Analysis of Summation Algorithms

Manuel Kauers

Input:

$$F(n) = \sum_{k} \binom{n}{k} \binom{2n}{2k}$$

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Output:

$$\begin{aligned} &(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ &+ (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ &- (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0 \end{aligned}$$

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- How large can the output become?

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Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

degree

order

$$\begin{aligned} &(48n^{\boxed{3}} + 152n^2 + 144n + 40)\ F(n) \\ &+ (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ &- (6n^3 + 25n^2 + 32n + 12)\ F(n+\boxed{2}) = 0 \end{aligned}$$

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$$
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$$

Questions:

- How much time does this computation take?
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Input:

$$F(x) = \int_{\Omega} \sqrt{(2x-1)t+2}\, e^{xt^2}\, dt$$

Output:

$$\begin{aligned}(256x^6 - 256x^5 + 64x^3 - 16x^2)\ F''(x) \\ + (512x^5 + 256x^2 - 32x)\ F'(x) \\ + (48x^4 + 176x^3 + 84x - 3)\ F(x) = 0\end{aligned}$$

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- How much time does this computation take?
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Summation/Integration algorithms: (general principle)

$$
\begin{tikzcd}
\boxed{\sum \int\text{-problem}} \arrow[d, "\text{translate}"'] & & \boxed{\sum \int\text{-solution}} \\
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Analysis of the underlying linear algebra problem gives rise to

- existence results / bounds on the order
- bounds on degree and height / complexity estimates

$$\begin{pmatrix} 3x^2+3x+10 & 7x^2+3x+3 & 3x^2+4x+6 \\ 9x^2+9x+4 & 9x^2 & 6x^2+x+3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \overset{!}{=} 0$$

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- More variables than equations $\Rightarrow$ there is a nonzero solution.

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- There is a nonzero solution $(a_1, a_2, a_3) \in \mathbb{Z}[x]^3$ with degree at most 4 and height at most 100.

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- More variables than equations $\Rightarrow$ there is a nonzero solution.
- There is a nonzero solution $(a_1, a_2, a_3) \in \mathbb{Z}[x]^3$ with degree at most 4 and height at most 100.
- There are fast algorithms (Storjohann-Villard 2005).

Indefinite summation: Given $f(k)$, find $g(k)$ such that

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$$\big(p_0(n) + p_1(n)S_n + \cdots + p_r(n)S_n^r\big) \cdot f(n,k) = g(n,k+1) - g(n,k).$$

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Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n,k,S_n,S_k) = -\frac{k}{n+1-k}.$$

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A telescoper for $f(n,k)$ is an annihilator of $\sum_k f(n,k)$.

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Or: Apagodu-Zeilberger-style approach

- Easier to implement
- Easier to analyze

	order	degree	height
hypergeometric			
hyperexponential			
D-finite			

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	
D-finite	●		

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hypergeometric	●	●	●
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$f(n,k)$ is called proper hypergeometric if it can be written in the form

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

for a certain polynomial c, certain constants $p, q, a_i'', b_i'', u_i'', v_i''$ and certain fixed nonnegative integers $a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'$.

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Example: $f(n,k) = (n+k)2^n(-1)^k \dfrac{(n+k)!(2n-k)!(2n-2k)!}{(n+2k)!^2}$

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \le \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

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Usually there is no telescoper of lower order.

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&\vdots \\
f(n+r,k) &= \frac{(2n+k)\cdots\cdots\cdots\cdots\cdots\cdots(2n+k+(2r-1))}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)
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Choose $Q = \frac{q_0(n) + q_1(n)k + \cdots + q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-3))}$.

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Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-3))}$. Then:

$$(S_k-1)Q \cdot f(n,k) = \frac{q_0(n)\mathbf{pol}_0(n,k) + \cdots\cdots\cdots + q_{2r-2}(n)\mathbf{pol}_{2r-2}(n,k)}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$\deg_k \leq 2r$

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$

$$= \frac{p_0(n)\mathbf{poly}_0(n,k) + \cdots\cdots\cdots + p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

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Equating coefficients with respect to k gives a linear system with $(r+1)+(2r-2+1)$ variables and $2r+1$ equations. It has a nontrivial solution as soon as $r \geq 2$.

