Gelfand-Kirillov dimensions of differential difference modules via Gröbner bases

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Introduction. Differential-difference algebras were defined by Mansfield and Szanto in [5], which arose from the calculation of symmetries of discrete systems (c.f., [2]). Mansfield and Szanto developed the Gröbner basis theory of differential difference algebras over a field by using a special kind of left admissible orderings (which they called differential difference orderings). We generalize the main results of [5] to any left admissible ordering, and apply the generalized results to compute the Gelfand-Kirillov dimensions of cyclic differential difference modules.

Definition of differential difference algebras. Let k be a field, R be a k-algebra and integers $m, n \geq 1$. Suppose that $R[D; \mathrm{id}, \delta] = R[D_1; \mathrm{id}, \delta_1] \cdots [D_n; \mathrm{id}, \delta_n]$ and $R[S; \sigma, 0] = R[S_1; \sigma_1, 0] \cdots [S_m; \sigma_m, 0]$ are two Ore algebras ([5]) such that $\sigma_i \circ \delta_j = \delta_j \circ \sigma_i$ for $1 \leq i \leq m, 1 \leq j \leq n$. Furthermore, suppose that each $\sigma_i : R \to R$, $1 \leq i \leq m$, can be extended to a k-algebra automorphism $\sigma_i : R[D; \mathrm{id}, \delta] \to R[D; \mathrm{id}, \delta]$ such that $\sigma_i(D_j) = \sum_{l=1}^{n} a_{ijl} D_l, \;\; a_{ijl} \in R$. Let F be the free R-R bi-module with basis $\{S_1, \ldots, S_m, D_1, \ldots, D_n\}$, T be the tensor algebra on F over R, and K be the two-sided ideal in T generated by the set of the following elements of T:
(1) $D_i r - r D_i - \delta_i(r), 1 \leq i \leq n, r \in R$; (2) $S_i r - \sigma_i(r) S_i, 1 \leq i \leq m, r \in R$;
(3) $S_i S_j - S_j S_i, 1 \leq i, j \leq m$; (4) $D_i D_j - D_j D_i, 1 \leq i, j \leq n$;
(5) $D_i S_j - S_j \sigma_j(D_i), 1 \leq i \leq n, 1 \leq j \leq m$.
Then the R-algebra T/K, denoted by $R[D; \mathrm{id}, \delta][S; \sigma, 0]$, is called a *differential difference algebra* of type (m, n), or DD-algebras for short.

DD-algebras are generalizations of commutative polynomial algebras, Ore extensions, skew polynomials of derivation (or automorphism) type, and quantum planes. Since elements in S do not commute with those in D in general, DD-algebras are different from difference-differential rings (see, e.g., [6]). The following example distinguishes DD-algebras from algebras of solvable type [3], or PBW extensions [1], or G-algebras [4].
Example. Let $A = k[D; \mathrm{id}, 0][S; \sigma, 0]$ be a DD-algebra of type $(1, 2)$ with $\sigma_1(D_1) = D_2$ and $\sigma_1(D_2) = D_1$. Then $D_1 S_1 = S_1 D_2$ and $D_2 S_1 = S_1 D_1$. Hence A is not an algebra of solvable type (or a PBW extension, or a G-algebra).

Gröbner bases of DD-algebras. We only consider the special case when $R = k$. From now on, let $A = k[D; \mathrm{id}, \delta][S; \sigma, 0]$ be a DD-algebra. Then, it is easy to see that $\delta = 0$ and $\sigma|_k = \mathrm{id}$. Thus $A = k[D; \mathrm{id}, 0][S; \sigma, 0]$ and $\sigma|_k = \mathrm{id}$. One can prove that the set $\mathcal{M} = \{S^\alpha D^\beta : \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n\}$ is a k-basis of A. Let $u = S^\alpha D^\beta \in \mathcal{M}$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. Then the *(total) degree* of u is defined as $\deg(u) = \alpha_1 + \cdots + \alpha_m + \beta_1 + \cdots + \beta_n$, and the degree of u with respect to S_i (D_j, respectively) is defined as $\deg_{S_i} = \alpha_i$ ($\deg_{D_j} = \beta_j$, respectively).

For any given well ordering on $\mathcal{M}$ and $f = c_1u_1 + \cdots + c_tu_t \in A$ ($0 \neq c_i \in k$, $u_i \in \mathcal{M}$, $1 \leq i \leq t$) with $u_1 > \cdots > u_t$, the *leading monomial* of f is denoted by $\mathrm{lm}(f) = u_1$. A *DD-monomial ordering* on $\mathcal{M}$ is a well ordering $>$ on $\mathcal{M}$ such that if $S^\alpha D^\beta > S^{\alpha'} D^{\beta'}$ and $f \in A \setminus k$, then $\mathrm{lm}(fS^\alpha D^\beta) > \mathrm{lm}(fS^{\alpha'} D^{\beta'})$. Note that DD-monomial orderings are more general than differential difference orderings defined in [5].

Let $f, g \in A$. If there exists $h \in A$ such that $f = hg$, we say that f is *right divisible* by g.

Let $>$ be a DD-monomial ordering on $\mathcal{M}$ and I be a left ideal of A. A finite set $G \subseteq A$ is called a (finite) *left Gröbner basis* of I with respect to $>$ if G satisfies: (i) G generates I as a left ideal of A; and (ii) For any $0 \neq f \in I$, there exists $g \in G$ such that $\mathrm{lm}(f)$ is right divisible by $\mathrm{lm}(g)$.

Similarly as in [5], we can define reductions and S-polynomials. Then the reduction algorithm and the left Gröbner basis algorithm still work under a DD-monomial ordering. We have

Theorem 1 *Let $G \subseteq A$ be a finite set and I be the left ideal of A generated by G. Then G is a left Gröbner basis of I if and only if* $\mathrm{Spoly}(g_1, g_2) \to_G 0$ *for any $g_1, g_2 \in G$.*

It can be proved that the Hilbert basis theorem is valid for DD-algebras: every left ideal of A is finitely generated. Thus we have

Theorem 2 *Every left ideal of a DD-algebra $k[D; \mathrm{id}, \delta][S; \sigma, 0]$ has a (finite) left Gröbner basis.*

Gelfand-Kirillov dimension of cyclic A-modules. For convenience, let $x_i = S_i, x_{m+j} = D_j$ for $1 \leq i \leq m, 1 \leq j \leq n$ and let $l = m + n$. Denote $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_l^{\alpha_l}$ for $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$. Then $\mathcal{M} = \{X^\alpha : \alpha \in \mathbb{N}^l\}$. For $u = X^\alpha \in \mathcal{M}$ and $p \in \mathbb{N}$, define $\mathrm{top}_p(u) = \{i : 1 \leq i \leq l, \alpha_i \geq p\}$ and $\mathrm{sh}_p(u) = X^\beta$, where $\beta_i = \min\{p, \alpha_i\}, 1 \leq i \leq l$.

Then we have the following theorem which computes the Gelfand-Kirillov dimension of a cyclic DD-module.

Theorem 3 *Let I be a left ideal of A and G be a left Gröbner basis of I with respect to a total degree DD-monomial ordering. Set $p = \max\{\deg_{x_i}(\mathrm{lm}(g)) : g \in G, 1 \leq i \leq l\}$. Then*

$$\mathrm{GKdim}(M) = \max\{|\,\mathrm{top}_p(u)| : \mathrm{sh}_p(u) = u\}.$$

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