_m \in \mathbf{M}_n(\mathbf{K})$ satisfying (6) and (7).

Connections to analytic interpretations of isomonodromy can be found, e.g., in §5 of [5] and §6.2 of [10]. We will need the following definition in §3.2 (which is a weaker restriction than differentially closed) to state the main result.

Definition 2. For $\Delta' = \{\delta_1, \ldots, \delta_s\} \subset \Delta$, a Δ-field $\mathbf{k}$ is called *linearly Δ'-closed* if, for all $n \geqslant 1$ and $B_1, \ldots, B_s \in \mathbf{M}_n(\mathbf{k})$ such that

$$\delta_i(B_j) - \delta_j(B_i) = [B_i, B_j], \quad 1 \leqslant i, j \leqslant s,$$

there exists $Z \in \mathbf{GL}_n(\mathbf{k})$ such that

$$\delta_1(Z) = B_1Z, \ldots, \delta_s(Z) = B_sZ.$$

3.2 Statement and proof of the main result

In this section, we will state and prove our main result, Theorem 1, which allows us to reduce the number of integrability conditions to be tested. For this, we need to impose a restriction on the fields we consider, which we will describe now. Note that, if this restriction were not added, the conclusion of Theorem 1 would no longer be true (see §3.4).

Assumption 1. Let $\mathbf{K}$ be a $\{\Phi, \Delta\}$-field such that, after some renumbering of $\partial_1, \ldots, \partial_m$, for all k, $1 \leqslant k \leqslant m-1$,

$$\mathbf{K}^\Phi := \{a \in \mathbf{K} \mid \phi(a) = a, \ \phi \in \Phi\}$$

is linearly $\{\partial_1, \ldots, \partial_k\}$-closed.

Example 2. Let $\mathcal{U}$ be a linearly closed ∂_1-field and $\mathbf{K} := \mathcal{U}(x)$, the field of rational functions in x, with $\Phi = \{\phi\}$ and $\Delta = \{\partial/\partial x, \partial_1\}$, where $\phi(f)(x) = f(x+1)$, $f \in \mathbf{K}$, and $\partial_1(x) = 0$. Then $\mathbf{K}^\Phi = \mathcal{U}$ with $\{\partial/\partial x, \partial_1\}$, and $\mathbf{K}$ satisfies assumption 1 with the renumbering $\{\partial_1, \partial/\partial x\}$.

Theorem 1. *Let $\mathbf{K}$ satisfy assumption 1. Then, for all $A_1, \ldots, A_q \in \mathbf{GL}_n(\mathbf{K})$ satisfying (4), if there exist $B_1, \ldots, B_m \in \mathbf{M}_n(\mathbf{K})$ satisfying (6), then there exist $D_1, \ldots, D_m \in \mathbf{M}_n(\mathbf{K})$ such that all integrability conditions are satisfied, that is,*

$$\phi_j(D_i) = \partial_i(A_j)A_j^{-1} + A_jD_iA_j^{-1}, \ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant q, \\ \partial_i(D_j) - \partial_j(D_i) = [D_i, D_j], \quad 1 \leqslant i, j \leqslant m. \quad (8)$$

Proof. This will be done by induction. Suppose that a renumbering from assumption 1 has been performed. Let there exist $B_1, \ldots, B_m$ such that (6) is satisfied and let $k \leqslant m$. Suppose that there exist

$$D_1, \ldots, D_{k-1}$$

that satisfy both (6) and (7) ($k-1$ is substituted for m and the D_i's are substituted for the B_i's). We claim that there exists $E \in \mathbf{M}_n(\mathbf{K})$ such that

$$(D_1, \ldots, D_{k-1}, B_k + E)$$

satisfies both (6) and (7) (under the substitution as above), so that we can take

$$(D_1, \ldots, D_k) := (D_1, \ldots, D_{k-1}, B_k + E)$$

to satisfy (8) by induction. In order to show that $B_k + E$ satisfies (6) and (7), we need to show that

$$\phi_j(B_k + E) = \partial_k(A_j)A_j^{-1} + A_j(B_k + E)A_j^{-1} \quad (9)$$

and

$$\partial_i(B_k + E) - \partial_k(D_i) = [D_i, B_k + E] \quad (10)$$

for all i and j, $1 \leqslant j \leqslant q$, $1 \leqslant i < k$. However, expanding the left-hand side of (9) using (6), we see that

$$\phi_j(B_k + E) = \phi_j(B_k) + \phi_j(E) = \\ = \partial_k(A_j)A_j^{-1} + A_jB_kA_j^{-1} + \phi_j(E).$$