Theorem (Apagodu-Zeilberger)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \le \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'),\ \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

Theorem (Apagodu-Zeilberger; Chen-Kauers)
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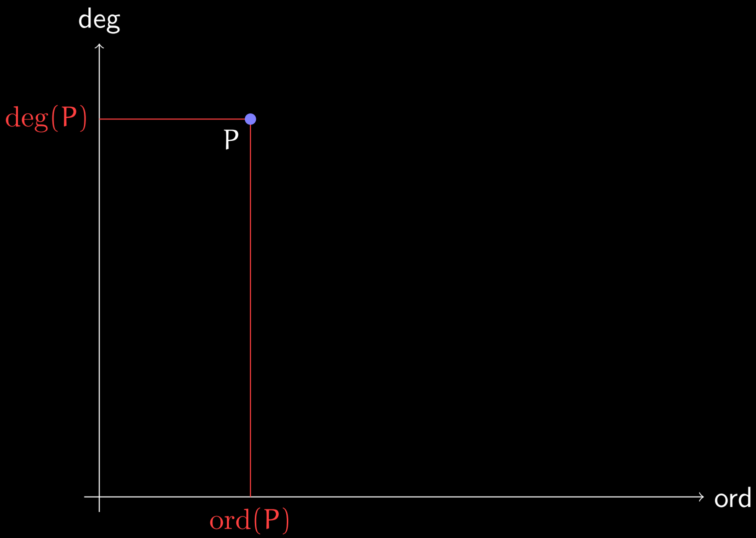
and

$$\deg(P) \le \left\lceil \tfrac{1}{2}\nu(2\delta + 2\nu\vartheta + |\mu| - \nu|\mu|) \right\rceil$$

where

- $\delta = \deg(c)$
- $\nu = \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'),\ \sum_{i=1}^{m} (u_i' + b_i') \right\}$
- $\vartheta = \max\left\{ \sum_{i=1}^{m} (a_i + b_i),\ \sum_{i=1}^{m} (u_i + v_i) \right\}$
- $\mu = \sum_{i=1}^{m} \big((a_i + b_i) - (u_i + v_i) \big)$


