Computer Proofs for Polynomial Identities in Arbitrarily Many Variables

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Motivation

- For every $n \in \mathbb{N}$, we have

$$\Bigl(\sum_{k=1}^{n} x_k\Bigr)^3 = \sum_{i=1}^{n} x_i^3 + 3\sum_{i=1}^{n}\sum_{j=1}^{i-1}(x_i^2 x_j + x_i x_j^2) + 6\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1} x_i x_j x_k$$

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- But how to prove the identity for *general* n?
- Can this been done algorithmically?

Overview

Admissible univariate sequences

Zero equivalence test for admissible sequences

Extension to arbitrarily many variables

Admissible univariate sequences

Nested Polynomial Recurrences

- A sequence is admissible if it satisfies a *(nested) polynomial recurrence.*
- Example: Definition of a sequence $(f_1(n))_{n=1}^{\infty}$

$$f_1(1), f_1(2), f_1(3)\text{: initial values of } f_1$$

$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$f_1(5)$	$f_1(6)$	$f_1(7)$
$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$	$f_2(6)$	$f_2(7)$
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$$p = \text{poly or } p = 1/\text{poly} \quad \text{fixed}$$

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- sequences like 2^{2^n}
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- indefinite sums and products of other admissible sequences
- indefinite continued fractions of other admissible sequences

Zero equivalence test for admissible sequences

Algebraic Representation of admissible Sequences

- Model admissible sequences by *difference algebra* concepts

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- Consider the $t_{i,j}$ as indeterminates of a polynomial ring
- The recurrence relations give rise to polynomial relations among these indeterminates.

Proving Zero Equivalence of Admissible Sequences

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- *Goal:* Show that $f_3(n) = 0$ for all $n \in \mathbb{N}$
- *Idea:* Use ideal arithmetic to construct an induction proof
- *Observation:* Every $t_{i,j}$ (j high enough) is "connected" with other indeterminates via a polynomial relation

$$\underbrace{t_{i,j} - \text{poly}}_{=:d(t_{i,j})} = 0 \quad \text{or} \quad \underbrace{\text{poly} \cdot t_{i,j} - 1}_{=:d(t_{i,j})} = 0$$

The polynomial $d(t_{i,j})$ is called the *defining relation* of $t_{i,j}$.

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$t_{1,0}$	$t_{1,1}$	$t_{1,2}$	$d(t_{1,3})$	$d(t_{1,4})$	$t_{1,5}$	$t_{1,6}$
$t_{2,0}$	$t_{2,1}$	$d(t_{2,2})$	$d(t_{2,3})$	$d(t_{2,4})$	$t_{2,5}$	$t_{2,6}$
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\hline
\multicolumn{4}{c}{\underbrace{t_{3,0} \qquad t_{3,1} \qquad t_{3,2} \qquad t_{3,3}}_{\overset{!}{=}0 \text{ by IH}}} & & &
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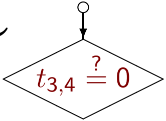
$$\underbrace{t_{3,0} \qquad t_{3,1} \qquad t_{3,2} \qquad t_{3,3}}_{\overset{!}{=}0 \text{ by IH}}$$



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- This can be decided by a radical membership test in $K[t_{1,0}, \ldots, t_{3,4}]$

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$t_{3,0}$	$t_{3,1}$	$t_{3,2}$	$t_{3,3}$	$d(t_{3,4})$	$t_{3,5}$	$t_{3,6}$

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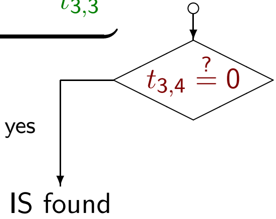

Proving Zero Equivalence of Admissible Sequences

- *Goal:* Show that $f_3(n) = 0$ for all $n \in \mathbb{N}$

$$\begin{array}{ccccccc}
t_{1,0} & t_{1,1} & t_{1,2} & d(t_{1,3}) & d(t_{1,4}) & t_{1,5} & t_{1,6} \\
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\hline
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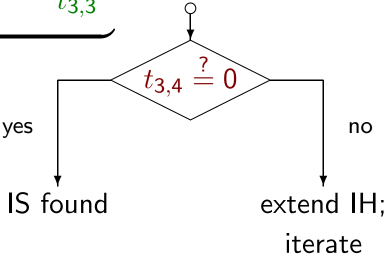

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t_{3,0} & t_{3,1} & t_{3,2} & t_{3,3} & d(t_{3,4}) & d(t_{3,5}) & d(t_{3,6})\\
\hline
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$t_{3,0}$	$t_{3,1}$	$t_{3,2}$	$t_{3,3}$	$d(t_{3,4})$	$d(t_{3,5})$	$\cdots$
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- Finally, check sufficiently many initial values

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\end{array}
$$

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- *Termination:* see paper

Extension to arbitrarily many variables

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- *Fix:* Put all $t_{i,j}$ without relations into the ground field

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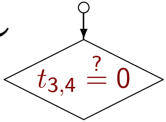

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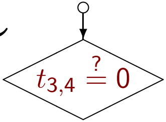