Moreover, rearranging the terms in (10), we see that we need to find $E \in \mathbf{M}_n(\mathbf{K})$ such that the following two conditions are satisfied:

$$\phi_j(E) = A_jEA_j^{-1} \quad (11)$$

and

$$\partial_i(E) + [E, D_i] = \partial_k(D_i) - \partial_i(B_k) + [D_i, B_k] \quad (12)$$

for all i and j, $1 \leqslant j \leqslant q$, $1 \leqslant i < k$. We now show that

$$\partial_k(D_i) - \partial_i(B_k) + [D_i, B_k]$$

satisfies (11) as an equation in E. We have:

$$\phi_j(\partial_k(D_i) - \partial_i(B_k) + [D_i, B_k]) = \\ = \phi_j(\partial_k(D_i)) - \phi_j(\partial_i(B_k)) + \phi_j(D_iB_k) - \phi_j(B_kD_i) = \\ = \partial_k(\phi_j(D_i)) - \partial_i(\phi_j(B_k)) + \\ + \phi_j(D_i)\phi_j(B_k) - \phi_j(B_k)\phi_j(D_i) = \\ = \partial_k\big(\partial_i(A_j)A_j^{-1}\big) + \partial_k\big(A_jD_iA_j^{-1}\big) - \\ - \partial_i\big(\partial_k(A_j)A_j^{-1}\big) - \partial_i\big(A_jB_kA_j^{-1}\big) + \\ + \big(\partial_i(A_j)A_j^{-1} + A_jD_iA_j^{-1}\big)\big(\partial_k(A_j)A_j^{-1} + A_jB_kA_j^{-1}\big) - \\ - \big(\partial_k(A_j)A_j^{-1} + A_jB_kA_j^{-1}\big)\big(\partial_i(A_j)A_j^{-1} + A_jD_iA_j^{-1}\big).$$

Using the relation

$$\partial_d\big(A_j^{-1}\big) = -A_j^{-1}\partial_d(A_j)A_j^{-1},$$

and after we expand and cancel out terms, we are left with

$$\begin{aligned}&A_j\partial_k(D_i)A_j^{-1} - A_j\partial_i(B_k)A_j^{-1}+\\&\quad + A_jD_iB_kA_j^{-1} - A_jB_kD_iA_j^{-1} =\\&= A_j(\partial_k(D_i) - \partial_i(B_k) + (D_iB_k - B_kD_i))A_j^{-1}\end{aligned}$$

as desired.

For any $Z \in \mathbf{M}_n(\mathbf{K})$ satisfying (11), we now show that

$$\partial_i(Z) + [Z, D_i]$$

also satisfies (11) for $1 \leqslant j \leqslant q$, $1 \leqslant i < k$. Indeed,

$$\begin{aligned}&\phi_j(\partial_i(Z) + [Z, D_i]) =\\&= \partial_i(\phi_j(Z)) + \phi_j(Z)\phi_j(D_i) - \phi_j(D_i)\phi_j(Z) =\\&= \partial_i\big(A_jZA_j^{-1}\big) + \big(A_jZA_j^{-1}\big)\big(\partial_i(A_j)A_j^{-1} + A_jD_iA_j^{-1}\big)-\\&\quad - \big(\partial_i(A_j)A_j^{-1} + A_jD_iA_j^{-1}\big)\big(A_jZA_j^{-1}\big) =\\&= \partial_i(A_j)ZA_j^{-1} + A_j\partial_i(Z)A_j^{-1} + A_jZ\partial_i\big(A_j^{-1}\big)+\\&\quad + A_jZA_j^{-1}\partial_i(A_j)A_j^{-1} + A_jZD_iA_j^{-1}-\\&\quad - \partial_i(A_j)ZA_j^{-1} - A_jD_iZA_j^{-1} =\\&= A_j\partial_i(Z)A_j^{-1} + A_jZD_iA_j^{-1} - A_jD_iZA_j^{-1} =\\&= A(\partial_i(Z) + [Z, D_i])A_j^{-1}.\end{aligned}$$

Since the set S of solutions of (11) inside $\mathbf{M}_n(\mathbf{K})$ is a finite-dimensional vector space over $\mathbf{K}^\Phi$, say, $\dim_{\mathbf{K}^\Phi} S = p$, there exists a $\mathbf{K}^\Phi$-basis of S and $C_{i,s} \in \mathbf{M}_p\big(\mathbf{K}^\Phi\big)$ for $s = 1, 2$, such that C_{i2} represents

$$\partial_k(D_i) - \partial_i(B_k) + [D_i, B_k]$$

in this basis and C_{i1} is such that the differential operator defined by

$$\partial_i(Z) + [Z, D_i],$$

when restricted to the matrices from S, has the form

$$\partial_i(Y) + C_{i1}Y = C_{i2} \tag{13}$$

for all i, $1 \leqslant i < k$. We will now show that these equations are consistent, that is, they have a common solution with entries in some Δ-field extension of $\mathbf{K}^\Phi$. For this, it is sufficient to show that (12) are consistent. By Proposition IV.6.3 of [25], to prove this, it is sufficient to show that (12) and the inductive hypothesis for (8) imply

$$\partial_i(\partial_j(E)) = \partial_j(\partial_i(E)), \quad 1 \leqslant i, j < k. \tag{14}$$

