Rainbow Cliques and the Classification of Small BLT-Sets

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ABSTRACT

In Finite Geometry, a class of objects known as BLT-sets play an important role. They are points on the $Q(4, q)$ quadric satisfying a condition on triples. This paper is a contribution to the difficult problem of classifying these sets up to isomorphism, i.e., up to the action of the automorphism group of the quadric. We reduce the classification problem of these sets to the problem of classifying rainbow cliques in graphs. This allows us to classify BLT-sets for all orders q in the range 31 to 67.

Categories and Subject Descriptors

G.2.1 [**Mathematics of Computing**]: Discrete Mathematics—*Combinatorics*; G.2.2 [**Mathematics of Computing**]: Discrete Mathematics—*Graph Theory*; I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*

Keywords

Isomorphism, Classification, Finite Geometry

1. INTRODUCTION

BLT-sets have been introduced in [1] in relation with the study of flocks of a quadratic cone in projective 3-space (cf. [15, 18]). A BLT-set of order q is a set S of $q+1$ points on the parabolic quadric $Q(4, q)$ such that $P^{\perp} \cap Q^{\perp} \cap R^{\perp}$ is empty for any three points P, Q, R in S. BLT-sets are important in finite geometry, due to the connections to translation planes and to generalized quadrangles. We refer to the introduction of [16] for more details. In a curious twist, BLT-sets predate themselves by six years, as they first arose in [7] under the name $(0, 2)$-sets.

It is known that BLT-sets of order q exist if and only if q is odd. Two BLT-sets S and T of order q are equivalent if there is a group element g in the automorphism group of $Q(4, q)$ such that $S^g = T$. It is an important problem to classify all BLT-sets of $Q(4, q)$ up to isomorphism, and the problem is open for $q \geq 31$. It is known that the quadric $Q(4, q)$ has $(q+1)(q^2+1)$ points and that its automorphism group is the orthogonal semilinear group $\mathrm{P\Gamma O}(5, q)$, of order $h(q^4-1)(q^2-1)q^4$, where $q = p^h$ for some prime p and some integer h. We let G denote the group $\mathrm{P\Gamma O}(5, q)$. In this paper, we classify all BLT-sets of orders $q = 31, 37, 41, 43, 47, 49, 53, 59, 61, 67$. The reason for stopping at $q = 67$ is that we have reached the limits of what can be computed using the resources available to us.

q	BLT	F	q	BLT	F
3	1	1	29	9	28
5	2	2	31	8	33
7	2	2	37	7	37
9	3	3	41	10	51
11	4	4	43	6	50
13	3	4	47	10	51
17	6	9	49	8	24
19	5	8	53	8	39
23	9	18	59	9	48
25	6	12	61	5	36
27	6	14	67	6	39

Table 1: The Number of Isomorphism Classes of BLT-Sets (BLT) and Flocks (F) of Order q (The Numbers for $31 \leq q \leq 67$ Are New)

THEOREM 1. *The number of isomorphism types of BLT-sets and of flocks of the quadratic cone in* $\mathrm{PG}(3, q)$ *for any given order* $q \leq 67$ *that is relevant is given in Table 1.*

PROOF. By computer, using the classification algorithm decribed in this paper. In Sections 2,3,4, and 6, we describe our algorithm to search for BLT-sets. In Section 5 we present our algorithm to solve the isomorphism problem for BLT-sets. In Section 7 we present the results from the search. This constitutes our proof of the theorem. A few remarks on the correctness of the result are contained in Section 8. ☐

A brief description of all BLT-sets in this range is given in Appendix A. In Section 9, we describe an invariant, called the plane type, that can distinguish between all isomorphism types of BLT-sets of order at most 67.

2. THE ALGORITHM

Let us now describe our search algorithm and the results of the search. This will constitute the basis for Theorem 1.

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The classification of BLT-sets up to isomorphism suffers from the fact that there is a large number of nonisomorphic partial objects, many of which either do not extend or are embedded in BLT-set in different ways. Thus, classifying partial objects becomes infeasible. On the other hand, doing a search without isomorph rejection is almost sure to fail, since the number of BLT-sets of $Q(4,q)$ (not considering isomorphism) is simply too large. In addition, checking BLT-sets for isomorphism is difficult.

Our approach is a combination of techniques from geometry, group theory and combinatorics. We reduce the problem to a search for rainbow cliques in colored graphs. While this is still NP-hard, it allows us to find and classify all BLT-sets for the orders relevant for Theorem 1.

Our main method is a technique called "breaking the symmetry." This is a strategy to attack classification problems involving symmetry and the resulting issues of isomorphism. The terminology seems to be due to [5], but the methodology is part of the folklore. It is the idea behind the "we may assume" term that is frequently found in proofs.

We separate the search into different stages. The goal is to find suitable *starter configurations* such that every BLT-set can be obtained (in the sense of a practical computation) from at least one of these starter configurations. A good choice for these starter configurations are partial BLT-sets of a certain size. A partial BLT-set is simply a set S of points on $Q(4,q)$ such that $P^\perp \cap Q^\perp \cap R^\perp$ is empty for any three points P, Q, R in S. Thus, a BLT-set is a partial BLT-set of size $q+1$ and every subset of a BLT-set is a partial BLT-set. We take as starter configurations the partial BLT-sets of size s, for some reasonably chosen integer s. In Section 3, the starter configurations are classified up to isomorphism. The orbits of starter configurations are called *starters,* and the chosen representatives of these orbits are called *starter sets.* The integer s is chosen so that the average starter configuration has only a very small automorphism group, but at the same time the number of starters is reasonably small ($s=5$ seems to work well). The fact that starter configurations have small average stabilizers explains the terminology of breaking the symmetry.

