Hypergeometric generating functions and series for $1/\pi$

James G. Wan
Singapore University of Technology and Design
james_wan@sutd.edu.sg

Introduction

Some truly innovative series for $1/\pi$, first discovered by Ramanujan and elucidated in [1], take the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(\frac{1}{2})_n(1-s)_n}{n!^3}(a+bn)z_0^n = \frac{1}{\pi}. \tag{1}$$

In other words, the constant $1/\pi$ can be written as a suitable linear combination of a hypergeometric function (in this case a ${}_3F_2$) and its derivative at some z_0. Such series have both theoretical and practical applications. In a recent preprint [3], some double sums are conjectured to also evaluate to $1/\pi$; we aim to prove them using the theory behind (1). Examples of these sums include

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\binom{2n-2k}{n-k}\binom{2k}{k}\binom{2n}{n}\frac{140n+19}{2^{6k}}\left(\frac{2}{17}\right)^{2n} = \frac{289}{3\pi}, \tag{2}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}^2\binom{2n}{n}\frac{12n+1}{6^{2k}}\left(\frac{3}{20}\right)^{2n} = \frac{75}{8\pi}. \tag{3}$$

Method

Instead of ${}_3F_2$'s, these conjectural series in [3] all contain functions of the form

$$G(x,z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} F(n,k)x^n z^k,$$

where F is a product of four or more binomial coefficients. It is routine to find a differential equation in x satisfied by G; however such ODEs have degrees ≥ 4 and current CAS struggle to find or rule out hypergeometric solutions implicitly required in (1). Our approach is to guess, based on numerical evidence, that x and z are connected by a simple algebraic relation r. For instance, we may guess that

$$G(x, r_{a,b}(x)) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} F(n,k)\frac{x^{k+n}}{(a+bx)^{2n+1}} \text{ or } \sum_{n=0}^{\infty}\sum_{k=0}^{n} F(n,k)\frac{(-1)^n x^{k+n}}{(a+bx)^{n+1/2}}, \tag{4}$$

for some a and b. We compute sufficiently many coefficients in the x-expansion of (4), and attempt to find a, b such that they satisfy a *three-term* recurrence (with polynomials of bounded degrees as coefficients). Such a recurrence corresponds to a degree 3 ODE satisfied by G. The key step then comes down to an easy problem in linear algebra of checking if a certain determinant is zero for some a and b.

Once suitable a and b are found, we need to solve the 3rd order ODE satisfied by $G(x, r_{a,b}(x))$; for this we have a more complete theory. E. g. in the case corresponding to (2), *Maple 13* is able to give the solution which can be rearranged into a ${}_3F_2$. In the case of (3), the ODE is of Heun type, and can be solved using [2, eqn. (3.5b)] followed by a transform due to E. Goursat; we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}^2\binom{2n}{n}\frac{x^{k+n}}{(1+4x)^{2n+1}} = \sum_{n=0}^{\infty}\frac{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}{n!^3}\big(108x^2(1-4x)\big)^n. \tag{5}$$

In either case the ${}_3F_2$ is of the type in (1), and the extensive theory for producing formulas of this type can be used to prove equations (2) and (3).

Some details

When we take the x-derivative of (5) (as is required in (1)), linear dependence on k appears on the left hand side, which is not found in (3). To cancel this k term, a vanishing, k-dependent identify (known as a 'satellite identity', coined in [4]) is required. For (5), the satellite identity is

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}^2\binom{2n}{n}\frac{x^{k+n}}{(1+4x)^{2n}}\big(4x+2k(4x+1)+n(4x-1)\big)=0. \tag{6}$$

Identity (6) was *guessed* as follows: pick a small, irrational x and compute $a_0 = \sum_{n,k} A(n,k,x)$, $a_1 = \sum_{n,k} A(n,k,x)k$, and $a_2 = \sum_{n,k} A(n,k,x)n$ (A being the summand), then use PSLQ to find a null integer linear combination among the elements of $\{a_0, a_1, a_2, a_0x, a_1x, a_2x, a_0x^2, a_1x^2, a_2x^2, \ldots\}$. Once found, the satellite identity can be proven by the multiple WZ algorithm. Similarly, (5) itself can be rigorously proven (as the 3rd order recursion was only a guess): write the coefficients of x on the LHS as a double sum, apply the multiple WZ algorithm to obtain a recursion, convert it to an ODE for the LHS, and finally check that the ODE annihilates the RHS. Many conjectures from [3] have been settled using our method, via the discovery of generating functions like (5).

Future work

Some conjectures in [3] do not fall into the type (4); perhaps more elaborate algebraic relations are needed – this could also anticipate more exotic generating functions. It would be illuminating to be able to find suitable a and b in (4) analytically (without extensive computer searches), and also to prove the existence of satellite identities whenever F is a hypergeometric term.

References

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