To do this, first note that (12) implies that

$$\partial_j\partial_i(E) = \partial_j\partial_k(D_i) - \partial_j\partial_i(B_k) + \partial_j[D_i, B_k + E]$$

and

$$\partial_i\partial_j(E) = \partial_i\partial_k(D_j) - \partial_i\partial_j(B_k) + \partial_i[D_j, B_k + E]$$

Since $\partial_j\partial_i(B_k) = \partial_i\partial_j(B_k)$ and $\partial_j\partial_k(D_i) = \partial_k\partial_j(D_i)$ and, by the inductive hypothesis for (8),

$$\begin{aligned}&\partial_j\partial_i(E) - \partial_i\partial_j(E) = \partial_k\left(\partial_j(D_i) - \partial_i(D_j)\right)+\\&\quad + \partial_j\left([D_i, B_k + E]\right) - \partial_i\left([D_j, B_k + E]\right) =\\&= \partial_k\left([D_j, D_i]\right) + \partial_j\left([D_i, B_k + E]\right) - \partial_i\left([D_j, B_k + E]\right) =\\&= \partial_k\left([D_j, D_i]\right) + [[D_j, D_i], B_k + E]+\\&\quad + [D_i, \partial_j(B_k + E)] - [D_j, \partial_i(B_k + E)].\end{aligned}$$

Now, substituting the expression for $\partial_i(E)$ from (12), and similarly for $\partial_j(E)$,

$$\begin{aligned}&\partial_j\partial_i(E) - \partial_i\partial_j(E) = \partial_k\left([D_j, D_i]\right) + [[D_j, D_i], B_k + E]+\\&\quad + [D_i, \partial_j(B_k)] - [D_j, \partial_i(B_k)]+\\&\quad + [D_i, \partial_k(D_j) - \partial_j(B_k) + [D_j, B_k + E]]-\\&\quad - [D_j, \partial_k(D_i) - \partial_i(B_k) + [D_i, B_k + E]].\end{aligned}$$

Rearranging terms,

$$\begin{aligned}&\partial_j\partial_i(E) - \partial_i\partial_j(E) = \partial_k\left([D_j, D_i]\right) + [D_i, \partial_k(D_j)]-\\&\quad - [D_j, \partial_k(D_i)] + [[D_j, D_i], B_k + E]+\\&\quad + [D_i, [D_j, B_k + E]] - [D_j, [D_i, B_k + E]]+\\&\quad + [D_i, \partial_j(B_k)] - [D_j, \partial_i(B_k)]+\\&\quad + [D_j, \partial_i(B_k)] - [D_i, \partial_j(B_k)] =\\&= 0 + [[D_j, D_i], B_k + E] + [D_i, [D_j, B_k + E]]-\\&\quad - [D_j, [D_i, B_k + E]] + 0 = 0\end{aligned}$$

by the Jacobi identity. Therefore, (14) holds and, thus, (13) is consistent.

Since $\mathbf{K}^\Phi$ is linearly $\{\partial_1, \ldots, \partial_{k-1}\}$-closed, there exists a solution $F \in \mathbf{M}_p\big(\mathbf{K}^\Phi\big)$ to (13), which implies the existence of $E \in \mathbf{M}_n(\mathbf{K})$ satisfying (11) and (12). Thus,

$$(D_1, \ldots, B_k + E)$$

satisfies both (6) and (7). $\square$

3.3 Using difference Picard–Vessiot theory

In this section, we will see in Proposition 2 how isomonodromicity can be detected using the methods of parameterized difference Picard–Vessiot (PPV) theory [13] in light of Theorem 1.

Recall that a Δ-field $\mathcal{U}$ is called differentially closed if any system of polynomial differential equations with coefficients in $\mathcal{U}$ having a solution with entries in some Δ-extension of $\mathcal{U}$ already has a solution with entries in $\mathcal{U}$ (for various equivalent versions of this definition, see Definition 3.2 in [5], Definition 4 in [37], and the references given there). Recall also that every Δ-field $\mathbf{k}$ is contained in some differentially closed Δ-field $\mathcal{U}$.