In a second step, described in Section 4, each starter set is considered in turn and all BLT-sets containing this set are constructed using a technique from graph theory, called rainbow cliques. Computationally, this is the dominant part of the algorithm. To this end, a graph Γ_S is defined in such a way that all BLT-sets B containing S correspond to cliques of a certain size in Γ_S. In fact, we can find a colored graph $\Gamma_{S,\ell}$ such that the BLT-sets B containing S correspond to the rainbow cliques in $\Gamma_{S,\ell}$.

In Section 5 we perform the isomorphism testing to classify the BLT-sets that arise by means of rainbow cliques. In Section 6 we describe a technique to speed up the search, using the lexicographical ordering of subsets.

3. STARTER CONFIGURATIONS

In order to compute the orbits of partial BLT-sets of a fixed size s under the symmetry group of $Q(4,q)$, we employ the orbit algorithm "Snakes and Ladders," due to Schmalz [22]. The algorithm builds up a data structure that stores all orbit representatives and the associated automorphism groups. We choose the orbit representative to be the lexicographically least sets in their respective orbits, relative to the ordering induced from some total ordering of the points of $Q(4,q)$. The algorithm also stores additional data that can be used to identify the representative R of any partial BLT-set S of size at most s quickly and to provide a group element $g \in G$ such that $S^g = R$.

q	$\lvert Q(4,q)\rvert$	$\lvert P\Gamma O(5,q)\rvert$	5-Orbits	Time
31	$30,784$	8.1×10^{14}	$2,693$	52 sec
37	$52,060$	4.8×10^{15}	$6,739$	2 min 52 sec
41	$70,644$	1.3×10^{16}	$11,478$	6 min 24 sec
43	$81,400$	2.1×10^{16}	$14,693$	8 min 33 sec
47	$106,080$	5.2×10^{16}	$23,312$	17 min 31 sec
49	$120,100$	1.5×10^{17}	$14,542$	14 min 10 sec
53	$151,740$	1.7×10^{17}	$43,465$	44 min 36 sec
59	$208,920$	5.1×10^{17}	$75,707$	1 hr 46 min
61	$230,764$	7.1×10^{17}	$89,954$	2 hrs 14 min
67	$305,320$	1.8×10^{18}	$146,009$	4 hrs 50 min

Table 2: Summary of the Classification of Starters

Table 2 gives information about the classification of partial BLT-sets. In the table, we list the number of 5-orbits in each of the cases $q = 31, \ldots, 67$. These are the orbits on partial BLT-sets of size 5.

4. RAINBOW CLIQUES

Given a partial BLT-set S, we employ techniques from graph theory to find all BLT-sets containing S. For each partial BLT-set S, we define a graph Γ_S. The vertices of Γ_S are the points of $Q(4,q)$ that are admissible. A point P is admissible if $S \cup \{P\}$ is a partial BLT-set also. The edges in Γ_S are between points P and Q such that $S \cup \{P, Q\}$ is partial BLT-set. It is clear that cliques of size $q+1-s$ in Γ_S are necessary for the existence of a BLT-set T containing the starter S. Interestingly, the existence of such cliques is also sufficient for the existence of a BLT-set T containing S:

LEMMA 1. *Let S be a partial BLT-set of size s in the $Q(4,q)$ quadric, and let Γ_S be the associated graph as defined above. Then*

1. *The stabilizer of S induces a group of automorphisms of Γ_S.*
2. *The BLT-sets T containing S are in one-to-one correspondence to the cliques of Γ_S of size $q+1-s$.*

PROOF. The first statement is clear, so we look at the second. It follows from the definition of the graph Γ_S that a BLT-set T containing S gives rise to a clique of size $q+1-s$ in Γ_S. For the converse, we need to show that a clique C in Γ_S of size $q+1-s$ defines a BLT-set $T := S \cup C$. For this purpose, we need to look at all triples $P, Q, R \in T$. The only interesting case is when $P, Q, R \in C$. In this case, pick a point $P_0 \in S$. The presence of the three edges PQ, PR, and QR in Γ_S implies that the triples P_0PQ, P_0PR and P_0QR are partial BLT-sets. By [2, Lemma 4.3], the triple PQR is partial BLT, too. $\square$

Thus we have reduced the problem of finding BLT-sets containing a given starter into the problem of finding certain

