Critical Point Methods and Effective Real Algebraic Geometry: New Results and Trends

Mohab Safey El Din
Université Pierre and Marie Curie
Institut Universitaire de France
INRIA Paris-Rocquencourt
CNRS – LIP6 UMR 7606
Mohab.Safey@lip6.fr

ABSTRACT

Critical point methods are at the core of the interplay between polynomial optimization and polynomial system solving over the reals. These methods are used in algorithms for solving various problems such as deciding the existence of real solutions of polynomial systems, performing one-block real quantifier elimination, computing the real dimension of the solution set, etc.

The input consists of s polynomials in n variables of degree at most D. Usually, the complexity of the algorithms is $(s\,D)^{O(n^\alpha)}$ where α is a constant. In the past decade, tremendous efforts have been deployed to improve the exponents in the complexity bounds. This led to efficient implementations and new geometric procedures for solving polynomial systems over the reals that exploit properties of critical points. In this talk, we present an overview of these techniques and their impact on practical algorithms. Also, we show how we can tune them to exploit algebraic and geometric structures in two fundamental problems.

The first one is real root finding of determinants of n-variate linear matrices of size $k \times k$. We introduce an algorithm whose complexity is polynomial in $\binom{n+k}{k}$ (joint work with S. Naldi and D. Henrion). This improves the previously known $k^{O(n)}$ bound. The second one is about computing the real dimension of a semi-algebraic set. We present a probabilistic algorithm with complexity $(s\,D)^{O(n)}$, that improves the long-standing $(s\,D)^{O(n^2)}$ bound obtained by Koiran (joint work with E. Tsigaridas).

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*algebraic algorithms*; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Non numerical algorithms and problems—*complexity of proof procedures*

General Terms

Theory, algorithms

Keywords

polynomial system solving, real roots, effective real algebraic geometry

Introduction. Many important results in combinatorial and computational geometry (see *e.g.* [12, 27]), in theoretical computer science (see *e.g.* results on non-negative matrix factorization [1] or game theory [18]) rely on effective real algebraic geometry. Polynomial system solving over the reals has also many applications in engineering sciences, *e.g.* in robotics [2], and control theory [11] among other areas.

Typical computational challenges in real algebraic geometry are: deciding the emptiness of semi-algebraic sets, performing geometric operations such as projection (quantifier elimination), answering connectivity queries (roadmaps), computing the real dimension or computing the Euler-Poincaré characteristic, Betti numbers, etc.

Huge efforts have been invested during the last 25 years to derive algorithms that improve the doubly exponential complexity in n of Cylindrical Algebraic Decomposition [13]. This has led to algorithms for deciding the emptiness of semi-algebraic sets (in time $(s\,D)^{O(n)}$) [7], performing one-block quantifier elimination [6], computing the real dimension [22], answering connectivity queries (in time $(s\,D)^{O(n^2)}$) [8, 12]; see [9] for a self-contained overview.

Critical point methods are at the heart of these results. They consist in extracting important properties of semi-algebraic sets from the *critical points* of a well-chosen map. These are points at which the differential (of the map) is not surjective; local extrema of the map are reached at its critical points. These methods were used in combination with the introduction of infinitesimals that deform the input. This allows us to obtain cheap reductions to smooth and bounded semi-algebraic sets but affects the cost of arithmetic operations and hence practical performance.

It has been a long-standing problem to obtain efficient implementations for real-world problems based on critical point methods. Indeed, it requires to improve the exponents in the complexity bounds by introducing new algebraic and geometric techniques to avoid the use of infinitesimals. One successful research direction is to identify properties of critical points or polar varieties and to exploit them computationally using algorithms of elimination theory.

This trend started with [3] and has been developed for a decade (*e.g.* [4, 15, 19, 23, 24] and references therein) to understand the properties of these sets of points. We refer to [5] for an exposition of properties of polar varieties. Once these properties are understood, they can be exploited to design geometric procedures for solving. For instance, the first improvement of the long-standing $O(n^2)$ exponent in the complexity of Canny's probabilistic algorithm [12] to $O(n^{3/2})$ is based on a new geometric connectivity result obtained by investigating properties of polar varieties in [25] (see also [10] for a further generalization to general algebraic sets).