Definition 3. A *Kolchin-closed* subset W of $\mathcal{U}^n$ defined over $\mathbf{k}$ is the set of common zeroes of a system of differential algebraic equations with coefficients in $\mathbf{k}$, that is, there exist $f_1, \ldots, f_r \in \mathbf{k}\{Y\}$ such that

$$W = \{a \in \mathcal{U}^n \mid f_1(a) = \ldots = f_r(a) = 0\}.$$

Its coordinate ring is defined to be

$$\mathbf{k}\{W\} := \mathbf{k}\{Y\}/\{f_1, \ldots, f_r\},$$

where $\{f_1, \ldots, f_r\}$ denotes the radical differential ideal generated by $f_1, \ldots, f_r$.

Definition 4. (Ch. II, §1, p. 905 of [3]) A *linear differential algebraic group* (LDAG) defined over $\mathbf{k}$ is a Kolchin-closed subgroup G of $\mathbf{GL}_n(\mathcal{U})$, over $\mathbf{k}$ that is, an intersection of a Kolchin-closed subset of $\mathcal{U}^{n^2}$ over $\mathbf{k}$ with $\mathbf{GL}_n(\mathcal{U})$, which is closed under the group operations.

We will now briefly recall difference PPV theory. Let $\mathbf{K}$ be a $\{\Phi, \Delta\}$-field and $A_1, \ldots, A_q \in \mathbf{M}_n(\mathbf{K})$ be given. Contrary to [13], we do not assume $\mathbf{K}^\Phi$ to be differentially closed. A *parameterized Picard-Vessiot ring (PPV-ring)* R of $\mathbf{K}$ associated with (3) is a $\{\Phi,\Delta\}$-$\mathbf{K}$-algebra R with no $\{\Phi,\Delta\}$-ideals such that there exists a $Z \in \mathbf{GL}_n(R)$ satisfying (3), $(\mathrm{Quot}(R))^{\Phi} = \mathbf{K}^{\Phi}$, and R is Δ-generated over $\mathbf{K}$ by the entries of Z and $1/\det Z$ (that is, $R = \mathbf{K}\{Z, 1/\det Z\}$).

$\mathbf{K}^{\Phi}$ is a Δ-field and, if it is differentially closed, a PPV-ring associated with (3) always exists and is unique up to Δ-$\mathbf{K}$-isomorphism (Propositions 6.14 and 6.16 of [13]). If $R = \mathbf{K}\{Z, 1/\det Z\}$ is a PPV-ring of $\mathbf{K}$, one defines the *parameterized Picard–Vessiot group (PPV-group)*, denoted by $\mathbf{Gal}(R/\mathbf{K})$, of R over $\mathbf{K}$ to be

$$\begin{aligned} G := \{\sigma : R \to R \mid\ & \sigma \text{ is an automorphism},\\ & \sigma\delta = \delta\sigma \text{ for all } \delta \in \{\Phi,\Delta\}, \text{ and}\\ & \sigma(a) = a,\ a \in \mathbf{K}\}. \end{aligned}$$

For any $g \in G$, one can show that there exists a matrix $C_g \in \mathbf{GL}_n(\mathbf{K}^{\Phi})$ such that

$$g(Z) = ZC_g,$$

and the map $\sigma \mapsto C_g$ is an isomorphism of G onto a differential algebraic subgroup of $\mathbf{GL}_n(\mathbf{K}^{\Phi})$. One can also give a functorial definition of $\mathbf{Gal}$ (as on pages 368–369 of [13]) turning $\mathbf{Gal}$ into a functor from $\mathbf{K}^{\Phi}$-Δ-algebras to groups: $A \mapsto \mathbf{Gal}(A)$ (each $g \in \mathbf{Gal}(A)$ becomes $\{\Phi,\Delta\}$-automorphism $R \otimes_{\mathbf{K}^{\Phi}} A \to R \otimes_{\mathbf{K}^{\Phi}} A$), that is, the A-points of $\mathbf{Gal}$ (see also page 420 of [32]). We will need the functorial definition of $\mathbf{Gal}$ in the proof of Proposition 2.

Proposition 2.9 of [13] gives a criterion for isomonodromicity via PPV theory, but requires $\mathbf{K}^{\Phi}$ to be differentially closed. This is an obstacle for potential applications, and our version of this result, Proposition 3, avoids it. Moreover, several recent results [9, 39] on differential PPV theory, in which the constants are not required to be differentially closed (weaker assumptions are made there), indicate that their difference analogues could be constructed using [24]. This encourages us to extend the result of [13] by not requiring that $\mathbf{K}^{\Phi}$ be differentially closed.

