Telescopers for Rational and Algebraic Functions via Residues

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ABSTRACT

We show that the problem of constructing telescopers for functions of m variables is equivalent to the problem of constructing telescopers for algebraic functions of $m-1$ variables and present a new algorithm to construct telescopers for algebraic functions of two variables. These considerations are based on analyzing the residues of the input. According to experiments, the resulting algorithm for rational functions of three variables is faster than known algorithms, at least in some examples of combinatorial interest. The algorithm for algebraic functions implies a new bound on the order of the telescopers.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*

General Terms

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1. INTRODUCTION

The problem of creative telescoping is to find, for a given "function" f in several variables $t_1, \ldots, t_n, x_1, \ldots, x_m$, linear differential operators L involving only the t_i and derivations with respect to the t_i, and some other "functions" $g_1, \ldots, g_m$ such that

$$L(f) = D_{x_1}(g_1) + \cdots + D_{x_m}(g_m),$$

where D_{x_j} denotes the derivative with respect to x_j. The main motivation for computing such operators L (called "telescopers" for f) is that, under suitable technical assumptions on f and the domain Ω, these operators have the definite integral

$$F(t_1, \ldots, t_n) = \int_\Omega f(t_1, \ldots, t_n, x_1, \ldots, x_m)\, dx_1 \cdots dx_m$$

as a solution. Once differential operators for F have been found, other algorithms can next be used for determining possible closed forms, or asymptotic information, or recurrence equations for the series coefficients of F.

There are general algorithms for computing telescopers when the input f is holonomic [25, 15, 24, 20, 9] as well as special-purpose algorithms designed for restricted input classes [25, 26, 5]. The focus in the present paper is on two such restricted input classes: rational and algebraic functions of several variables. Our first result is that an algorithm for computing telescopers for rational functions of m variables directly leads to an algorithm for computing telescopers for algebraic functions of $m-1$ variables and *vice versa* (Section 2). Our second result is a new algorithm for creative telescoping of algebraic functions of two variables (Section 3), which, by the equivalence, also implies a new algorithm for creative telescoping of rational functions of three variables. The algorithm for algebraic functions is mainly interesting because it implies a new bound on the order of the telescoper in this case (Theorem 15), while the implied algorithm for rational functions is mainly interesting because at least for some examples it provides an efficient alternative to other methods (Section 4).

For a precise problem description, let k be a field of characteristic zero, and $k(t, \mathbf{x})$ be the field of rational functions in t and $\mathbf{x} = (x_1, \ldots, x_m)$ over k. Let $\hat{\mathbf{x}}_m$ denote the $m-1$ variables $x_1, \ldots, x_{m-1}$. The algebraic closure of a field K will be denoted by $\overline{K}$. The usual derivations ∂/∂_t and ∂/∂_{x_i} are denoted by D_t and D_{x_i}, respectively. Let $k(t)\langle D_t\rangle$ be the ring of linear differential operators in t with coefficients in $k(t)$. Then we are interested in the following two problems:

Problem 1. *Given $f \in k(t, \mathbf{x})$, find a nonzero operator $L \in k(t)\langle D_t\rangle$ such that*

$$L(f) = D_{x_1}(g_1) + \cdots + D_{x_m}(g_m) \quad \textit{for some } g_j \in k(t, \mathbf{x}).$$

Such an L is called a telescoper *for f, and the rational functions $g_1, \ldots, g_m$ are called* certificates *of L.*

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Problem 2. *Given $\alpha \in \overline{k(t, \hat{\mathbf{x}}_m)}$, find a nonzero operator $L \in k(t)\langle D_t \rangle$ such that*

$$L(\alpha) = D_{x_1}(\beta_1) + \cdots + D_{x_{m-1}}(\beta_{m-1}) \text{ for some } \beta_j \in \overline{k(t, \hat{\mathbf{x}}_m)}.$$

Such an L is called a telescoper *for α, and the algebraic functions $\beta_1, \ldots, \beta_{m-1}$ are called* certificates *of L.*

Both the equivalence of these two problems and the new algorithm for Problem 2 (when $m = 2$) are based on the general idea of eliminating residues in the input. As an introduction to this approach, let us consider the problem of finding a telescoper and certificate for a rational function in two variables, that is, given a *rational* function $f \in k(t, x)$, we want to find a nonzero $L \in k(t)\langle D_t \rangle$ such that $L(f) = D_x(g)$ for some $g \in k(t, x)$. We may consider f as an element of $\overline{K}(x)$, where $K = k(t)$, and as such we may write

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(x - \beta_i)^j}, \tag{1}$$

where $p \in K[x]$, the β_i are the roots in $\overline{K}$ of the denominator of f and the $\alpha_{i,j}$ are in $\overline{K}$. We refer to the element $\alpha_{i,1}$ as the *residue* of f at β_i. Using Hermite reduction, one sees that a rational function $h \in K(x)$ is of the form $h = D_x(g)$ for some $g \in K(x)$ if and only if all residues of h are zero. Therefore to find a telescoper for f it is enough to find a nonzero operator $L \in K\langle D_t \rangle$ such that $L(f)$ has only zero residues. For example assume that f has only simple poles, i.e., $f = \frac{a}{b}, a, b \in K[x]$, $\deg_x a < \deg_x b$ and b squarefree. We then know that the Rothstein-Trager resultant [23, 19]

$$R := \mathrm{resultant}_x(a - zD_x(b), b) \in K[z]$$

is a polynomial whose roots are the residues at the poles of f. Given a squarefree polynomial in $K[z] = k(t)[z]$, differentiation with respect to t and elimination allow one to construct a nonzero linear differential operator $L \in k(t)\langle D_t \rangle$ such that L annihilates the roots of this polynomial. Applying L to each term of (1) one sees that $L(f)$ has zero residues at each of its poles. Applying Hermite reduction to $L(f)$ allows us to find a g such that $L(f) = D_x(g)$.

The main idea in the method described above is that nonzero residues are the obstruction to being the derivative of a rational function and one constructs a linear operator to remove this obstruction. Understanding how residues form an obstruction to integrability and constructing linear operators to remove this obstruction will be the guiding principal that motivates the results which follow.