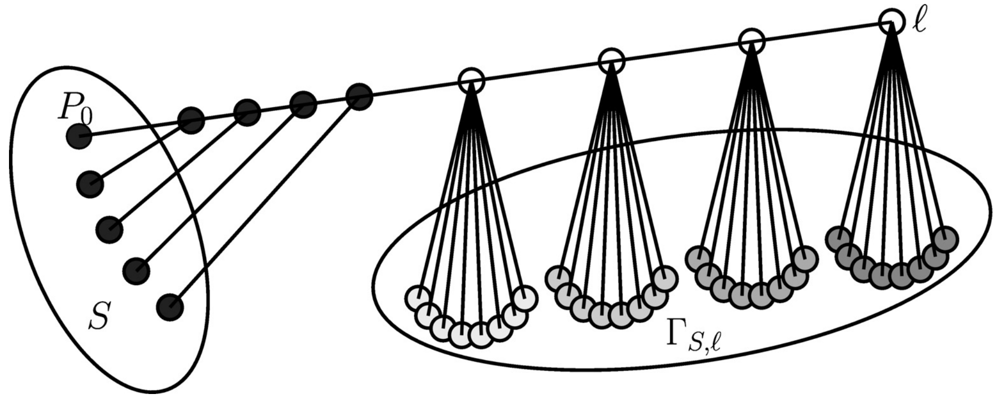

Figure 1: The Colored Graph $\Gamma_{S,\ell}$ (Edges Not Shown)

cliques in a certain graph. We will now consider a refinement of this technique, based on a coloring of the graph Γ_S. Since $Q(4,q)$ is a generalized quadrangle [19], a point not on a line is collinear with exactly one point on that line (this is true for any non-incident point/line pair). We can define a colored graph $\Gamma_{S,\ell}$ as follows. Suppose we fix a point $P_0 \in S$, together with a line ℓ of $Q(4,q)$ that passes through P_0. Let $\mathcal{C}$ be the points on ℓ that are not collinear to any of the points of S other than P_0. Any point P in Γ_S is collinear to exactly one point $Q \in \mathcal{C}$. So, by labeling the point P with the color Q, we find that Γ_S can be colored with the points in $\mathcal{C}$. This gives a new graph $\Gamma_{S,\ell}$ (cf. Fig. 1). The next result shows that the cliques T containing S are in one-to-one correspondence to the rainbow cliques in $\Gamma_{S,\ell}$. Here, a rainbow clique is a clique that intersects each color class in exactly one element.

Lemma 2. *Let S be a partial BLT-set of size s in the $Q(4,q)$ quadric, and let Γ_S be the associated graph as defined above. Let $P_0 \in S$ and let ℓ be a line of $Q(4,q)$ containing P_0. Let $\Gamma_{S,\ell}$ be the graph Γ_S after coloring the vertices according to the line ℓ as described above. Then*

1. *The stabilizer of S induces a group of automorphisms of $\Gamma_{S,\ell}$ that permutes the color classes among themselves.*
2. *The BLT-sets T containing S are in one-to-one correspondence to the cliques of $\Gamma_{S,\ell}$ of size $q+1-s$ with one vertex from each color class.*

Proof. The first statement is clear. Regarding the second, we observe that the collinearity relation establishes a bijection between the $s-1$ points of $S \setminus \{P_0\}$ and some $s-1$ points on $\ell \setminus \{P_0\}$. For this, observe that S is partial BLT, and hence no two points P_i, P_j of $S \setminus \{P_0\}$ are collinear to a point $Q \in \ell \setminus \{P_0\}$, for otherwise $Q \in P_i^\perp \cap P_j^\perp \cap P_0^\perp$. Thus, the set $\mathcal{C}$ of points on $\ell \setminus \{P_0\}$ *not* collinear with S has size $q+1-s$. In any BLT-set T containing S, the relation of collinearity induces a bijection between the points on the line ℓ different from P_0 and the points in $T \setminus \{P_0\}$. The points in $S \setminus \{P_0\}$ are paired to some $s-1$ points on $\ell \setminus \{P_0\}$. Thus, the points in $T \setminus S$ are paired to the points in $\mathcal{C}$. This shows that $T \setminus S$ (thought of as a set of vertices in the graph $\Gamma_{S,\ell}$) intersects each color class in exactly one element. By Lemma 1, we know that T is a clique. Therefore, T is a rainbow clique of size $q+1-s$ in $\Gamma_{S,\ell}$. $\square$

Rainbow cliques are easier to search for than ordinary cliques. One reason is that the size of the clique is known beforehand, and thus a condition on the number of neighbors of any live point in the search can be facilitated. The second reason is that the color classes can be used to cut down the size of the choice set encountered at each stage of the backtracking. In Section 6 below, we will use the lexicographic ordering to improve the search algorithm yet again.

We observe:

Corollary 1. *There is no BLT-set T containing the partial BLT-set S if, for some line ℓ with $|\ell \cap S| = 1$, the graph $\Gamma_{S,\ell}$ has an empty color class.*

5. ISOMORPH TESTING

Let G be a group acting on finite sets $\mathcal{X}$ and $\mathcal{Y}$, and let $\mathcal{R}$ be a relation between $\mathcal{X}$ and $\mathcal{Y}$ (i.e., a subset of $\mathcal{X} \times \mathcal{Y}$). Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be the orbits of G on $\mathcal{X}$ and let $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ be the orbits of G on $\mathcal{Y}$. Let P_i be a representative of orbit $\mathcal{P}_i$ $(i = 1, \ldots, m)$ and let Q_j be a represenative of orbit $\mathcal{Q}_j$ $(j = 1, \ldots, n)$. Let $\mathcal{T}_{i,1}, \ldots, \mathcal{T}_{i,u_i}$ be the orbits of $\mathrm{Stab}(P_i)$ on the set $\{Y \in \mathcal{Y} \mid (P_i, Y) \in \mathcal{R}\}$, with representatives $T_{i,j}$ for $1 \le i \le m$ and $1 \le j \le u_i$. Let $\mathcal{S}_{j,1}, \ldots, \mathcal{S}_{j,v_j}$ be the orbits of $\mathrm{Stab}(Q_j)$ on the set $\{X \in \mathcal{X} \mid (X, Q_j) \in \mathcal{R}\}$, with representatives $S_{j,i}$ for $1 \le j \le n$ and $1 \le i \le v_j$.