PROPOSITION 2. *Let the system of difference equations* (3) *be such that difference integrability conditions* (4) *are satisfied. Assume also that there exists a PPV-ring of* $\mathbf{K}$ *for* (3) *and* G *is its PPV Galois group.*

Then (3) *is isomonodromic if and only if, for all* i, $1 \leqslant i \leqslant m$, *there exists* $D_i \in \mathbf{M}_n(\mathbf{K}^{\Phi})$ *satisfying*

$$\partial_u(D_v) - \partial_v(D_u) = [D_u, D_v], \quad 1 \leqslant u, v \leqslant m \tag{15}$$

such that, for all r *and* s, $1 \leqslant r, s \leqslant n$, *the following equation is in the defining ideal of* G:

$$\partial_i(x_{rs}) + [(x_{rs}), D_i] = 0. \tag{16}$$

Moreover, if (16) *is in the defining ideal of* G, *then there exists a finitely generated* Δ-*field extension* F *of* $\mathbf{K}^{\Phi}$ *and* $C_i \in \mathbf{GL}_n(F)$ *such that, for all* r *and* s, $1 \leqslant r, s \leqslant n$,

$$\partial_i\left(C_i^{-1}(x_{rs})C_i\right) = 0 \tag{17}$$

is in the defining ideal of G, *that is,* G *is conjugate over* F *to a group of matrices with* ∂_i-*constant entries.*

PROOF. Let R be a PPV-ring of $\mathbf{K}$ for (3) and $Z \in \mathbf{GL}_n(R)$ be a fundamental solution matrix.

Let there exist $B_i \in \mathbf{GL}_n(\mathbf{K})$ such that (6) and (7) are satisfied. For all j, $1 \leqslant j \leqslant q$, we have

$$\begin{aligned} &\phi_j\big(\partial_i(Z) - B_iZ\big) = \partial_i(\phi_j(Z)) - \phi_j(B_i)\phi_j(Z) =\\ &= \partial_i(AZ) - \big(\partial_i(A)A^{-1} + AB_iA^{-1}\big)AZ = A(\partial_i(Z) - B_iZ). \end{aligned}$$

Therefore, since $R^{\Phi} = \mathbf{K}^{\Phi}$, there exists $D_i \in \mathbf{M}_n(\mathbf{K}^{\Phi})$ such that

$$\partial_i(Z) = B_iZ - ZD_i.$$

Therefore,

$$\begin{aligned} \partial_j(\partial_i(Z)) &= \partial_j(B_i)Z + B_i(B_jZ - ZD_j) +\\ &\quad + (B_jZ - ZD_j)D_i + Z\partial_j(D_i), \end{aligned} \tag{18}$$

$$\begin{aligned} \partial_i(\partial_j(Z)) &= \partial_i(B_j)Z + B_j(B_iZ - ZD_i) +\\ &\quad + (B_iZ - ZD_i)D_j + Z\partial_i(D_j). \end{aligned} \tag{19}$$

Hence, by (7), we have

$$Z\partial_j(D_i) - ZD_jD_i = Z\partial_i(D_j) - ZD_iD_j,$$

which implies (15) since Z is invertible. Now, for every $\mathbf{K}^{\Phi}$-Δ-algebra L and $g \in G(L)$, let $C_g \in \mathbf{GL}_n(L)$ be such that $g(Z) = ZC_g$. Then, on the one hand,

$$g(\partial_i(Z)) = g(BZ - ZD_i) = BZC_g - ZC_gD_i.$$

On the other hand,

$$g(\partial_i(Z)) = \partial_i(g(Z)) = \partial_i(ZC_g) = BZC_g - ZD_iC_g + Z\partial_i(C_g).$$

Therefore, for all $g \in G$, we have

$$\partial_i(C_g) + [C_g, D_i] = 0, \quad 1 \leqslant i \leqslant m,$$

showing (16). To show that (17) is in the defining ideal of G, let F be the Δ-field generated over $\mathbf{K}^{\Phi}$ by the entries of an invertible solution C of

$$\partial_i(Y) = D_iY, \quad 1 \leqslant i \leqslant m,$$

inside of a differential closure of $\mathbf{K}^{\Phi}$ ((15) is satisfied). Then,

$$\begin{aligned} \partial_i\big(C^{-1}C_gC\big) &= -C^{-1}D_iCC^{-1}C_gC -\\ &\quad - C^{-1}[C_g, D_i]C + C^{-1}C_gD_iC = 0. \end{aligned}$$

Suppose now that, for all $\mathbf{K}^{\Phi}$-∂-algebras L and $g \in G(L)$,

$$\partial_i(C_g) + [C_g, D_i] = 0, \quad 1 \leqslant i \leqslant m,$$

where $C_g := Z^{-1}g(Z)$, and (15) is satisfied. Let

$$B_i := \partial(Z)Z^{-1} + ZD_iZ^{-1}.$$

Then, for all $g \in G(L)$,

$$\begin{aligned} &g(B_i) = \partial_i(ZC_g)C_g^{-1}Z^{-1} + ZC_gD_iC_g^{-1}Z^{-1} =\\ &= \partial_i(Z)Z^{-1} - Z[C_g, D_i]C_g^{-1}Z^{-1} + ZC_gD_iC_g^{-1}Z^{-1} = B_i. \end{aligned}$$