deg
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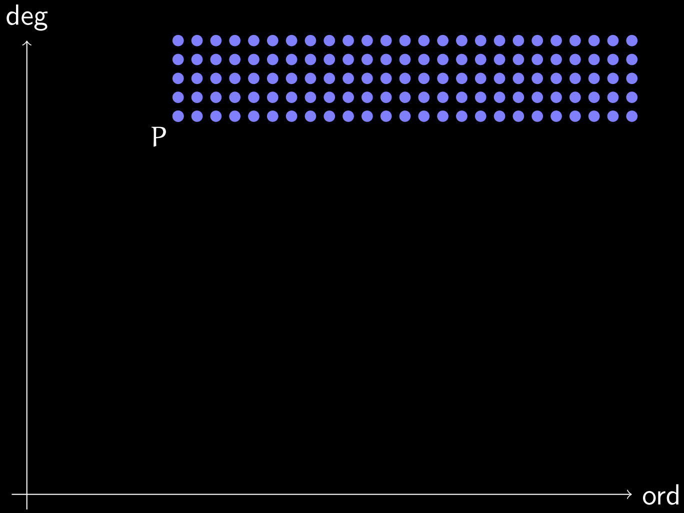
P
ord
ord(P)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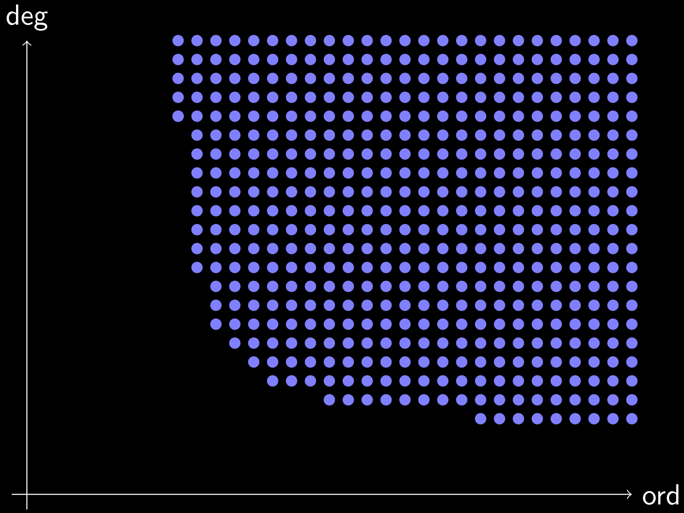
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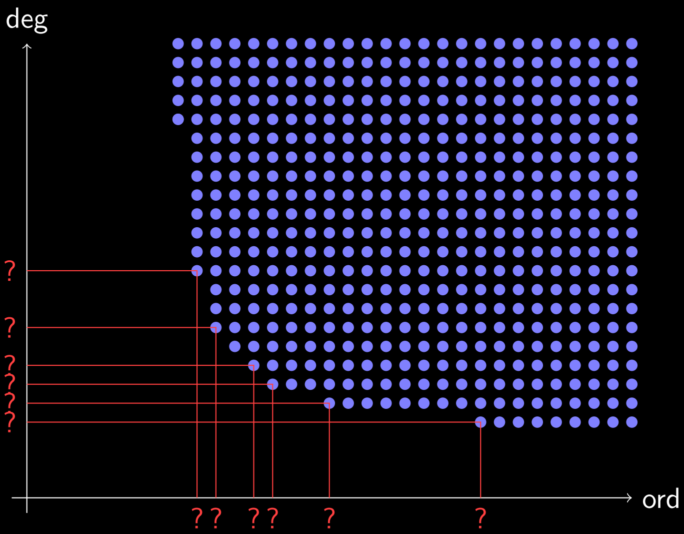
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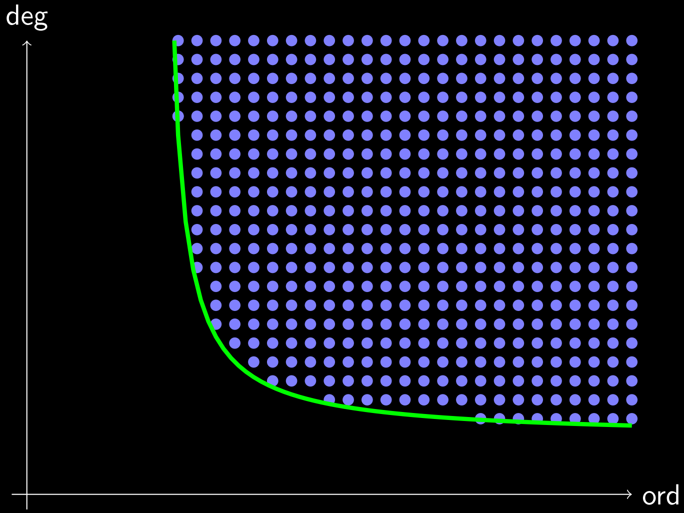
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there exist telescopers P with $\operatorname{ord}(P) \le r$ and $\deg(P) \le d$ for all $(r,d) \in \mathbb{N}^2$ with

$$r \ge \nu \text{ and } d > \frac{(\vartheta\nu - 1)r + \frac{1}{2}\nu\big(2\delta + |\mu| + 3 - (1 + |\mu|)\nu\big) - 1}{r - \nu + 1}.$$