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2. TELESCOPERS FOR RATIONAL FUNCTIONS

2.1 Rational and algebraic integrability

In this section, we give a criterion which decides whether or not 1 is a telescoper for a rational function in $k(t, \mathbf{x})$. Again, let $K = k(t)$. A rational function $f \in K(\mathbf{x})$ is said to be *rational integrable* with respect to $\mathbf{x}$ if $f = \sum_{j=1}^{m} D_{x_j}(g_j)$ for some $g_j \in K(\mathbf{x})$. An algebraic function $\alpha \in \overline{K(\hat{\mathbf{x}}_m)}$ is said to be *algebraic integrable* with respect to $\hat{\mathbf{x}}_m$ if $\alpha = \sum_{j=1}^{m-1} D_{x_j}(\beta_j)$ for some $\beta_j \in \overline{K(\hat{\mathbf{x}}_m)}$. By taking traces, one can show that if α is algebraic integrable with respect to $\hat{\mathbf{x}}_m$, then an antiderivative of α already exists in the field $K(\hat{\mathbf{x}}_m)(\alpha)$.

For a rational function $f \in K(\mathbf{x})$, Hermite reduction with respect to x_m decomposes f into

$$f = D_{x_m}(r) + \frac{a}{b}, \tag{2}$$

where $r \in K(\mathbf{x})$ and $a, b \in K(\hat{\mathbf{x}}_m)[x_m]$ such that $\deg_{x_m}(a) < \deg_{x_m}(b)$ and b is squarefree with respect to x_m. It is clear that f is rational integrable with respect to $\mathbf{x}$ if and only if a/b in (2) is rational integrable with respect to $\mathbf{x}$. Over the field $\overline{K(\hat{\mathbf{x}}_m)}$, one can write a rational function $f \in K(\mathbf{x})$ as

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{ij}}{(x_m - \beta_i)^j},$$

where $p \in K(\hat{\mathbf{x}}_m)[x_m]$ and the α_{ij}, β_i are in $\overline{K(\hat{\mathbf{x}}_m)}$. We call α_{i1} the *x_m-residue* of f at β_i, denoted by $\mathrm{residue}_{x_m}(f, \beta_i)$.

Proposition 3. *Let $f \in K(\mathbf{x})$ and $\beta \in \overline{K(\hat{\mathbf{x}}_m)}$. Then*

(i) $\mathrm{residue}_{x_m}(f, \beta) = 0$ *if $f = D_{x_m}(g)$ for some $g \in K(\mathbf{x})$*

(ii) $D_{x_j}(\mathrm{residue}_{x_m}(f, \beta)) = \mathrm{residue}_{x_m}(D_{x_j}(f), \beta)$ *for all j with $1 \leq j \leq m - 1$.*

Proof. The first assertion follows by observing the effect of D_{x_m} on each term in the partial fraction decomposition of g. By Hermite reduction, we can decompose f into

$$f = D_{x_m}(r) + \sum_{i=1}^{n} \frac{\alpha_i}{x_m - \beta_i}.$$

By the first assertion, either $\mathrm{residue}_{x_m}(f, \beta) = \alpha_i$ if $\beta = \beta_i$ or $\mathrm{residue}_{x_m}(f, \beta) = 0$ if $\beta \neq \beta_i$ for all $i = 1, \ldots, n$. Applying D_{x_j} to the two sides of the equation above yields

$$\begin{aligned} D_{x_j}(f) &= D_{x_j}(D_{x_m}(r)) + \sum_{i=1}^{n} \left(\frac{D_{x_j}(\alpha_i)}{x_m - \beta_i} + \frac{\alpha_i D_{x_j}(\beta_i)}{(x_m - \beta_i)^2} \right) \\ &= D_{x_m}\left(D_{x_j}(r) - \sum_{i=1}^{n} \frac{\alpha_i D_{x_j}(\beta_i)}{x_m - \beta_i} \right) + \sum_{i=1}^{n} \frac{D_{x_j}(\alpha_i)}{x_m - \beta_i}. \end{aligned}$$

Then we have either $\mathrm{residue}_{x_m}(D_{x_j}(f), \beta) = D_{x_j}(\alpha_i)$ if $\beta = \beta_i$ or $\mathrm{residue}_{x_m}(D_{x_j}(f), \beta) = 0$ if $\beta \neq \beta_i$ for all $i = 1, \ldots, n$. The second assertion follows. ∎

If f is written as the form in (2), then we have

$$\mathrm{residue}_{x_m}(f, \beta_i) = \frac{a}{D_{x_m}(b)}\Big|_{x_m = \beta_i} \in K(\hat{\mathbf{x}}_m)(\beta_i).$$

Therefore, all the x_m-residues of f are roots of the Rothstein-Trager resultant (see [19, 23])

$$R := \mathrm{resultant}_{x_m}(b, a - zD_{x_m}(b)) \in K(\hat{\mathbf{x}}_m)[z].$$

Lemma 4. *Let $f \in K(\mathbf{x})$. Then f is rational integrable with respect to $\mathbf{x}$ if and only if all the x_m-residues of f are algebraic integrable with respect to $\hat{\mathbf{x}}_m$.*

Proof. By the Hermite reduction and partial fraction decomposition, f can be written as

$$f = D_{x_m}(r) + \sum_{i=1}^{n} \frac{\alpha_i}{x_m - \beta_i},$$

where $r \in K(\mathbf{x})$, $\alpha_i, \beta_i \in \overline{K(\hat{\mathbf{x}}_m)}$ and the β_i are pairwise distinct.

Suppose that all the x_m-residues α_i of f are algebraic integrable with respect to $\hat{\mathbf{x}}_m$, i.e., $\alpha_i = \sum_{j=1}^{m-1} D_{x_j}(\gamma_{i,j})$ for some $\gamma_{i,j} \in K(\hat{\mathbf{x}}_m)(\alpha_i)$. Note that for each j we have

$$\frac{D_{x_j}(\gamma_{i,j})}{x_m - \beta_i} = D_{x_j}\left(\frac{\gamma_{i,j}}{x_m - \beta_i}\right) + D_{x_m}\left(\frac{\gamma_{i,j} D_{x_j}(\beta_i)}{x_m - \beta_i}\right).$$

Then we get

$$\frac{\alpha_i}{x_m - \beta_i} = \sum_{j=1}^{m-1} D_{x_j}\left(\frac{\gamma_{i,j}}{x_m - \beta_i}\right) + D_{x_m}\left(\sum_{j=1}^{m-1} \frac{\gamma_{i,j} D_{x_j}(\beta_i)}{x_m - \beta_i}\right).$$

Therefore, f is rational integrable with respect to $\mathbf{x}$ by taking

$$g_j = \sum_{i=1}^{n} \frac{\gamma_{i,j}}{x_m - \beta_i} \quad \text{and} \quad g_m = r + \sum_{i=1}^{n}\sum_{j=1}^{m-1} \frac{\gamma_{i,j} D_{x_j}(\beta_i)}{x_m - \beta_i}.$$

Note that all the g_j and g_m are in $K(\mathbf{x})$ because $\gamma_{i,j} \in K(\hat{\mathbf{x}}_m)(\beta_i)$ and β_i are roots of a polynomial in $K(\hat{\mathbf{x}}_m)[x_m]$.