Hence, by Lemma 6.19 of [13], $B_i \in \mathbf{GL}_n(\mathbf{K})$. Moreover, for all j, $1 \leqslant j \leqslant q$, we have

$$\begin{aligned} &\phi_j(B_i) = \partial_i(\phi_j(Z))(\phi_j(Z))^{-1} + \phi_j(Z)D_i(\phi_j(Z))^{-1} =\\ &= \partial_i(A_jZ)(A_jZ)^{-1} + A_jZD_i(A_jZ)^{-1} =\\ &= \partial_i(A_j)A_j^{-1} + A_j\partial_i(Z)Z^{-1}A_j^{-1} + A_jZD_iZ^{-1}A_j^{-1} =\\ &= \partial_i(A_j)A_j^{-1} + A_jB_iA_j^{-1}, \end{aligned}$$

showing (6). To show (7), we proceed as in (18) and (19). $\square$

Remark 1. It follows from Theorem 1 that, if $\mathbf{K}$ satisfies assumption 1, then one can remove requirement (15) from the statement of Proposition 2.

3.4 Example

We will now give an example that shows that the linearly differentially closed restriction in the statement of Theorem 1 cannot be waived.

Example 3. This example is based on Example 6.7 from [10], but requires a modification described below. Starting with $\mathbf{K} := \overline{\mathbb{Q}(t_1,t_2)}$ (the algebraic closure), $\Delta := \{\partial_{t_1}, \partial_{t_2}\}$, and $\Phi = \{\phi\}$ with ϕ acting as identity on $\mathbf{K}$, we let

$$F := \mathbf{K}\left(\phi^i \partial_{t_1}^{j_1} \partial_{t_2}^{j_2} I_k,\ j_1, j_2 \geqslant 0,\ i \in \mathbb{Z},\ k = 1,2\right),$$

be the field of $\{\phi, \partial_{t_1}, \partial_{t_2}\}$-rational functions in the difference-differential indeterminates I_1 and I_2 over $\mathbf{K}$. Notice that $\mathbf{K}$ is neither ∂_{t_1}- nor ∂_{t_2}-linearly closed as $\partial_{t_i}(y) = y$ has no non-zero solutions in $\mathbf{K}$, $i = 1,2$. Let S be a PPV-ring of F for the difference equation

$$\phi(y) - y = (\phi(I_1) - I_1) \cdot I_2, \tag{20}$$

$L := \operatorname{Quot}(S)$, and $I \in L$ be a solution of (20) (the existence of S is shown as in Proposition 5.4 of [10], in particular,

$$L = \operatorname{Quot}\left(F\{x\}_\Delta / [D_1(x) - a_1, \ldots, D_r(x) - a_r]\right),$$

where

$$\phi(x) := x + (\phi(I_1) - I_1) \cdot I_2,$$

$a_1, \ldots, a_r \in F$, and the D_i's are some homogeneous linear Δ-operators, possibly, all equal to zero). We now show that there are no $a \in F$ and non-zero homogeneous linear Δ-operator D with coefficients in $\mathbf{k}$ such that

$$\phi(a) - a = D((\phi(I_1) - I_1) \cdot I_2). \tag{21}$$

For this, let $\frac{P}{Q} \in F$ be such that P and Q are relatively prime polynomials. Then $\frac{\phi(P)}{\phi(Q)} - \frac{P}{Q}$ is a polynomial if and only if Q is a non-zero constant. Indeed, suppose

$$\frac{\phi(P)}{\phi(Q)} - \frac{P}{Q} = R,$$

with $R \in F$ is a polynomial. Rearranging the terms, we have

$$\phi(P) = \phi(Q)R + \frac{\phi(Q)P}{Q}.$$

Since $\phi(P)$ is a polynomial and P and Q are relatively prime, $\frac{\phi(Q)}{Q}$ must be a polynomial. Let $\phi(Q) = Q\tilde{R}$, $\tilde{R} \in F$ a polynomial. Suppose that Q is nonconstant and take a term of maximal total degree in Q. The corresponding term of $\phi(Q)$ is of the same degree and is a term of maximal degree in $\phi(Q)$. Thus, $\tilde{R}$ must be constant and, therefore, Q must be a constant multiple of $\phi(Q)$. But this is only possible if Q is a constant. The other direction is automatic.