Lemma 3. *There is a bijection between the orbits*

$$\{\mathcal{S}_{j,i} \mid j = 1, \ldots, n,\ i = 1, \ldots, v_j\}$$

and the orbits

$$\{\mathcal{T}_{i,j} \mid i = 1, \ldots, m,\ j = 1, \ldots, u_i\}.$$

Proof. Consider the action of G on $\mathcal{R}$. Both sets are in one-to-one correspondence to the orbits of G on pairs $(X,Y) \in \mathcal{X} \times \mathcal{Y}$ with $(X,Y) \in \mathcal{R}$. $\square$

The *decomposition matrix* is the $m \times n$ integer matrix whose (i,j) entry counts the number of h such that $\mathcal{T}_{i,h}$ is paired with an orbit $\mathcal{S}_{j,k}$ for some k.

For the purposes of this paper, *two relations* are important. The first one, denoted as $\mathcal{R}$, is between partial BLT-sets of size s and partial BLT-sets of size $q+1$. The relation is inclusion of sets. The group $G = \mathrm{P\Gamma O}(5, q)$ from above acts on $\mathcal{R}$ component-wise. The second relation will be introduced in Section 6.

Suppose that, for a fixed order q, we have classified the G-orbits $\mathcal{P}_1, \ldots, \mathcal{P}_m$ of partial BLT-sets of $Q(4, q)$ of size s (with representatives $P_i \in \mathcal{P}_i$). Suppose further that we have computed for each i with $1 \leq i \leq m$ the orbits $\mathcal{T}_{i,1}, \ldots, \mathcal{T}_{i,u_i}$ of $\mathrm{Aut}(P_i)$ on the BLT-sets containing P_i, with representatives $T_{i,j} \in \mathcal{T}_{i,j}$. The following algorithm computes the orbits $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ of BLT-sets, with representatives $Q_i \in \mathcal{Q}_i$ and stabilizers $\mathrm{Aut}(Q_i)$.

ALGORITHM 1. **(Classification)**
Input: *Orbits $\mathcal{P}_1, \ldots, \mathcal{P}_m$ of partial BLT-sets of size s, with representatives P_i and automorphism groups $\mathrm{Aut}(P_i)$.*
For each $i = 1, \ldots, m$, the set of orbits $\mathcal{T}_{i,j}$ $(1 \leq j \leq u_i)$ of BLT-sets containing P_i under the group $\mathrm{Aut}(P_i)$. Representatives $T_{i,j}$ of the orbit $\mathcal{T}_{i,j}$ and groups $\mathrm{Stab}_{\mathrm{Aut}(P_i)}(T_{i,j})$ for $j = 1, \ldots, u_i$.
Output: *Representatives $Q_1, \ldots, Q_n$ for the G-orbits of BLT-sets of $Q(4, q)$, together with their stabilizers.*

1. *Initialize by marking all orbits $\mathcal{T}_{i,j}$ as unprocessed.*
2. *Consider the first/next unprocessed $\mathcal{T}_{i,j}$.*
3. *Mark $\mathcal{T}_{i,j}$ as processed.*
4. *Define a new isomorphism type $\mathcal{Q}$ of BLT-sets represented by $Q = T_{i,j}$. Record $\mathrm{Stab}_{\mathrm{Aut}(p_i)}(T_{i,j})$ as subgroup of $\mathrm{Aut}(Q)$.*
5. *Loop over all s-subsets of $T_{i,j}$*
6. *Let S be the first/next unprocessed s-subset of $T_{i,j}$.*
7. *Determine the index a such that the partial BLT-set S lies in the orbit $\mathcal{P}_a$.*
8. *Determine a group element $g_1 \in G$ such that $S^{g_1} = P_a$.*
9. *Determine the index $b \leq u_a$ such that $T_{i,j}^{g_1}$ is contained in the orbit $\mathcal{T}_{a,b}$ of BLT-sets containing P_a.*
10. *Determine a group element $g_2 \in \mathrm{Aut}(P_a)$ such that $T_{i,j}^{g_1 g_2} = T_{a,b}$.*
11. *If $a = i$ and $b = j$, record $g_1 g_2$ as generator for $\mathrm{Aut}(Q)$.*
12. *Otherwise, mark $T_{a,b}$ as processed. Store the index pair (i, j) and the group element $\theta_{a,b} := g_1 g_2$ with it.*
13. *Continue with the next s-subset of $T_{i,j}$ in step 6.*
14. *Continue with the next unprocessed orbit $\mathcal{T}_{i,j}$ in step 2.*

In practice, Steps 7 and 8 will be performed in parallel, as well as Steps 9 and 10. After the algorithm terminates, the $Q := T_{i,j}$ associated to $\mathcal{T}_{i,j}$ considered in Step 2 form a system of representatives for the G-orbits on BLT-sets. The automorphismgroup $\mathrm{Aut}(Q)$ is obtained by extending $\mathrm{Stab}_{\mathrm{Aut}(P_i)}(T_{i,j})$ from Step 4 with all elements $\theta_{a,b}$ encountered in Step 11.