Suppose now that f is rational integrable with respect to $\mathbf{x}$, i.e., $f = \sum_{j=1}^{m} D_{x_j}(g_j)$ for some $g_j \in K(\mathbf{x})$. For any $i \in \{1, 2, \ldots, n\}$, taking the x_m-residues of f and $\sum_{j=1}^{m} D_{x_j}(g_j)$, respectively, and using Proposition 3 we get

$$\text{residue}_{x_m}(f, \beta_i) = \alpha_i = \sum_{j=1}^{m-1} D_{x_j}(\text{residue}_{x_m}(g_j, \beta_i)),$$

which implies that α_i is algebraic integrable with respect to $\hat{\mathbf{x}}_m$. ∎

Example 5. *Let $f = 1/(x_1 + x_2)$. Then the x_2-residue of f at $-x_1$ is 1. Since $1 = D_{x_1}(x_1)$, f is rational integrable with respect to x_1 and x_2. More precisely,*

$$f = D_{x_1}\left(\frac{x_1}{x_1 + x_2}\right) + D_{x_2}\left(-\frac{x_1}{x_1 + x_2}\right).$$

Example 6. *Let $f = 1/(x_1 x_2)$. Then the x_2-residue of f at 0 is $1/x_1$. Since $1/x_1$ has no antiderivative in $\overline{K(x_1)}$, f is not rational integrable with respect to x_1 and x_2.*

2.2 Equivalence

Theorem 7. *Let $f \in k(t, \mathbf{x})$. Then $L \in k(t)\langle D_t \rangle$ is a telescoper for f if and only if L is a telescoper for every x_m-residue of f.*

Proof. By a similar calculation as in the proof of Proposition 3, we have

$$L(\text{residue}_{x_m}(f, \beta)) = \text{residue}_{x_m}(L(f), \beta) \tag{3}$$

for any $L \in k(t)\langle D_t \rangle$ and $\beta \in \overline{k(t, \hat{\mathbf{x}}_m)}$. If $L \in k(t)\langle D_t \rangle$ is a telescoper for f, then $L(f) = \sum_{j=1}^{m} D_{x_j}(g_j)$ for some $g_j \in k(t, \mathbf{x})$. By Proposition 3 and Equation (3), for the x_m-residue $\alpha := \text{residue}_{x_m}(f, \beta)$ at any pole β of f with respect to x_m, we have

$$L(\alpha) = \sum_{j=1}^{m-1} D_{x_j}(\text{residue}_{x_m}(g_j, \beta)).$$

So L is a telescoper for α. Conversely, assume that L is a telescoper for any x_m-residue of f. Note that any x_m-residue of $L(f)$ is of the form $L(\text{residue}_{x_m}(f, \beta))$, which is algebraic integrable by assumption. Then $L(f)$ is rational integrable by Lemma 4. Therefore, L is a telescoper for f. ∎

Now we can present an explicit translation between the two telescoping problems by using Theorem 7.

If we can solve Problem 2, then for a rational function $f \in k(t, \mathbf{x})$, first, we can perform Hermite reduction to decompose f into $f = D_{x_m}(r) + a/b$; second, we compute the resultant $R := \text{resultant}_{x_m}(a - zD_{x_m}(b), b) \in k(t, \hat{\mathbf{x}}_m)[z]$; finally, we get a telescoper for f by constructing telescopers for all the roots of R in $\overline{k(t, \hat{\mathbf{x}}_m)}$ and taking their least common left multiple.

On the other hand, if we can solve Problem 1, then for an algebraic function $\alpha \in \overline{k(t, \hat{\mathbf{x}}_m)}$ with minimal polynomial $F \in k[t, \hat{\mathbf{x}}_m, x_m]$, we compute a telescoper L for the rational function $f = x_m D_{x_m}(F)/F$. Note that α is the x_m-residue of f at α. Therefore, L is a telescoper for α.

Example 8. *Consider the rational function*

$$f = \frac{2y(1-x)x(x+1)(x+2)(t+x)(xy - y - t^4)}{1 - x(2 - x + (x+1)(x+2)(t+x)(xy - y - t^4)^2)}.$$

In order to find a telescoper for f, we view f as a rational function in y with coefficients in $k(t, x)$ and determine its residues in $\overline{k(t, x)}$. Write a and b for the numerator and denominator of f. Since b is squarefree, the residues z of f are precisely the roots of the the Rothstein-Trager resultant $\text{resultant}_y(a - zD_y(b), b) \in k(t, x)[z]$. *In the present example, these are*

$$\frac{t^4}{x-1} \pm \frac{1}{\sqrt{x(x+1)(x+2)(x+t)}}.$$

According to Theorem 7, it now suffices to find a telescoper for this algebraic function. This problem is discussed in the following section.

3. TELESCOPERS FOR ALGEBRAIC FUNCTIONS

We showed above how focusing on residues can yield a technique to find telescopers of rational functions by reducing this question to a similar one for algebraic functions. In this section we describe an algorithm to solve this latter problem for algebraic functions of two variables. In what follows, the term "algebraic function" will always refer to functions of two variables t and x. When one tries to use residues to solve the problem of finding telescopers for algebraic functions one must deal with several complications. The first is a technical complication. One does not have a global way of expressing a function similar to partial fractions and so must rely on local expansions. This forces one to look at differentials rather than functions in order to define the notion of residue in a manner that is independent of local coordinates. The second complication is a more substantial one. There are differentials αdx having zero residues everywhere that are not of the form $d\beta = D_x(\beta)dx$, i.e. α is not the derivative of an algebraic function. Nonetheless, one knows that there must exist an operator $L \in k(t)\langle D_t \rangle$ of order equal to twice the genus of the curve associated to f such that $L(\alpha)dx = d\beta$ for some algebraic β. This will force us to add an additional step to find our desired telescoper.