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Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term $f(n,k)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper P with $\operatorname{ord}(P) \le \nu$ and

$$\begin{aligned} \operatorname{ht}(P) \le \ & \max\bigl\{|p|^\nu, |q|+1\bigr\} \operatorname{ht}(c)^{\nu+1} (\delta + \vartheta\nu + 1)!^{\nu+1} (\nu+1)^{\delta(\nu+1)} \\ & \times (|y|+1)^{\delta+(\vartheta-1)\nu+1} \delta!^{2(\nu+1)} |x|^{\nu^2} \\ & \times (\delta+\vartheta\nu+1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2} \\ & \times (2(\nu+2)\Omega-2)^{(\delta+\vartheta+1)\nu+(2\vartheta-1)\nu^2} \end{aligned}$$

where ν, ϑ, δ are as before, and

$$\Omega = \max_{i=1}^{m}\bigl\{|a_i|, |a_i'|, |a_i''|, |b_i|, |b_i'|, |b_i''|, |u_i|, |u_i'|, |u_i''|, |v_i|, |v_i'|, |v_i''|\bigr\}.$$

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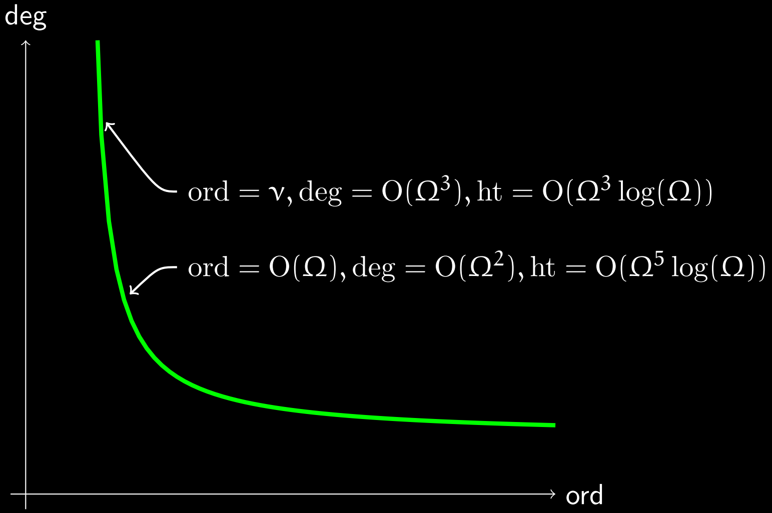
Summary: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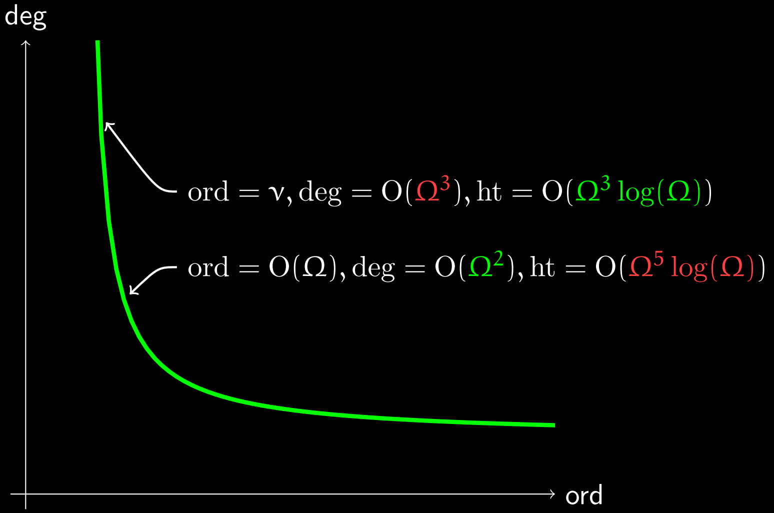
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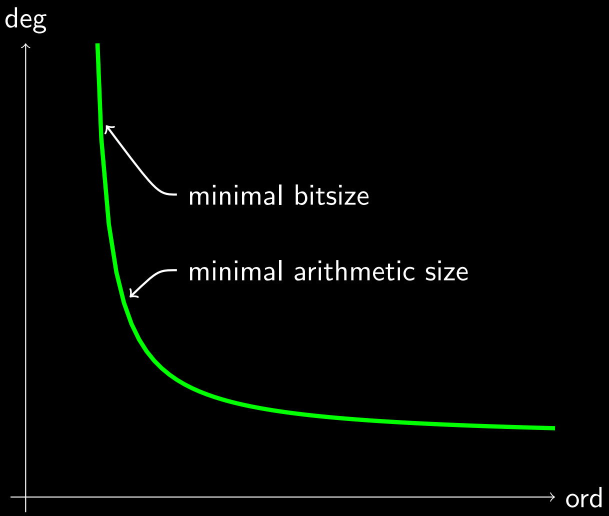