In Step 5, we loop over all subsets of the new BLT-set $Q = T_{i,j}$. This can be improved as follows. Consider the orbits of the group $\mathrm{Stab}_{\mathrm{Aut}(P_i)}(T_{i,j})$ as computed in Step 4 on s-subsets of Q. The loop in Step 5 is over the orbit representatives under this group. In fact, as this group gets extended by automorphisms $g_1 g_2$ in Step 11, we take orbit representatives under that larger group. This will help reduce the number of s-subsets that need to be considered, in particular for BLT-sets with a large automorphism group.

6. LEX-REDUCTION

We can use the lexicographic ordering of points of the $Q(4, q)$ quadric to reduce the number of times that representatives from the same isomorphism type of BLT-sets are constructed. Choose any total ordering of points of the $Q(4, q)$ quadric. Consider the lexicographic ordering of subsets induced from this total ordering of points.

Let us introduce the following two notations.

Let G act on the finite, totally ordered set X. Let S and T be subsets of X with $S \subseteq T$. Let $\max S$ be the largest element of S. The pair (S, T) is *admissible* if $T \setminus S$ intersects trivially the $\mathrm{Aut}(S)$-orbits of the points $1, \ldots, \max S$. Also, if A is a subset of X, the *prefix of size s* of A is the lex-least s-subset of A. Thus, if $A = \{a_1, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$, then the prefix of size s of A is $\{a_1, \ldots, a_s\}$.

Observe that (S, T) is not admissible if and only if there is an element $x \in T \setminus S$ such that $x^g < \max S$ for some $g \in \mathrm{Stab}(S)$.

We consider the relation $\mathcal{R}_2$ consisting of pairs (S, T) with S and T partial BLT-sets of size s and $q+1$ respectively, and (S, T) admissible. The relation $\mathcal{R}_2$ is obtained by restricting $\mathcal{R}$ to admissible pairs. However, $\mathcal{R}_2$ is no longer G-invariant. We will see that we can modify Algorithm 1 to classify orbits on BLT-sets of size $q + 1$.

Let Γ_S^* be the graph whose vertices are the vertices $x \in \Gamma_S$ such that for no group element $g \in \mathrm{Aut}(S)$ we have $x^g < \max S$. In the same way, let $\Gamma_{S,\ell}^*$ be the graph obtained from $\Gamma_{S,\ell}$. We say that Γ_S^* ($\Gamma_{S,\ell}^*$, resp.) is obtained from Γ_S ($\Gamma_{S,\ell}$, resp.) by *lex-reduction.*

If lex-reduction is used, not all clique orbits associated to a given starter are present. Therefore, it is necessary to modify Algorithm 1. We replace Steps 7-10 of Algorithm 1 by the following steps. The idea is the following. Given a BLT-set T, determine the prefix S of T (assume that S is a representative, otherwise move T using a group element $g \in G$). If T cannot be found among the stored BLT-sets associated to S, then (S, T) is not admissible, and we can find a group element $g \in \mathrm{Aut}(S)$ such that T^g has a lexicographically smaller prefix S' and we repeat (with Step 3). This procedure must terminate because the number of sets that preceed S in the lexicographic order is finite.

1. Let T be the set $T_{i,j}$. Let $h := id$.
2. Repeat the following loop:
3. Let S be the prefix of size s of T.
4. Determine the index a such that $S \in \mathcal{P}_a$. Let g_1 be a group element such that $S^{g_1} = P_a$.

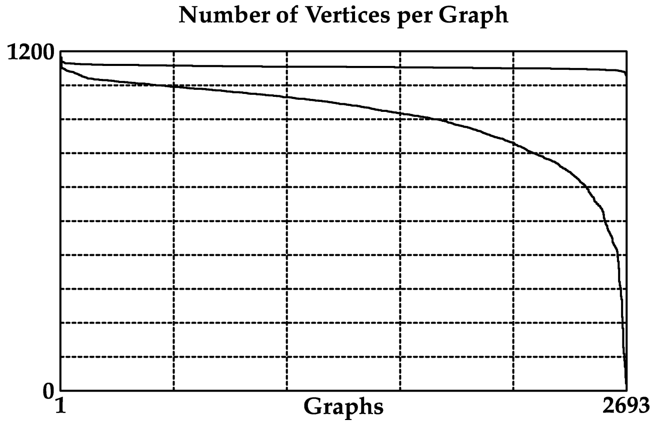

Figure 2: Distribution of the Number of Vertices per Graph when $q = 31$: Top Curve is Without Lex-Reduction, Bottom Curve is With Lex-Reduction

5. Try to find the orbit $\mathcal{T}_{a,b}$ containing T with $1 \le b \le u_a$.
6. If that orbit exists, break out of the loop and go to step 8.
7. If no such orbit exists, T must contain an element x such that $x^g < \max S$ for some $g \in \mathrm{Aut}(P_a)$. Replace T by T^g and h by hg and repeat with Step 3.
8. Replace g_1 by hg_1 and find a group element g_2 such that $T_{i,j}^{g_1 g_2}$ is the orbit representative $T_{a,b}$.