In Section 3.1, we will gather some facts concerning differentials in function fields of one variable that will be needed in our algorithm. In Section 3.2 we describe the algorithm.

3.1 Derivations and Differentials

In this section we review some notation and facts concerning function fields of one variable (cf. [2, 4, 8, 11, 16]). In the previous section the results and calculations depended heavily on the notion of the residue of a rational function of y at an algebraic function β_i of x. In the present section we shall also need to use the notion of a residue but since we are dealing with algebraic functions instead of rational functions, the appropriate notion is that of a residue of a differential ω at a place $\mathcal{P}$ of the associated function field E. We will denote this by $\text{residue}_{\mathcal{P}}\,\omega$ and refer to the above mentioned books for basic definitions and properties. We note that when $f \in E = \overline{K(x)}(y)$, and $\beta_i \in \overline{K(x)}$, then $\text{residue}_y(f, \beta_i) = \text{residue}_{\mathcal{P}}\,\omega$, where $\omega = f dx$ and $\mathcal{P}$ is the place $(y - \beta_i)$ of E.

Let K be a differential field of charactersitic zero with derivation denoted by D_t (for example, $K = k(t)$ with D_t as above). Let x be transcendental over K and $E = K(x, y)$ an algebraic extension of $K(x)$. We may extend the derivation D_t to a derivation D_t^x on $K(x)$ by first letting $D_t^x(x) = 0$ and then taking the unique extension to E. We define a derivation D_x on $K(x)$ by letting D_x be zero on K, $D_x(x) = 1$ and taking the unique extension of D_x from $K(x)$ to E. We shall also assume that the constants $E^{D_x} = \{c \in E \mid D_x(c) = 0\}$ are precisely K. This is equivalent to saying that the minimal polynomial of y over $K(x)$ is absolutely irreducible (cf. [10]). In [8], Chapter VI, §7, Chevalley shows that D_t^x can be used to define a map (which we denote again by D_t^x) on differentials such that $D_t^x(f dx) = (D_t^x(f))dx$. The map D_t^x furthermore has the following properties:

1. $D_t^x(dg) = d(D_t^x g)$ for any $g \in E$, and
2. for any place $\mathcal{P}$ of E and any differential ω,
$$\text{residue}_{\mathcal{P}}(D_t^x\omega) = D_t^x(\text{residue}_{\mathcal{P}}(\omega)).$$

Given $\alpha \in E$ we will want to find an operator $L \in K\langle D_t^x\rangle$ and an element $\beta \in E$ such that $L(\alpha) = D_x(\beta)$. In terms of differentials, this latter equation may be written as $L(\omega) = d\beta$, where $\omega = \alpha dx$.

We shall have occasion to write our field E as $E = K(\bar{x}, \bar{y})$ for some other $\bar{x}$ which is transcendental over K and $\bar{y}$ algebraic over $K(\bar{x})$ and work with the derivation $D_t^{\bar{x}}$ defined in a similar manner as above. We will need to know that if we can find a telescoper with respect to the derivation $D_t^{\bar{x}}$ then we can convert this into a telescoper with respect to D_t^x. The following lemma and proposition allow us to do this.

Lemma 9. *Let x and $\bar{x}$ be as above and let ω be a differential of E. For any $i = 1, 2, \ldots$ there exists $u_i \in E$ such that*
$$(D_t^{\bar{x}})^i(\omega) - (D_t^x)^i(\omega) = du_i. \tag{4}$$

Proof. Write $\omega = \bar{\alpha} d\bar{x}$. Lemma 1 of [16] (see also Lemma 3 in Chapter VI, §7 of [8]) implies that
$$D_t^{\bar{x}}(\omega) - D_t^x(\omega) = -d(\bar{\alpha} D_t^x(\bar{x})). \tag{5}$$
Letting $u_1 = -\bar{\alpha} D_t^x(\bar{x})$, we have equation (5) for $i = 1$. One can verify by induction that (5) holds for $u_{i+1} = D_t^{\bar{x}}(u_i) - v_i D_t^x(\bar{x})$, where $v_i = (D_t^x)^i[\bar{\alpha} D_x \bar{x}] \cdot D_{\bar{x}}(x)$. ■

Proposition 10. *Let $\alpha \in E$, $\omega = \alpha dx$,*
$$(D_t^{\bar{x}})^n + a_{n-1}(D_t^{\bar{x}})^{n-1} + \ldots + a_0 \in K\langle D_t^{\bar{x}}\rangle,$$
and $\bar{\beta} \in E$ such that
$$\big((D_t^{\bar{x}})^n + a_{n-1}(D_t^{\bar{x}})^{n-1} + \ldots + a_0\big)(\omega) = d\bar{\beta}.$$
One can effectively find $\beta \in E$ such that
$$\big((D_t^x)^n + a_{n-1}(D_t^x)^{n-1} + \ldots + a_0\big)(\alpha) = D_x(\beta).$$

Proof. From Lemma 9 we have that
$$\begin{aligned}&\big((D_t^{\bar{x}})^n + a_{n-1}(D_t^{\bar{x}})^{n-1} + \ldots + a_0\big)(\omega)\\ &= ((D_t^x)^n(\omega) + du_n) + a_{n-1}((D_t^x)^{n-1}(\omega) + du_{n-1})\\ &\quad + \ldots + a_0\omega.\end{aligned}$$
Therefore, taking into account that the a_i belong to K,
$$\begin{aligned}&\big((D_t^x)^n + a_{n-1}(D_t^x)^{n-1} + \ldots + a_0\big)(\omega)\\ &= d\big(\bar{\beta} - u_n - a_{n-1}u_{n-1} - \ldots - a_1u_1\big),\end{aligned}$$
which implies the conclusion of the proposition with $\beta = \bar{\beta} - u_n - a_{n-1}u_{n-1} - \ldots - a_1 u_1$. ■

In the algorithm described in the next section, we will consider a differential ω in $E = K(x, y)$ and assume that

1. ω has no poles at any place above the place of $K(x)$ at infinity, and
2. the places where ω does have a pole are all unramified above places of $K(x)$.

