Schützenberger's factorization on $q-$stuffle Hopf algebra

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Schützenberger's monoidal factorization [9] has been introduced and plays a central role in the renormalization [7] of associators which are formal power series in non commutative variables[1]. The coefficients of these power series are polynomial at positive integral multi-indices of Riemann's zêta function[2] [5, 10] and they satisfy quadratic relations [1] which can be explained through Lyndon words. These relations can be obtained by identification of the local coordinates on a bridge equation connecting the Cauchy and Hadamard algebras of polylogarithmic functions and use the factorizations of the non commutative generating series of polylogarithms [6] and of harmonic sums [7]. This equation is mainly a consequence of the double isomorphy between these structures to respectively the shuffle [6] and stuffle [3] algebras both admitting the Lyndon words as a transcendence basis.

Symbolic computation allows us to introduce a formal variable q in order to better understand the mechanisms of the shuffle and to obtain algorithms on stuffles. We will then examine the q-stuffle interpolating between the shuffle [9], stuffle [8] and minus-stuffle [3]. In particular, we will give an effective construction of pair of bases in duality. It uses essentially an adapted version of the Eulerian projector in order to obtain the primitive elements of the q-stuffle Hopf algebra and they are obtained thanks to the computation of the logarithm of the diagonal series. This study completes the treatment for the stuffle [7] and boils down to the shuffle [9].

More precisely, let $Y = \{y_s\}_{s\geq 1}$ be an alphabet with the total order $y_1 > y_2 > \cdots$. Let also $\mathbf{k}$ be a unitary $\mathbb{Q}$-algebra containing q. One defines the q-stuffle, or its dual co-product, as follows, for any $y_s, y_t \in Y$ and $u, v \in Y^*$,

$$u \uplus_q 1_{Y^*} = 1_{Y^*} \uplus_q u = u \quad \text{and} \quad y_s u \uplus_q y_t v = y_s(u \uplus_q y_t v) + y_t(y_s u \uplus_q v) + q y_{s+t}(u \uplus_q v), \tag{1}$$

$$\Delta_{\uplus_q}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*} \quad \text{and} \quad \Delta_{\uplus_q}(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + q \sum_{s_1+s_2=s} y_{s_1} \otimes y_{s_2}. \tag{2}$$

This product is commutative, associative and unital. With the co-unit defined by, $\epsilon(P) = \langle P \mid 1_{Y^*}\rangle$, for $P \in \mathbf{k}\langle Y\rangle$, one gets $\mathcal{H}_{\uplus_q} = (\mathbf{k}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_q}, \epsilon)$ and $\mathcal{H}^{\vee}_{\uplus_q} = (\mathbf{k}\langle Y\rangle, \uplus_q, 1_{Y^*}, \Delta_{\mathtt{conc}}, \epsilon)$ which are mutually dual bialgebras and, in fact, Hopf algebras because they are $\mathbb{N}$-graded by the weight.

Group-like elements, redefined below, form a group for which the log-exp correspondence is explained by as follows

Lemma 1 (q-extended Friedrichs criterium) *Let $S \in \mathbf{k}\langle\langle Y\rangle\rangle$ (for 2., we suppose in addition that $\langle S \mid 1_{Y^*}\rangle = 1$).*

1. *S is* primitive, *i.e.* $\Delta_{\uplus_q} S = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S$, *if and only if, for any $u, v \in Y^+$, $\langle S \mid u \uplus_q v\rangle = 0$.*
2. *S is* group-like, *i.e.* $\Delta_{\uplus_q} S = S \otimes S$, *if and only if, for any $u, v \in Y^+$, $\langle S \mid u \uplus_q v\rangle = \langle S \mid u\rangle\langle S \mid v\rangle$.*
3. *S is group-like if and only if $\log S$ is primitive.*

Proposition 1 *Let $\mathcal{D}_Y = \sum_{w\in Y^*} w \otimes w$ be the diagonal series over Y. Then*

1. $\log \mathcal{D}_Y = \displaystyle\sum_{w\in Y^+} w \otimes \pi_1(w)$, *where* $\pi_1(w) = w + \displaystyle\sum_{k\geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 \uplus_q \ldots \uplus_q u_k\rangle u_1 \ldots u_k.$
2. *For any $w \in Y^*$, we have* $w = \displaystyle\sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 \uplus_q \ldots \uplus_q u_k\rangle \pi_1(u_1) \ldots \pi_1(u_k).$

[1]These associators were introduced in quantum field theory by Drinfel'd and the universal associator, *i.e.* Φ_{KZ}, was obtained with explicit coefficients which are polyzêtas and regularized polyzêtas [5].

[2]These values are usually abbreviated MZV's by Zagier [10] and are also called polyzêtas by Cartier [1].

Let $\mathcal{P} = \{P \in \mathbb{Q}\langle Y\rangle \mid \Delta_{\uplus_q} P = P \otimes 1 + 1 \otimes P\}$ be the set of primitive polynomials. Since, in virtue of $\Delta_{\uplus_q}\pi_1(w) = \pi_1(w) \otimes 1 + 1 \otimes \pi_1(w)$, $\mathrm{Im}(\pi_1) \subseteq \mathcal{P}$, we can state the following

Theorem 1 ([2]) *1. Let $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ be defined by, for any $y_k \in Y, \Pi_{y_k} = \pi_1(y_k)$ and for any $l \in \mathcal{L}ynX$ of standard factorization $l = (s, r), \Pi_l = [\Pi_s, \Pi_r]$. Then $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ forms a basis of $\mathcal{P}$.*

2. Let $\{\Pi_w\}_{w\in Y^*}$ be defined by, for any $w \in Y^*$ such that $w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY$, $\Pi_w = \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}$. Then $\{\Pi_w\}_{w\in Y^*}$ forms a basis of $\mathbf{k}\langle Y\rangle$.