We illustrate the effect of lex-reduction by an example. We consider the partial BLT-sets of size $s = 5$ of $Q(4, 31)$. Recall from Table 2 that we have 2,693 starters in this case, each associated with one graph. In Figure 2, we plot the distribution of the number of vertices in these graphs (we arrange the graphs in order of increasing number of vertices). The top curves shows the number of vertices in the graphs $\Gamma_{S,\ell}$. The bottom curve shows the number of vertices in the graphs $\Gamma^*_{S,\ell}$, i.e., after the lex-reduction has been applied to the vertex set. In Figure 3, we plot the distribution of the number of rainbow cliques in these graphs (we arrange the graphs in order of increasing number of cliques). Again, the top curve corresponds to the graphs $\Gamma_{S,\ell}$ while the bottom curve corresponds to the graphs $\Gamma^*_{S,\ell}$ obtained from lex-reduction. The total number of cliques is 180,816 in the first case and 19,989 in the latter. These numbers can be thought of as the area under the curves. Without lex-reduction, all but one graph have cliques. With lex-reduction, 960 graphs have no cliques. In each of the two cases, one graph has a large number of cliques. Without lex-reduction, this graph has 485 cliques. After lex-reduction, 416 cliques remain.

Since the lex-reduction removes whole orbits under $\mathrm{Aut}(S)$, the stabilizer of S, the group $\mathrm{Aut}(S)$ acts as a group of symmetries on the resulting graphs Γ^*_S and $\Gamma^*_{S,\ell}$. Thus, $\mathrm{Aut}(S)$ also acts on the set of cliques (rainbow cliques, respectively) of these graphs. Considering the case $q = 31$ once again, Figure 4 shows the distributions of the number of orbits under $\mathrm{Aut}(S)$ on rainbow cliques per graph over all starter sets S. The top curve is without lex-reduction, while the bottom

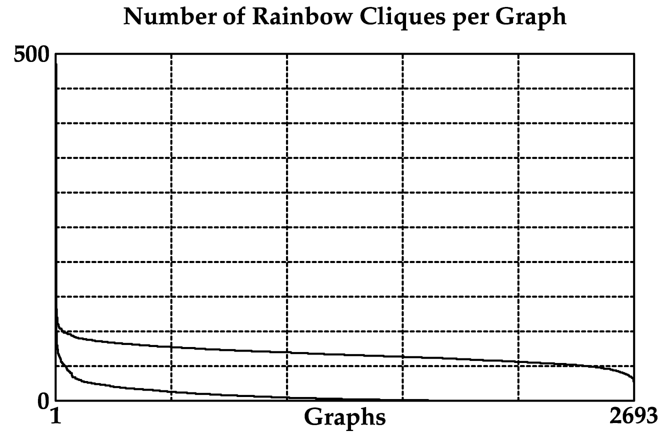

Figure 3: Distribution of the Number of Rainbow Cliques per Graph when $q = 31$: Top Curve is Without Lex-Reduction, Bottom Curve is With Lex-Reduction

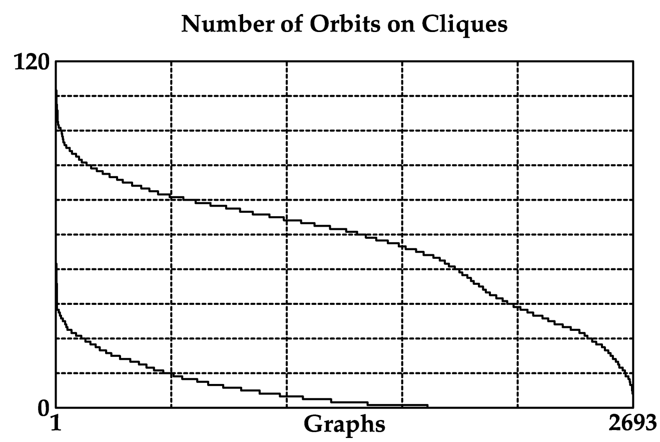

Figure 4: Distribution of the Number of Orbits on Rainbow Cliques per Graph when $q = 31$: Top Curve is Without Lex-Reduction, Bottom Curve is With Lex-Reduction

q	Cliques	Time
31	19,989	9 min 8 sec
37	39,969	1 hr 13 min
41	82,156	4 hrs 58 min
43	94,797	10 hrs 5 min
47	154,377	13 days
49	62,618	11 days
53	126,824	85 days
59	249,466	473 days
61	208,710	921 days
67	304,604	16 years

Table 3: Cumulative Number of Rainbow Cliques Obtained by Exhaustive Search Over All Graphs $\Gamma^*_{S,\ell}$ Associated to Starter Sets S (i.e., with Lex-Reduction)

q	Orbits on Rainbow Cliques
31	15,893
37	32,743
41	68,078
43	79,746
47	131,728
49	51,565
53	107,409
59	216,140
61	181,460
67	265,461

Table 4: Cumulative Number of Stabilizer Orbits on Rainbow Cliques with Lex-Reduction

curve is with lex-reduction. In total, there are $152,402$ orbits without lex-reduction and $15,893$ orbits with lex-reduction.

7. RESULTS OF THE SEARCH

Table 3 displays the results of the search for rainbow cliques from the starters computed in Section 3, using the lexicographic condition. For each order q, the table shows the total number of cliques that arise from the graphs $\Gamma^*_{S,\ell}$, where S runs over a system of representatives of starters and ℓ is a line on a point $P_0 \in S$. The table also shows the time to compute these cliques.

Table 4 shows the cumulative number of orbits of the groups $\mathrm{Aut}(S)$ on the cliques associated with $\Gamma^*_{S,\ell}$, where S ranges over all starter sets.