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

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Hypergeometric summation exploits the fact that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k)$$
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It's sufficient when $f(n,k)$ lives in some finite-dimensional $\mathbb{Q}(n,k)$-vector space which is closed under shifts.

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Indeed, we have

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$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
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f(n,k) & f(n,k+1) & f(n,k+2) & f(n,k+3) \\
f(n+1,k) & f(n+1,k+1) \longleftarrow & f(n+1,k+2) & f(n+1,k+3) \\
f(n+2,k) & f(n+2,k+1) & f(n+2,k+2) & f(n+2,k+3) \\
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$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
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$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
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Of course you are free to work with different bases, if you wish.

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The shift actions with respect to n and k can be encoded by matrices $M_n, M_k \in \mathbb{Q}(n,k)^{d\times d}$ such that for the function

$$f(n,k) \cong (\, u_1(n,k), \ldots, u_d(n,k)\,)$$

we have

$$\begin{aligned} f(n+1,k) &\cong (\, u_1(n+1,k), \ldots, u_d(n+1,k)\,) \cdot M_n \\ f(n,k+1) &\cong (\, u_1(n,k+1), \ldots, u_d(n,k+1)\,) \cdot M_k. \end{aligned}$$

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Example: For $B = \left\{ 2^{n-k}, \binom{n}{k} \right\}$ we have

$$M_n = \begin{pmatrix} 2 & 0 \\ 0 & \frac{n+1}{n+1-k} \end{pmatrix} \quad \text{and} \quad M_k = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{n-k}{k+1} \end{pmatrix}.$$

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The usual bounds only apply to "proper" hypergeometric terms.

Question: What is a "proper" D-finite function?

Hypergeometric means that

$$\begin{aligned} f(n+1,k) &= \mathbf{rat}_1(n,k)\, f(n,k),\\ f(n,k+1) &= \mathbf{rat}_2(n,k)\, f(n,k) \end{aligned}$$

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Proper hypergeometric means (essentially) that the denominators of these rational functions have only integer-linear factors.

Definition (Chen-Kauers-Koutschan) A D-finite function $f(n, k)$ is called **proper D-finite** if it lives in a vector space which admits a basis B such that

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- the coordinates of $f(n, k)$ with respect to B are polynomials.
- the shift matrices M_n, M_k with respect to B are such that the common denominator of all their entries has only integer-linear factors.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.

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Then there exists a telescoper P for $f(n,k)$ with $\boxed{\operatorname{ord}(P) \leq |B|r + d}$.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n,k)$ be proper D-finite.

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- Let B be a basis of the vector space and M_n, M_k be the shift matrices with respect to B.
- Write $M_k = \frac{1}{h}H$ for a polynomial matrix H and a polynomial h of the form $h = \prod_{i=1}^{m}(a_i n + b_i k + c_i)^{\overline{b_i}}(a_i' n - b_i' k + c_i')^{\underline{b_i'}}$ for nonnegative integers a_i, b_i, a_i', b_i'. Let

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- Let d be the dimension of the $\mathbb{Q}(n)$-subspace of all vectors v with $S_k \cdot v = v$.

Then there exists a telescoper P for $f(n,k)$ with $\boxed{\operatorname{ord}(P) \le |B|r + d}$.

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
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