This talk presents an overview of critical point methods. We highlight recent advances that lead to practically fast algorithms for deciding the existence of real solutions of polynomial systems. We describe new ways of exploiting structural properties of critical points and we introduce new geometric procedures to improve the complexity bounds for solving two important problems: *(i)* real root finding of determinants of matrices whose entries are linear

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forms (linear matrices) and *(ii)* computing the real dimension of a semi-algebraic set.

Real root finding of determinants of linear matrices. Let $\mathsf{M}_0, \dots, \mathsf{M}_n$ be matrices of size $k \times k$ with rational entries, $X_1, \dots, X_n$ be variables and $\mathsf{M} = \mathsf{M}_0 + X_1\mathsf{M}_1 + \cdots + X_n\mathsf{M}_n$. We consider the problem of finding real roots of the determinant of M. This is a generalization of the eigenvalue problem. It is also related to simultaneous stabilization problems in control theory (*e.g.* [11]). Moreover, if M is symmetric and of full rank, then $\det(\mathsf{M})$ shapes the boundary of the feasible solution set of the Linear Matrix Inequality $\mathsf{M} \succeq 0$. In this case, exact algorithms for finding real points in the feasible set start by computing points on its boundary.

To compute real roots of the determinant of M, the traditional procedure consists in applying algorithms for deciding the emptiness of the real solution set of the equation $\det(\mathsf{M}) = 0$. Using this strategy, the cost is $k^{O(n)}$ arithmetic operations. Additionally, the equation $\det(\mathsf{M}) = 0$ defines a hypersurface with *generic* singularities (corresponding to rank deficiencies greater than 1).

Our approach consists in studying the variety defined by the bilinear system $\mathsf{M}.\mathbf{Y} = 0$ where $\mathbf{Y}$ is a vector of new homogeneous variables. Under some genericity assumptions on the entries of M, this new set of bi-linear equations defines a smooth algebraic set. We show how to reduce our problem to global optimization problems that preserve the bi-linear structure of the system. We model the global optimization problems using Lagrange multipliers that in turn leads to solve multi-linear polynomial systems. The multihomogeneous bound associated to these systems is dominated by $\binom{n+k}{k}^2$. Using algebraic elimination routines which take advantage of multi-linear structures (*e.g.* [14, 17]), we obtain algorithms whose complexity is polynomial in this quantity. Hence, families of problems where k remains constant can be solved in polynomial time. Preliminary implementations allow to handle problems involving 20 variables (for 6×6 matrices).

Computation of the dimension of semi-algebraic sets. For computing the real dimension of a semi-algebraic set, the best previously known complexity bound, due to Koiran, was $(s\,D)^{O(n^2)}$ [22] in the worst case (see also [28] for a partial improvement). It is based on quantifier elimination techniques, see [9, Alg. 14.10] and references therein. On the other hand, in the complex case, it is well understood that we can compute the (Krull) dimension of an algebraic variety over algebraically closed fields in time $D^{O(n)}$ [16], see also [21].

It is of great interest to know if the problem of computing the dimension admits the same complexity bound in the real case and in the algebraically closed case. This problem also finds applications in geometric modeling and mechanics (see *e.g.* [20]).

We present a probabilistic algorithm for computing the real dimension of a semi-algebraic set $\mathscr{S} \subset \mathbb{R}^n$ in time $(s\,D)^{O(n)}$ [26]. First, we perform a classical reduction to the case of bounded semi-algebraic sets. Next, we consider the critical loci W_i of the restrictions of projections $(x_1, \dots, x_n) \to (x_1, \dots, x_i)$ to a smooth deformation of $\mathscr{S}$. It turns out that these critical loci coincide with $\mathscr{S}$ when $i \geq \dim(\mathscr{S})+1$. The algorithm exploits this structural property. It uses a subroutine that finds the largest integer i such that $W_i \neq W_{i-1}$ in time $(s\,D)^{O(n)}$; this integer equals $\dim(\mathscr{S}) + 1$. Finally, we obtain an algorithm which computes the real dimension of a semi-algebraic set in time $(s\,D)^{O(n)}$.

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