8. CORRECTNESS OF THE RESULTS

Lemma 3 allows for an easy check of correctness of the classification when lex-reduction is *not* used. In terms of the decomposition matrix (see Section 5), we sum up the entries in two different ways (by rows and by columns). We simply check if the cumulative number of $\mathrm{Aut}(S)$-orbits on cliques over all starter sets S of size s equals the cumulative number of orbits of $\mathrm{Aut}(T)$ on s-subsets of T where T runs through a set of representatives of the BLT-sets obtained from the classification in Section 7. For instance, when $q = 31$, the number of 5-orbits of $\mathrm{Aut}(T)$ for each of the 8 BLT-sets T is shown in the middle column of Table 5. The fact that we

BLT-set	5-Orbits	Special 5-Orbits
31#1	11	1
31#2	387	70
31#3	2,490	269
31#4	50,449	5343
31#5	25,277	2573
31#6	50,344	5114
31#7	20,245	2111
31#8	3,199	412
Total:	152,402	15,893

Table 5: Orbits on 5-Subsets for each of the BLT-sets of $Q(4,31)$ (Labeling of BLT-sets according to Appendix A)

have $152,402$ orbits in total, which is the same as the number of orbits on rainbow-cliques as described in Section 6 (when lex-reduction is not used) corroborates the correctness of our classification in this case.

If lex-reduction is used, we must proceed differently. An s-orbit represented by an s-subset O of a BLT-set T is called *special* if it has the following property: Let $g \in G$ be such that $O^g = S$ is one of the chosen starter sets (such an element g exists since O is a partial BLT-set of size s and since the starters provide an exhaustive list of all partial BLT-sets of size s). The s-orbit O is special if $(S, (T \setminus O)^g)$ is admissible (in the sense of Section 6). In the third column of Table 5, we list the number of special 5-orbits for each of the 8 BLT-sets of order 31. The total number of these special orbits is $15,893$, which equals the number of orbits on rainbow cliques when lex-reduction is used (as shown in Section 6). This is evidence that the classification algorithm is correct in the case that lex-reduction is used.

9. THE PLANE TYPE

It is helpful to have invariants to distinguish between non-isomorphic BLT-sets. Let B be a BLT-set of $Q(4,q)$ inside $\mathrm{PG}(4,q)$. Let a_i be the number of planes π of $\mathrm{PG}(4,q)$ such that $|\pi \cap B| = i$. The plane type of B is the vector $(a_0, \ldots, a_{q+1})$. Counting all planes yields $\sum_{i=0}^{q+1} a_i = (q^4+q^3+q^2+q+1)(q^2+1)$. For this reason, we may omit the coefficient a_0. In addition, it suffices to list only the non-zero coefficients among the $(a_1, \ldots, a_{q+1})$. We list these coefficients in exponential notation i^{a_i}. For instance (a^m, b^n, c^o) means that there are m planes intersecting in a points, n planes intersecting in b points and o planes intersecting in c points. Several families of BLT-sets can be identified by their plane type. For instance, the linear BLT-set consists of a conic on a plane. The Fisher BLT-set consists of two halved conics on two planes. The FTWKB BLT-set has no 4 points coplanar.

10. FINAL REMARKS

It is possible to formulate the problem of finding all BLT-sets above a starter set as an exact cover problem, and then hand it over to Knuth's dancing links (DLX) algorithm [14]. It should be observed though that the size of the system tends to get large. The rows correspond to the points on all lines through points of the starter sets that are not collinear to a point of the starter set. The columns correspond to the vertices in the graph Γ_S. It seems that the large input size of this system makes this approach impractical. Also, if only one line would be used, then the covering might not be a BLT-set, since the system would not have sufficient information to tell whether two points can be added simultaneously. We have not yet discussed how to choose the line ℓ that gave us the colored graphs. We did not try very hard: We chose the first line on the first point in the starter set, using a certain total order on the lines of $Q(4, q)$. We do not know if a clever choice of line would make the algorithm faster.

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APPENDIX

A. TABLES OF BLT-SETS

In Tables 5-6, we list all BLT-sets of orders 31-67. For each BLT-set $\mathcal{B}$, we indicate: 1. A unique identifier of the form $q\#i$, with q the order and i a number that we assign to distinguish BLT-sets of order q. 2. The order of the automorphism group $A = \mathrm{Aut}(\mathcal{B})$ of $\mathcal{B}$. 3. The orbit structure of A on $\mathcal{B}$. 4. The plane type of $\mathcal{B}$ (omitting 3-planes, 2-planes and 1-planes, as they can be computed). 5. The common name of $\mathcal{B}$ (if there is one). We use the following shortcuts: Orb = Orbit structure on points, Ago = Automorphism group order, PT = Plane type, L = Linear, Fi = [8], DCH = [6], PR = [21], K1 = [12], K2 = [13], K3 = GJT / [12], FTWKB = Fisher, Thas [8] / Kantor [11] / Walker [23] / Betten [4], LP = [17], KS = Kantor semifield, G = [9], GJT = [10], DCP = De Clerck, Penttila, unpublished result.