We describe below an algorithm that allows one to select an $\bar{x} \in E$ such that $E = K(\bar{x}, y)$ and such that ω satisfies conditions 1. and 2. above with respect to $K(\bar{x})$. The algorithm of Section 3.2 can be used to produce a telescoper with respect to $D_t^{\bar{x}}$ and Proposition 10 allows one to convert this telescoper to a telescoper with respect to D_t^x. In the following proposition, the proof that condition 2. can be fulfilled was outlined to us by Barry Trager [21, 22].

Proposition 11. *Let ω be a differential in $E = K(x, y)$. One can effectively find an $\bar{x} \in E$ such that $E = K(\bar{x}, y)$ and*

1. *ω has no poles at any place above the place of $K(\bar{x})$ at infinity, and*
2. *the places where ω does have a pole are all unramified above places of $K(\bar{x})$.*

Proof. If 1. does not hold, let $c \in K$ be selected so that ω has no poles above $x = c$, let
$$\bar{x} = \frac{cx}{x - c}.$$
This change of variables interchanges c and the point at infinity, so 1. is now satisfied with respect to $K(\bar{x})$ and we shall henceforth abuse notation and assume that 1. is satisfied with respect to $K(x)$.

Let $\mathcal{C}$ be a nonsingular curve that is a model of E. The elements of E can be considered as functions on $\mathcal{C}$. As noted in [21, p. 63], ramification occurs when the line of projection from the curve down to the x-axis is tangent to the curve and, for each pole of ω, there are only a finite number of projection directions that are tangent to the curve at this pole. Therefore for all but finitely many choices of an integer m, if we let $\bar{x} = x + my$, ω will satisfy 2. with respect to $K(\bar{x})$. One can refine this argument and produce a finite set of integers m that are to be avoided. This is done in the following way.

Let M be an indeterminate and consider the field $E_1 = E(M) = k_1(\bar{x}, y)$, where $k_1 = K(M)$ and $\bar{x} = x + My$. Let $\mathfrak{o} = K[M]$ and assume that (after a possible change of y), y satisfies a monic polynomial over $\mathfrak{o}[\bar{x}]$. The behavior of various objects in E_1 when one reduces $\mathfrak{o}$ modulo a prime ideal of $\mathfrak{o}$ is considered in [11, Chapter III, §6]. We shall be interested in reducing modulo ideals of the form $(M - m)$, where m is an integer. One can effectively calculate an integral basis $\{w_i(M)\}$ of the integral closure of $k_1[\bar{x}]$ in E_1 (cf. [12, 21]) and from this a complementary basis $\{w'_i(M)\}$ ([2, Chapter 5, §2], [4, §22]). In Chapter III §6.2 of [11], Eichler gives a method that will produce a finite set $S \subset \mathbb{Z}$ such that for $m \notin S$, the set $\{w_i(m)\}$ is again an integral basis of the integral closure of $K[\bar{x}]$ in E. This method can be refined (and the set S slightly increased if need be) so that $\{w'_i(m)\}$ is also a complementary basis. Expressing ω in terms of this complementary basis,

$$\omega = \frac{1}{b(\bar{x})} \sum_{i=1}^{n} p_i(M, \bar{x}) w'_i(M) d\bar{x},$$

one sees that ω will have poles precisely at the zeroes of $b(\bar{x})$. If one selects $m \in \mathbb{Z}$ such that $b(\bar{x})$ is relatively prime to $D(\bar{x})$, the discriminant of the integral basis $\{w_i(m)\}$, then ω will not have poles at ramification points. The finitely many values of m that do not satisfy this latter condition are roots of

$$\begin{aligned} S(M) = \mathrm{resultant}_X(\, &\mathrm{resultant}_Y(b(X+MY), F(X,Y)), \\ &\mathrm{resultant}_Y(D(X+MY), F(X,Y))), \end{aligned}$$

where $F \in K[X, Y]$ is the minimal polynomial of y over $K(x)$. ■

3.2 An Algorithm to Calculate Telescopers for Algebraic Functions

We assume we are given a function field of one variable $E = K(x, y)$ and a differential ω in E. We shall furthermore assume that ω satisfies conditions 1. and 2. of Proposition 11. We will describe an algorithm to find $a_0, \ldots, a_n \in K$, not all zero, and $\beta \in E$ such that

$$(a_n(D^x_t)^n + a_{n-1}(D^x_t)^{n-1} + \ldots + a_0)(\omega) = d\beta.$$

If $\omega = \alpha dx$, then $L = a_n(D^x_t)^n + a_{n-1}(D^x_t)^{n-1} + \ldots + a_0$ is a telescoper for α with certificate β. The algorithm has two steps. The first step finds an operator L_1 such that applying this operator to ω results in a differential $L_1(\omega)$ with only zero residues. The second step finds an operator L_2 of order at most twice the genus of E and an element $\beta \in E$ such that $L_2(L_1(\omega)) = d\beta$.

Step 1. We will describe two methods for constructing an operator that annihilates the residues of ω. The first one requires one to calculate in algebraic extensions of K while the second only requires calculations in K. Throughout, let $F(x, Y) \in K[x, Y]$ be a minimal polynomial of y over $K(x)$ and let

$$\omega = \alpha dx = \frac{A}{B} dx$$

for some $A \in K[x, y]$ with no finite poles and $B \in K[x]$.

Method 1. We make no assumptions concerning ramification at the poles but for convenience we do assume that the poles of ω only occur at finite points. Let $a \in \overline{K}$ be a root of B. For any branch of $F(x, Y) = 0$ at $x = a$, we may write

$$\omega = p_a(z) dz,$$

where $z = (x - a)^{1/m}$ for some positive integer m and p_a is a Laurent series in z with coefficients in $\overline{K}$. One can calculate the coefficient of $1/z$ in p_a and this will be the residue of ω at this place. In this way, one can calculate the possible residues $\{r_1, \ldots, r_s\}$ of ω. Let K_1 be a Galois extension of K containing $\{r_1, \ldots, r_s\}$. Let C be the field of D_t-constants in K_1 and $\{\tilde{r}_1, \ldots, \tilde{r}_\ell\}$ be a C-basis of $Cr_1 + \ldots + Cr_s$. Let $L(Y) = wr(Y, \tilde{r}_1, \ldots \tilde{r}_\ell)$ where $wr(\ldots)$ is the Wronskian determinant. One sees that $L(Y)$ is a nonzero linear differential polynomial with coefficients in K_1 such that $L_1(r_i) = 0$ for $i = 1, \ldots, s$. Define

$$L_1(Y) = \mathrm{lclm}\{L^\sigma(Y) \mid \sigma \in G\},$$

where G is the Galois group of K_1 over K, $L^\sigma(Y)$ denotes the linear differential polynomial resulting from applying σ to each coefficient of L and lclm denotes the least common left multiple. We then have that $L_1(Y)$ has coefficients in K and annihilates the residues of ω.