Now we prove the claim. If $a \in F$ and D satisfy (21), since the right-hand side of (21) is a polynomial, by the above, a is also a polynomial. Then,

$$\phi(a) - a = D\left((\phi(I_1) - I_1) \cdot I_2\right) = \tilde{D}\left(\phi(I_1) - I_1\right) \cdot I_2 + R,$$

where $\tilde{D}$ is another Δ-operator with coefficients in $\mathbf{K}$ and $R \in F$ a polynomial with no terms containing $\phi^i(I_2)$, $i \in \mathbb{Z}$ (including I_2 itself). Thus, $\phi(a)$ or a has a term of the form $c \cdot I_2$, with c a polynomial in F not containing I_2. Suppose this term is in a (the other case can be treated similarly). Then it can be seen by induction that $\phi(a)$ must have a term $\phi^n(cI_2)$ for all $n \in \mathbb{N}$, which contradicts $\phi(a) \in F$.

Since $\mathbf{K}$ is algebraically closed, Proposition 3.1 and the descent argument from the proof of Corollary 3.2 of [13] imply that the elements

$$\partial_{t_1}^{i_1} \partial_{t_2}^{i_2} I \in L, \quad i_1,\, i_2 \geqslant 0,$$

are algebraically independent over F. Let $\mathbf{K} \subset L$ be the $\{\Phi, \Delta\}$-subfield generated by

$$\begin{aligned}
&\phi(I_k) - I_k,\ \partial_{t_1} I_m,\ \partial_{t_2} I_m, \quad k = 1,2,\\
&J_1 := \partial_{t_1} I - \partial_{t_1} I_1 \cdot I_2 - I_2/t_1,\\
&J_2 := \partial_{t_2} I - \partial_{t_2} I_1 \cdot I_2 + I_1/t_2.
\end{aligned}$$

Since I satisfies (20) and $J_1, J_2 \in \mathbf{K}$, for all $(i, j_1, j_2) \neq (0,0,0)$, we have

$$(\phi - \mathrm{id})^i \partial_{t_1}^{j_1} \partial_{t_2}^{j_2}(I) \in \mathbf{K}(I_1, I_2).$$

Indeed, we will show this by induction on the triples (i, j_1, j_2), $i, j_1, j_2 \geqslant 0$ ordered degree-lexicographically. We denote the n^{th} triple by a_n, with $a_1 = (0,0,1)$. For $n = 1$,

$$\begin{aligned}
(\phi - \mathrm{id})^0 \partial_{t_1}^0 \partial_{t_2}(I) &= \partial_{t_2}(I) = \\
&= J_2 + \partial_{t_2}(I_1) \cdot I_2 - I_1/t_2 \in \mathbf{K}(I_1, I_2).
\end{aligned}$$

Supposing that the result holds for a_{n-1}, and letting $a_n = (i, j_1, j_2)$, first note that, for $n \geqslant 2$,

$$a_{n-1} = \begin{cases} (i, j_1, j_2 - 1) & \text{if} \quad j_2 \neq 0 \\ (i, j_1 - 1, j_2) & \text{if} \quad j_2 = 0 \text{ and } j_1 \neq 0 \\ (i-1, j_1, j_2) & \text{if} \quad j_1 = j_2 = 0 \text{ and } i \neq 0 \end{cases}$$

In the first case, if $a_{n-1} = (i, j_1, j_2 - 1)$, let

$$(\phi - \mathrm{id})^i \partial_{t_1}^{j_1} \partial_{t_2}^{j_2 - 1}(I) = \frac{P(I_1, I_2)}{Q(I_1, I_2)} \in K(I_1, I_2),$$

where $P, Q \in \mathbf{K}[I_1, I_2]$ and $Q \neq 0$. Then

$$\begin{aligned}
(\phi - \mathrm{id})^i \partial_{t_1}^{j_1} \partial_{t_2}^{j_2}(I) &= \partial_{t_2}\left(\frac{P(I_1, I_2)}{Q(I_1, I_2)}\right) = \\
&= \frac{\partial_{t_2}\left(P(I_1, I_2)\right) \cdot Q(I_1, I_2) - P(I_1, I_2) \cdot \partial_{t_2}\left(Q(I_1, I_2)\right)}{Q(I_1, I_2)^2},
\end{aligned}$$

which is in $\mathbf{K}(I_1, I_2)$ since $\partial_{t_2}(I_k) \in \mathbf{K}$, $k = 1,2$. In the second case, when $a_{n-1} = (i, j_1 - 1, j_2)$, it is similarly shown that

$$(\phi - \mathrm{id})^i \partial_{t_1}^{j_1} \partial_{t_2}^{j_2}(I) \in \mathbf{K}(I_1, I_2).$$