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ID	Ago	Orb	PT	Ref
31#1	1904640	(32)	(32^1)	L
31#2	2048	(32)	$(4^{64}, 16^2)$	Fi
31#3	96	$(24, 6, 2)$	$(4^{342}, 5^8, 8^1)$	LP96
31#4	4	$(4^7, 2^2)$	$(4^{142}, 5^4)$	LP4b
31#5	8	$(8^3, 4^2)$	$(4^{117}, 5^8)$	LP8
31#6	4	(4^8)	$(4^{136}, 5^4)$	LP4a
31#7	10	$(10^2, 5^2, 2)$	(4^{128})	LP10
31#8	64	(32)	(4^{145})	[20]
37#1	3846816	(38)	(38^1)	L
37#2	2888	(38)	(19^2)	Fi
37#3	4	$(4^9, 2)$	$(4^{190}, 5^8)$	LP4a
37#4	4	$(4^9, 1^2)$	(4^{230})	LP4b
37#5	4	$(4^9, 2)$	(4^{210})	New
37#6	72	$(36, 2)$	(4^{216})	K2
37#7	72	$(36, 2)$	$(4^{270}, 5^{12})$	LP72
41#1	5785920	(42)	(42^1)	L
41#2	3528	(42)	(21^2)	Fi
41#3	2	(2^{21})	$(4^{258}, 5^6)$	New
41#4	3	(3^{14})	$(4^{210}, 5^9)$	New
41#5	8	$(8^5, 2)$	$(4^{276}, 5^{16}, 6^2)$	LP
41#6	24	$(24, 12, 6)$	$(4^{228}, 5^{12}, 6^1)$	LP
41#7	60	$(30, 12)$	$(4^{210}, 5^{30})$	LP
41#8	84	(42)	$(4^{126}, 7^6)$	[3]
41#9	84	(42)	(4^{147})	[20]
41#10	68880	(42)	$()$	FTWKB
43#1	6992832	(44)	(44^1)	L
43#2	3872	(44)	$(4^{121}, 22^2)$	Fi
43#3	2	$(2^{21}, 1^2)$	$(4^{297}, 5^9)$	New
43#4	4	$(4^{10}, 2^2)$	(4^{275})	New
43#5	4	(4^{11})	(4^{306})	LP
43#6	84	$(42, 2)$	(4^{210})	K
47#1	9962496	(48)	(48^1)	L
47#2	4608	(48)	$(4^{144}, 24^2)$	Fi
47#3	2304	(48)	$(4^{1656}, 6^{32}, 8^{18})$	DCP
47#4	2	$(2^{23}, 1^2)$	$(4^{371}, 5^{10})$	New
47#5	3	(3^{16})	$(4^{276}, 5^{15})$	LP
47#6	8	$(8^5, 4^2)$	$(4^{327}, 5^4)$	New
47#7	12	$(12^3, 6^2)$	$(4^{267}, 5^6, 6^2)$	New
47#8	24	(24^2)	(4^{384})	LP
47#9	92	$(46, 2)$	(4^{207})	K
47#10	103776	(48)	$()$	FTWKB
49#1	23520000	(50)	(50^1)	L
49#2	10000	(50)	(25^2)	Fi
49#3	940800	(50)	(8^{350})	KS
49#4	20	$(20^2, 5^2)$	(4^{420})	LP
49#5	40	$(40, 10)$	(4^{450})	LP
49#6	8	$(8^6, 2)$	$(4^{356}, 5^8)$	New
49#7	8	$(8^6, 2)$	(4^{408})	New
49#8	200	(50)	$(4^{350}, 5^{10})$	[20]

Figure 5: The BLT-sets, Part I

ID	Ago	Orb	PT	Ref
53#1	16072992	(54)	(54^1)	L
53#2	5832	(54)	(27^2)	Fi
53#3	3	(3^{18})	$(4^{387}, 5^{12})$	New
53#4	12	$(12^4, 6)$	$(4^{381}, 5^{24})$	LP
53#5	24	$(24^2, 6)$	$(4^{540}, 6^1)$	LP
53#6	8	$(8^6, 4, 2)$	$(4^{414}, 5^{20})$	New
53#7	104	$(52, 2)$	(4^{390})	K2
53#8	148824	(54)	$()$	FTWKB
59#1	24638400	(60)	(60^1)	L
59#2	7200	(60)	$(4^{225}, 30^2)$	Fi
59#3	8	$(8^7, 4)$	$(4^{537}, 5^8, 6^2)$	New
59#4	3	(3^{20})	$(4^{453}, 5^{24})$	New
59#5	120	(60)	$(4^{240}, 6^{30})$	LP
59#6	120	(60)	$(4^{390}, 5^{12})$	[20]
59#7	5	(5^{12})	(4^{600})	LP
59#8	24	$(24^2, 12)$	(4^{564})	LP
59#9	205320	(60)	$()$	FTWKB
61#1	28138080	(62)	(62^1)	L
61#2	7688	(62)	(31^2)	Fi
61#3	4	$(4^{15}, 2)$	(4^{582})	New
61#4	4	$(4^{15}, 1^2)$	$(4^{572}, 5^{12})$	New
61#5	124	(62)	(4^{496})	[20]
67#1	40894656	(68)	(68^1)	L
67#2	9248	(68)	$(4^{289}, 34^2)$	Fi
67#3	4	$(4^{16}, 2)$	(4^{761})	New
67#4	4	(4^{17})	(4^{686})	New
67#5	68	(68)	(4^{1122})	New
67#6	132	$(66, 2)$	(4^{462})	K2

Figure 6: The BLT-sets, Part II

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