Method 2. We now assume that ω has poles only at finite places and that there is no ramification at the poles. This implies that at any place corresponding to a pole, we may write $\alpha = \sum_{i \geq i_0} \alpha_i (x - x_0)^i$ for some $\alpha_i \in \bar{K}$. Therefore the residue of ω at this place is

$$\alpha_{-1} = \left(D_x[(x - x_0)^{-i_0 - 1}\alpha]\right)_{x = x_0}.$$

This is the key to the following, parts of which in a slightly different form appear in [7].

Proposition 12. *Given ω as above, one can compute a polynomial $R \in K[Z]$ of degree*

$$m := \deg_Z(R) \leq \deg_Y(F) \deg_x(B^*),$$

with B^* the square free part of B, such that if a is a nonzero residue of ω then $R(a) = 0$. Furthermore, one can compute a nonzero operator $L_1 = a_m(D^x_t)^m + a_{m-1}(D^x_t)^{m-1} + \ldots + a_0 \in K\langle D^x_t \rangle$ such that $\tilde{\omega} := L_1(\omega)$ has residue zero at all places.

Proof. We may write

$$\alpha dx = \frac{A}{B} dx = \frac{A_1}{B_1} dx + \frac{A_2}{B_2^2} dx + \cdots + \frac{A_\ell}{B_\ell^\ell} dx,$$

where the $A, A_i \in K(x, y)$ are regular at finite places and $B = B_1 B_2^2 \cdots B_\ell^\ell \in K[x]$ is the squarefree decomposition of B. To achieve our goal it is therefore enough to prove the claim for a differential of the form $\alpha dx = \frac{A}{B^n} dx$, where $A \in K(x, y)$ is regular at finite places and $B \in K[x]$ is squarefree. Following [7], we let u be a differential indeterminate and let

$$h = \frac{(Au^{-n})^{(n-1)}}{(n-1)!} \in K(x, y)\langle u \rangle,$$

where $K(x, y)\langle u \rangle$ is the ring of differential polynomials in u with coefficients in $K(x, y)$ and $(\ldots)^{(i)}$ denotes i-fold differentiation with respect to x. Let $\mathcal{P}$ be a place where α has a pole and let a and b denote the values of x and y at the place. We note that since A is regular at $\mathcal{P}$ and $\mathcal{P}$ is not ramified, any derivative of A is also regular at $\mathcal{P}$ (one needs the hypothesis that these places are unramified to make this claim). Taking into account the rules of differentiation, we see that

$$h = \frac{p(x, y, u, u', \ldots, u^{(n-1)})}{q(x)u^t},$$

where $p(x, Y, z_0, z_1, \ldots, z_{n-1}) \in K[x, Y, z_0, z_1, \ldots, z_{n-1}]$, t is some positive integer and $q(x) \in K[x]$ does not vanish at $\mathcal{P}$, i.e. $q(a) \neq 0$. Let

$$\tilde{p} = p(x, Y, B', \tfrac{1}{2}B'', \tfrac{1}{3}B^{(3)}, \ldots, \tfrac{1}{n}B^{(n)}) \in K[x, Y]$$

and

$$\tilde{q} = q(x)(B')^t \in K[x].$$

One then shows, as in [7], that $\tilde{p}(a, b)/\tilde{q}(a)$ is the residue of $\frac{A}{B^n}dx$ at $\mathcal{P}$.

The above argument shows that the polynomial

$$R = \mathrm{resultant}_x\big(\mathrm{resultant}_Y(\tilde{p} - Z\tilde{q}, F), B\big) \in K[Z]$$

vanishes at the residues of αdx. The degree estimate for R follows from the general degree estimate for resultants which states for any $S, T \in K[u, v]$ that $\deg_u(\mathrm{resultant}_v(S, T))$ is at most

$$\deg_u(S)\deg_v(T) + \deg_v(S)\deg_u(T).$$

This implies first that the inner resultant in the definition of R has Z-degree at most $\deg_Y(F)$. (Note that no degree estimates for $\tilde{p}$ and $\tilde{q}$ are needed because $\deg_Z(F) = 0$.) Applying the rule again to the outer resultant gives the desired bound $\deg_Y(F)\deg_x(B)$.

Let $R \in K[Z]$ be the polynomial above. If necessary, we may replace R by a squarefree polynomial having the same nonzero roots so we shall assume that R is squarefree and of degree m. Using the fact that R and $\frac{dR}{dZ}$ are relatively prime, there exist polynomials $R_i \in K[Z]$ of degree at most $m - 1$ such that if γ is a root of R, then $D_t^i(\gamma) = R_i(\gamma)$ for $i = 0, 1, \ldots$. Since each R_i has degree at most $m - 1$, there exist $a_m, \ldots, a_0 \in K$, not all zero, such that $(a_m(D_t^x)^m + a_{m-1}(D_t^x)^{m-1} + \ldots + a_0)(\gamma) = 0$ for any root γ of R. Using the fact that $\mathrm{residue}_{\mathcal{P}}(D_t^x\omega) = D_t^x(\mathrm{residue}_{\mathcal{P}}(\omega))$ for any place $\mathcal{P}$, one sees that for $L_1 = a_m(D_t^x)^m + a_{m-1}(D_t^x)^{m-1} + \ldots + a_0$, $\tilde{\omega} = L_1(\omega)$ has zero residue at any place. ■

Remark. Although Method 2 does not require calculations in an algebraic extension of K, one needs the condition on ramification to prove that it is correct. This condition is painful to verify and although Propositions 10 and 11 imply that we can make a transformation, if necessary, to guarantee that the differential has poles at places that are not ramified, making such a transformation can increase the complexity of the data. In practice, one should calculate the operator L_1 above without testing if the places at poles are ramified, calculate the operator L_2 as in step 2 below (which requires no assumption concerning ramification) and then test to see if the resulting operator $L_2 \circ L_1$ is a telescoper by checking if the identity $L_2(L_1(\alpha)) = D_x(\beta)$ holds, a simple calculation in $K(x, y)$. If this equality does not hold, then one can make a change of variable $\bar{x} := x + my$ for a random m and try again. Proposition 11 guarantees that after a finite number of trials one will succeed.