In the third case, when $a_{n-1} = (i - 1, j_1, j_2)$,

$$\begin{aligned}
(\phi - \mathrm{id})^i \partial_{t_1}^{j_1} \partial_{t_2}^{j_2}(I) &= (\phi - \mathrm{id})\left(\frac{P(I_1, I_2)}{Q(I_1, I_2)}\right) = \\
&= \frac{P(\phi(I_1), \phi(I_2))}{Q(\phi(I_1), \phi(I_2))} - \frac{P(I_1, I_2)}{Q(I_1, I_2)} \in \mathbf{K}(I_1, I_2),
\end{aligned}$$

since $\phi(I_k) - I_k \in \mathbf{K}$, $k = 1,2$. When $i < 0$, a similar argument applies. Thus,

$$L = \mathbf{K}(I_1, I_2, I).$$

It follows from Lemma 3.6 of [8] (see also [26, 25]) that I_1, I_2, I are algebraically independent over $\mathbf{K}$ using a characteristic set argument with respect to any difference-differential ranking with $I > I_1 > I_2$ and $\phi > \partial_{t_1} > \partial_{t_2}$ (the corresponding characteristic set C will have leaders that are strictly greater than I in ranking and, therefore, no polynomial in I, I_1, I_2 can be reduced to zero with respect to C). Put

$$f_i := \phi(I_i) - I_i \in \mathbf{K}, \quad i = 1,2$$

and consider the equation

$$\phi(Y) = AY, \quad A := \begin{pmatrix} 1 & f_1 & 0 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Then

$$S := \begin{pmatrix} 1 & I_1 & I \\ 0 & 1 & I_2 \\ 0 & 0 & 1 \end{pmatrix}$$

is a fundamental matrix for equation (22). Hence, L is a PPV extension of $\mathbf{K}$ for equation (22). Let U be the differential algebraic group of matrices of the form

$$g(u_1, u_2, v) = \begin{pmatrix} 1 & u_1 & v \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and G be its differential algebraic subgroup defined by the equations:

$$\partial_{t_i} u_j = 0,\ i = 1, 2,\ j = 1, 2,\ \partial_{t_1} v = \frac{1}{t_1} u_2,\ \partial_{t_2} v = -\frac{1}{t_2} u_1.$$

If we let

$$D_1 := \begin{pmatrix} 0 & 1/t_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/t_2 \\ 0 & 0 & 0 \end{pmatrix},$$

then the defining equations for G turn into

$$\partial_{t_i}(x_{rs}) + [(x_{rs}), D_i], \quad i = 1, 2. \tag{23}$$

Note that $\mathbf{Gal}(L/\mathbf{K})$ is a differential algebraic subgroup in U, where U acts on S by multiplication from the right. For all $g(u_1, u_2, v) \in G$, we have

$$\begin{aligned}
&g(u_1, u_2, v)(\partial_{t_i}(I_i)) = \partial_{t_i}(g(u_1, u_2, v)(I_i)) = \\
&= \partial_{t_i}(I_i + u_i) = \partial_{t_i}(I_i), \\
&g(u_1, u_2, v)(\phi(I_i) - I_i) = \phi(I_i + u_i) - I_i - u_i = \\
&= \phi(I_i) - I_i,\ i = 1, 2, \\
&g(u_1, u_2, v)(J_1) = \partial_{t_1}(I + I_1 u_2 + v) - \\
&\quad - \partial_{t_1}(I_1 + u_1)(I_2 + u_2) - (I_2 + u_2)/t_1 = J_1, \\
&g(u_1, u_2, v)(J_2) = \partial_{t_2}(I + I_1 u_2 + v) - \\
&\quad - \partial_{t_2}(I_1 + u_1)(I_2 + u_2) + (I_1 + u_1)/t_2 = J_2.
\end{aligned}$$

Therefore, $K \subset L^G$. Since

$$\mathbf{k}\{G\} \cong \mathbf{k}[u_1, u_2, v] \quad \text{and} \quad \operatorname{tr.deg}_{\mathbf{K}} L = 3$$

by the above, we have $\mathbf{Gal}(L/\mathbf{K}) = G$ (see Lemma 6.19 and Proposition 6.26 in [13]). Therefore, by Example 4.7 of [10] and Proposition 2, equation (22) is not isomonodromic. On the other hand, by (23) and Proposition 2, equation (22) is isomonodromic with respect to each ∂_{t_i}, $i = 1, 2$, separately with the corresponding matrices given by

$$B_i := S \cdot D_i \cdot S^{-1} + \partial_{t_i} S \cdot S^{-1}, \quad i = 1, 2,$$

that is,

$$B_1 = \begin{pmatrix} 0 & 1/t_1 + \partial_{t_1} I_1 & J_1 \\ 0 & 0 & \partial_{t_1} I_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & \partial_{t_2} I_1 & J_2 \\ 0 & 0 & 1/t_2 + \partial_{t_2} I_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

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