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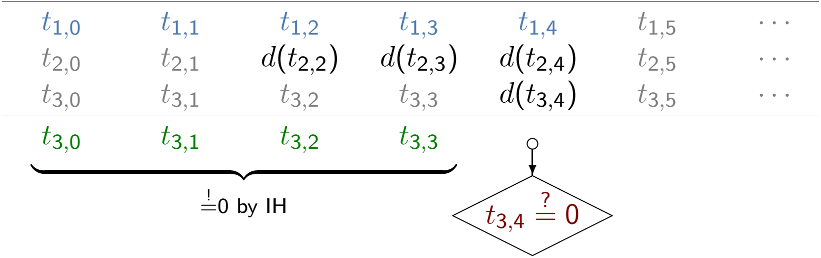

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- This can be decided by a radical membership test in $K(t_{1,0}, \ldots, t_{1,4})[t_{2,0}, \ldots, t_{3,4}]$
- Everything else carries over literally

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Prove: $$\sum_{k=0}^{n}\sum_{i=0}^{k} x_i = (n+1)\sum_{k=0}^{n} x_k - \sum_{k=0}^{n} k\,x_k.$$

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$f_3(n) = \sum_{k=0}^{n} k x_k$ $f_3(0) = 0, f_3(n+1) = f_3(n) + f_0(n+1) f_1(n+1)$

$f_4(n) =$ lhs $f_4(0) = x_0, f_4(n+1) = f_4(n) + f_2(n+1)$

$f(n) =$ lhs − rhs $f(0) = 0, f(n) = f_4(n) - (f_1(n)+1) f_2(n) - f_3(n)$

Simple Example

Prove: $\displaystyle \sum_{k=0}^{n}\sum_{i=0}^{k} x_i = (n+1)\sum_{k=0}^{n} x_k - \sum_{k=0}^{n} k\,x_k.$

Step 1 Translate recurrences to defining relations

$$
\begin{array}{rl}
f_0(n) \sim t_{0,0} & \text{none} \\
f_1(n) \sim t_{1,0} & t_{1,1} - t_{1,0} - 1 \\
f_2(n) \sim t_{2,0} & t_{2,1} - t_{2,0} - t_{0,1} \\
f_3(n) \sim t_{3,0} & t_{3,1} - t_{3,0} - t_{0,1}t_{1,1} \\
f_4(n) \sim t_{4,0} & t_{4,1} - t_{4,0} - t_{2,1} \\
f(n) \sim t_{5,0} & t_{5,0} - t_{4,0} + (t_{1,0} + 1)t_{2,0} + t_{3,0}
\end{array}
$$

Let D be the set of defining relations.

Simple Example

Prove: $\displaystyle\sum_{k=0}^{n}\sum_{i=0}^{k} x_i = (n+1)\sum_{k=0}^{n} x_k - \sum_{k=0}^{n} k\,x_k.$

Step 2 Find the induction step

$$t_{5,1} \in \mathrm{Rad}(\langle\{t_{5,0}\} \cup D\rangle)$$

This means $\forall\ n \in \mathbb{N} : f(n) = 0 \ \Rightarrow\ f(n+1) = 0$.
(No iteration necessary in this example.)

Simple Example

Prove: $\displaystyle\sum_{k=0}^{n}\sum_{i=0}^{k} x_i = (n+1)\sum_{k=0}^{n} x_k - \sum_{k=0}^{n} k\, x_k.$

Step 3 Check initial conditions: $f(0) = 0$. $\square$

Further Examples

- *Christoffel-Darboux identity:* For each $(c_n)_{n=1}^{\infty}$, $(\lambda_n)_{n=1}^{\infty}$ the recurrence

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$$

defines a family of orthogonal polynomials. We can prove

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(u)}{\prod_{i=1}^{k+1} \lambda_i} = \frac{P_{n+1}(x)P_n(u) - P_n(x)P_{n+1}(u)}{(x-u)\prod_{i=1}^{n+1} \lambda_i}$$

$$\sum_{k=0}^{n} \frac{P_k(x)^2}{\prod_{i=1}^{k+1} \lambda_i} = \frac{P_n(x)P'_{n+1}(x) - P_{n+1}(x)P'_n(x)}{\prod_{i=1}^{n+1} \lambda_i}$$

for general $(c_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$.

Further Examples

- *A hypergeometric identity for general $_mF_n$* Defining the multivariate sequences $f(n,m)$ and $g(n,m)$ by

$$f(n,m) = F\left(\begin{matrix} a_1, a_1 + \frac{1}{2}, \dots, a_m, a_m + \frac{1}{2} \\ b_1, b_1 + \frac{1}{2}, \dots, b_n, b_n + \frac{1}{2}, \frac{1}{2} \end{matrix} \,\middle|\, (2^{m-n-1}z)^2\right)$$

$$g(n,m) = \frac{1}{2}\left[F\left(\begin{matrix} 2a_1, \dots, 2a_m \\ 2b_1, \dots, 2b_n \end{matrix} \,\middle|\, z\right) + F\left(\begin{matrix} 2a_1, \dots, 2a_m \\ 2b_1, \dots, 2b_n \end{matrix} \,\middle|\, -z\right)\right],$$

we can prove

$$f(n,m) = g(n,m)$$

for general n and m (see paper).

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- . . . but is there any use of all this?