Example 13 (continuing Ex. 8)**.** *Let $F = y^2 - x(x+1)(x+2)(x+t)$ and consider*

$$\omega = \Big(\frac{t^4}{x-1} + \frac{1}{y}\Big)dx = \frac{u}{v}dx,$$

where $u = (x-1)y + t^4x(x+1)(x+2)(t+x)$ and $v = x(x+1)(x+2)(x+t)(x-1)$. The only pole of ω is a simple pole at $x = 1$, so the residues of ω are the roots of

$$\mathrm{residue}_x(\mathrm{residue}_y(u - zD_x(v), F), v) = (\ldots)(z - t^4)^2z^8,$$

where $(\ldots)$ stands for some factors which are free of z and therefore irrelevant here. The only nonzero residue t^4 is annihilated by $L_1 := tD_t - 4$, so

$$\tilde{\omega} = (tD_t - 4)(\omega) = -\frac{(9t + 8x)y}{2x(x+1)(x+2)(x+t)^2}dx$$

has no nonzero residues.

Remark.

1. In [21], Trager develops a Hermite reduction method for algebraic functions which, when applied to the differential ω above, shows how one can write $\omega = (D_x(g_1) + g_2)dx$, where $g_1, g_2 \in E$ and g_2 has only simple poles at finite points. Regretably, g_2 may have poles (of higher order) at infinity. Nonetheless, it would be interesting to see if Trager's procedure can be used to increase efficiency in our algorithm.

2. The above argument strongly relies on the fact that we are assuming that the places where ω has poles are not ramified above places in $K(x)$. It would be of interest to give a method to calculate an operator L_1 satisfying the conclusion of Proposition 12 without this assumption.

Step 2. Let $\tilde{\omega}$ be as in the conclusion of Proposition 12. Again using the fact that $\mathrm{residue}_{\mathcal{P}}(D_t^x\tilde{\omega}) = D_t^x(\mathrm{residue}_{\mathcal{P}}(\tilde{\omega}))$ for any place $\mathcal{P}$, we have for all $i \in \mathbb{Z}$ that $(D_t^x)^i(\tilde{\omega})$ is again a differential with zero residues at all places. Such a differential is called a *differential of the second kind* ([8], p. 50) and a differential of the form $d\gamma, \gamma \in E$ is called an *exact differential*. Note that any exact differential is a differential of the second kind. Corollary 1 of ([8], p. 130) states that the factor space of the space of differentials of the first kind by the space of exact differentials is a K-vector space of dimension equal to $2G$, where G is the genus of E. Therefore, there exist $\tilde{a}_{2G}, \ldots, \tilde{a}_0 \in K$, not all zero, such that for $L_2 = \tilde{a}_{2G}(D_t^x)^{2G} + \tilde{a}_{2G-1}(D_t^x)^{2G-1} + \ldots + \tilde{a}_0$, $L_2(\tilde{\omega}) = d\beta$ for some $\beta \in E$. Such L_2 and β can be found as follows.

Let $\tilde{\omega} = \tilde{\alpha}dx$ and let $[E : K(x)] = m$. For each $i \geq 0$, there exist $\alpha_{i,0}, \ldots, \alpha_{i,m-1} \in K(x)$ such that

$$(D_t^x)^i(\tilde{\alpha}) = (y, \ldots, y^{m-1})\begin{pmatrix} \alpha_{i,0} \\ \vdots \\ \alpha_{i,m-1} \end{pmatrix}.$$

In addition, there exists an $m \times m$ matrix A with entries in $K(x)$ such that

$$(D_x(y), \ldots, D_x(y^{m-1})) = (y, \ldots, y^{m-1})A.$$

Let $a_0, \ldots, a_{2G}$ be elements of K and $\beta_0, \ldots, \beta_{m-1}$ elements of $K(x)$. Letting $\beta = \beta_0 + \beta_1y + \ldots + \beta_{m-1}y^{m-1}$, the equation

$$d\beta = (a_{2G}(D_t^x)^{2G}(\tilde{\alpha}) + \ldots + a_0\tilde{\alpha})dx$$

is equivalent to

$$D_x \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{m-1} \end{pmatrix} + A \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{m-1} \end{pmatrix}$$
$$= a_{2G} \begin{pmatrix} \alpha_{2G,0} \\ \vdots \\ \alpha_{2G,m-1} \end{pmatrix} + \ldots + a_0 \begin{pmatrix} \alpha_{0,0} \\ \vdots \\ \alpha_{0,m-1} \end{pmatrix}. \quad (6)$$

In [3], Barkatou describes a decision procedure for deciding if there exist nontrivial $\beta_0, \ldots, \beta_{m-1} \in K(x)$ and $a_0, \ldots, a_{2G} \in K$ satisfying (6) when K is a computable field (i.e., the arithmetic operations and derivation are computable and one has an algorithm to factor polynomials over K). Therefore one can apply this to $K = k(t)$, where k is a computable field of characteristic zero to produce a desired L_2 and β.

Example 14 (continuing Ex. 13)**.** *Let again $F = y^2 - x(x+1)(x+2)(x+t)$ and consider the differential*

$$\tilde{\omega} = -\frac{(9t + 8x)y}{2x(x+1)(x+2)(x+t)^2}dx.$$

Since the field E has genus 1 and $\tilde{\omega}$ has only zero residues, there exists a telescoper for $\tilde{\omega}$ of order 2. Indeed, the algorithm outlined above finds that $L_2(\tilde{\omega}) = d\beta$, where

$$L_2 = 4(99t^5 - 540t^4 + 1055t^3 - 870t^2 + 256t)D_t^2$$
$$+ 4(297t^4 - 1269t^3 + 1900t^2 - 1152t + 256)D_t$$
$$+ 3(99t^3 - 306t^2 + 307t - 96)$$

and

$$\beta = \frac{3(429t^3 + 330t^2x - 891t^2 - 648tx + 384t + 256x)y}{(t+x)^3}.$$

For the differential ω from Example 13, it follows that we have $L\omega = d\beta$ with

$$L = L_2 \circ (tD_t - 4) = 4(t-2)(t-1)t^2(99t^2 - 243t + 128)D_t^3$$
$$+ 4t(99t^4 - 189t^3 - 210t^2 + 588t - 256)D_t^2$$
$$- 3(1089t^4 - 4770t^3 + 7293t^2 - 4512t + 1024)D_t$$
$$- 12(99t^3 - 306t^2 + 307t - 96).$$

By Theorem 7, this operator L is also a telescoper for the trivariate rational function f from Example 8. Certificates g, h with

$$L(f) = D_x(g) + D_y(h)$$

can be obtained from β following the calculations in the proof of Lemma 4. They are however too long to be printed here.