3. Let $\{\Sigma_w\}_{w\in Y^*}$ be the family of the quasi-shuffle algebra obtained by duality with $\{\Pi_w\}_{w\in Y^*}$. Then $\{\Sigma_w\}_{w\in Y^*}$ generates freely the quasi-shuffle algebra.

4. The family $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ forms a transcendence basis of $(\mathbf{k}\langle Y\rangle, \uplus_q)$.

We now give formulas which permit to compute the basis $\{\Sigma_w\}_{w\in Y^*}$ without inverting a huge Gram matrix.

Theorem 2 (q-extended Schützenberger's factorization, [2]) *1. For any $y \in Y$, $\Sigma_y = y$.*

2. For any $y_{s_1} \ldots y_{s_k} \in \mathcal{L}ynX$, $$\Sigma_{y_{s_1}\ldots y_{s_k}} = \sum_{\substack{\{s'_1,\cdots,s'_i\}\subset\{s_1,\cdots,s_k\}, l_1\geq\cdots\geq l_n\in\mathcal{L}ynY \\ (y_{s_1}\cdots y_{s_k})\overset{*}{\Leftarrow}(y_{s'_1},\cdots,y_{s'_n},l_1,\cdots,l_n)}} \frac{q^{i-1}}{i!} y_{s'_1+\cdots+s'_i} \Sigma_{l_1\cdots l_n}.$$

3. For any $w = l_1^{i_1} \ldots l_k^{i_k}$, with $l_1, \ldots, l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$, $$\Sigma_w = \frac{\Sigma_{l_1}^{\uplus_q i_1} \uplus_q \ldots \uplus_q \Sigma_{l_k}^{\uplus_q i_k}}{i_1! \ldots i_k!}.$$

4. $$\mathcal{D}_Y = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l \otimes \Pi_l).$$

Theorems 1.1 and 2.2 are based mainly on respectively the logarithm of the diagonal series $\mathcal{D}_Y$ and the standard sequences [9, 2] and lead to simplified algorithms getting bases in duality as shown in the following

Example 1

$$\begin{array}{rcl}
\Pi_{y_2} &=& y_2 - \frac{q}{2} y_1^2, \\
\Pi_{y_2y_1} &=& y_2y_1 - y_1y_2, \\
\Pi_{y_3y_1y_2} &=& y_3y_1y_2 - \frac{q}{2} y_3y_1^3 - qy_2y_1^2y_2 + \frac{q^2}{4} y_2y_1^4 - y_1y_3y_2 + \frac{q}{2} y_1y_3y_1^2 + \frac{q}{2} y_1^2y_2^2 - \frac{q^2}{2} y_1^2y_2y_1^2 - y_2y_3y_1 \\
&+& \frac{q}{2} y_2^2y_1^2 + y_2y_1y_3 + \frac{q}{2} y_1^2y_3y_1 - \frac{q}{2} y_1^3y_3 + \frac{q^2}{4} y_1^4y_2, \\
\Pi_{y_3y_1y_2y_1} &=& y_3y_1y_2y_1 - y_3y_1^2y_2 - \frac{q}{2} y_2y_1^2y_2y_1 - y_1y_3y_2y_1 + y_1y_3y_1y_2 + \frac{q}{2} y_1^2y_2^2y_1 - \frac{q}{2} y_1^2y_2y_1y_2 - y_2y_1y_3y_1 \\
&+& \frac{q}{2} y_2y_1y_2y_1^2 + y_2y_1^2y_3 + y_1y_2y_3y_1 - \frac{q}{2} y_1y_2^2y_1^2 - y_1y_2y_1y_3 + \frac{q}{2} y_1y_2y_1^2y_2. \\
\Sigma_{y_2} &=& y_2, \\
\Sigma_{y_2y_1} &=& y_2y_1 + \frac{q}{2} y_3, \\
\Sigma_{y_3y_2y_1} &=& y_3y_1y_2 + y_3y_2y_1 + qy_3^2 + \frac{q}{2} y_4y_2 + \frac{q^2}{3} y_6 + \frac{q}{2} y_5y_1, \\
\Sigma_{y_3y_1y_2y_1} &=& 2y_3y_2y_1^2 + qy_3y_2^2 + y_3y_1y_2y_1 + \frac{3q}{2} y_3^2y_1 + \frac{q}{2} y_3y_1y_3 + \frac{q^2}{2} y_3y_4 + \frac{q}{2} y_4y_2y_1 + \frac{q^2}{4} y_4y_3 + qy_5y_1^2 + \frac{q^2}{2} y_5y_2 + \frac{q^2}{2} y_6y_1 + \frac{q^3}{8} y_7.
\end{array}$$

In conclusion, since the pioneering works of Schützenberger and Reutenauer [9], the question of computing bases in duality (maybe at the cost of a more involved procedure, but without inverting a Gram matrix) remained open in the case of cocommutative deformations of the shuffle product. We have given such a procedure allowing a great simplification for an interpolation between shuffle and stuffle. In the next framework, this product will be continuously deformed, in the most general way while remaining commutative [4].

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