Remark. Telescopers and certificates for *holomorphic* differentials arise in Manin's solution of Mordell's Conjecture [16, 17] and Step 2 of our procedure is just an effective version of considerations that appear in these papers. Telescopers for holomorphic differentials are also referred to as *Gauss-Manin Connections.*

Combining the estimates on the order of the operators computed in steps 1 and 2 gives the following bound on the order of telescopers for algebraic functions. It can be viewed as a generalization of Corollary 14 in [5], which says that for every rational function $f = A/B \in K(x)$ there exists a telescoper of order at most $\deg_x B^*$, where B^* is the square free part of B.

Theorem 15. *Let E be an algebraic extension of $K(x)$, $\alpha = A/B \in E$ so that A is regular at finite places and $B \in K[x]$. Let B^* be the square free part of B. Then there exists $\beta \in E$ and a nonzero operator $L \in K\langle D_t \rangle$ with $L(\alpha) = D_x(\beta)$ and*

$$\deg_{D_t}(L) \leq [E : K(x)] \deg_x(B^*) + 2\,\mathrm{genus}(E).$$

4. IMPLEMENTATION AND OTHER EXAMPLES

We have produced a prototype implementation of the algorithms described above on top of Koutschan's Mathematica package "HolonomicFunctions.m" [13] and compared the performance to the built-in creative telescoping implementations of this package. In order to make the comparison as fair as possible, we have tried to reuse as much code from Koutschan's package as possible, so that the timings will not implicitly compare two different implementations of some subroutine but reflect as closely as possible the speed-up (or slow-down) offered by the ideas presented above.

Five different methods to solve the creative telescoping problem for a rational function $f \in k(t, x, y)$ were considered: **(CC)** first use Chyzak's algorithm [9] to find a holonomic system S of operators in $k(t, x)\langle D_t, D_x \rangle$ such that for all $L \in S$ there exists a rational function $g \in k(t, x, y)$ with $L(f) = D_y(g)$, afterwards apply the same algorithm to S to obtain a telescoper $L \in k(t)\langle D_t \rangle$ for f; **(CK)** first compute $S \subseteq k(t, x)\langle D_t, D_x \rangle$ as in variant (CC), then apply Koutschan's ansatz [14] to S to obtain a telescoper L for f; **(K)** compute a telescoper for f directly with Koutschan's ansatz; **(EC)** use the reduction from Section 2, then apply Chyzak's algorithm to the resulting algebraic functions, and then take the least common left multiple of the results; **(EA)** use the reduction from Section 2, then apply the algorithm from Section 3 to the resulting algebraic functions, and then take the least common left multiple of the results.

Table 1 shows the performance of these five approaches for the following examples.

1. The rational function f from Example 8 above. This example is not representative but was particularly designed to be easy for our algorithms and difficult for the known ones.

2. Here $f := \frac{1}{xy}h(\frac{t}{xy}, x, y)$ with $h(t, x, y) = \big(1 - \frac{x}{1-x} - \frac{y}{1-y} - \frac{t}{1-t} - \frac{xy}{1-xy} - \frac{xt}{1-xt} - \frac{yt}{1-yt} - \frac{xyt}{1-xyt}\big)^{-1}$. This is the problem of enumerating diagonal 3D-Queens walks raised in [6]. Our calculation confirms the correctness of the telescoper conjectured there.

3. Let now $h(t, x, y) = \big(1 - \frac{xy}{1-xy} - \frac{xt}{1-xt} - \frac{yt}{1-yt} - \frac{xyt}{1-xyt}\big)^{-1}$ and $f = \frac{1}{xy}h(\frac{t}{x^2y}, x, y)$. This is a variation of the previous problem, with the points $(2n, n, n)$ replacing the diagonal and now allowing steps along the axes.

4. The rational function $h(t, x, y) = 2t^2/((1-t)(3-(x+y+t+xy+xt+yt)+3xyt))$ appears in [18] as the generating function for the probability of certain structures in random groves. See [18] for details on the combinatorial background. Here we compute the diagonal series coefficients of f by applying creative telescoping to $f = \frac{1}{xy}h(\frac{t}{xy^3}, x, y)$. As can be seen in this example, our algorithms are in some cases not superior.

	CC	CK	K	EC	EA	telescoper statistics order	degree	bytecount
1	>150h	4000.89	469.03	1.30	1.04	3	6	3464
2	16029.55	40043.01	>100h	1390.14	1646.53	6	71	76472
3	>150h	350495.88	>150h	203.44	328.08	9	93	140520
4	638.70	1099.08	>40Gb	37606.28	216201.88	10	32	41840
5	23823.70	676.13	19085.67	1114.34	3117.43	7	27	25320

Table 1: Runtime comparison for the examples described in the text.

5. With h as before, we now consider $f = \frac{1}{xy}h(\frac{1}{x^2y^2}, x, y)$. Note the large difference between CC and CK.

We have put timings for a number of additional examples on the website [1]. Also our code and the certificates for Example 13 can be found there. The examples we tested suggest that the reduction from rational functions to algebraic functions can cause a decent speed-up. It does seem to depend on whether the Rothstein-Trager resultant of the input factors into several small factors or not. If it does, it is advantageous because solving several small instances of Problem 2 is cheaper than solving a single big one. Whether after the reduction, the algorithm of Section 3 or some other method is applied to the resulting algebraic functions, makes usually not much of a difference. Our algorithm tends to be faster when Step 1 in Section 3.2 already finds a great part of the telescoper, leaving only a small coupled differential system to be solved in Step 2.

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