Computing Limits of Sequences

Manuel Kauers*

Research Institute for Symbolic Computation
Johannes Kepler Universität Linz
4040 Linz, Austria

June 12, 2003

1 Introduction

Automated asymptotic methods only recently became subject of symbolic computation. Contributions like [3, 12, 5, 10, 13] discuss a variety of aspects concerning the asymptotic analysis of continuous, real-valued functions. First approaches employed *generalized series expansion* [5, 3], later methods are based on the theory of *Hardy fields* [6].

So far, it seems that more emphasis was laid on the treatment of continuous functions, and although some mathematical fundaments are already available [1, 2], algorithmic approaches for the discrete case seem rare.

Our poster focuses on limit computation of *sequences* in $\mathbb{Q}$, more specifically, of sequences that can be defined by $\Pi\Sigma$-expressions [7]. We will present a rather simple approach which avoids the use of heavy theory but whose first applications already give promising fine results.

2 A discrete Analogon to l'Hospital's Rule

As is pointed out, e.g., in [4], the difference operator Δ, defined by $\Delta a_n := a_{n+1} - a_n$ can be viewed as a discrete analogon to the differential operator $D\colon f \mapsto f'$ of the continuous world. For instance, the discrete product rule

$$\Delta(a_n b_n) := (\Delta a_n) b_n + a_{n+1} \Delta b_n$$

closely resembles the product rule for differentiation.

Analogously, the summation operator Σ, defined by $\Sigma a_n := \sum_{k=1}^{n} a_k$, corresponds to indefinite integration in the continuous setting.

This correspondence between discrete and continuous operators has been exploited in algorithms for difference equations and indefinite summation (see, e.g., [7, 11]). Following this spirit, the following theorem may be regarded as a discrete analogon to l'Hospital's rule.

*This work was supported by the Austrian Science Foundation FWF

Theorem (Discrete L'Hospital's Rule) Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ be real sequences, both tending to infinity as $n \to \infty$ and suppose that (b_n) is asymptotically strictly monotonous. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\Delta a_n}{\Delta b_n},$$

provided that the limit on the right-hand side exists. □

The theorem follows immediately from statement I.1.70 in [9] or II §8, ex. 5 in [8] (see also Lemma 8.7 of [2]).

As an example,

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{\sum\limits_{k=1}^{n} \binom{2k}{k}} = \lim_{n\to\infty} \frac{\binom{2(n+1)}{n+1} - \binom{2n}{n}}{\binom{2(n+1)}{n+1}} = \lim_{n\to\infty} \left(1 - \frac{n+1}{2(2n+1)}\right) = \tfrac{3}{4}.$$

Note that this limit is found neither by Maple nor by Mathematica because standard limit computation techniques cannot deal with the summation sign in the denominator.

3 Towards an Algorithm

We consider terms that are built from rational functions, indefinite sums, and indefinite products over an indeterminate n. Such expressions are called $\Pi\Sigma$-expressions. A $\Pi\Sigma$-expression is called *admissible,* if the scopes of all occuring product quantifiers are asymptotically postive. It can be shown that every admissible term is asymptotically monotonous, so we need not worry about a decision procedure for monotonicity.

Examples for admissible terms include expressions involving exponentials a^n $(a \geq 0)$, factorials $n!$, binomial coefficients $\binom{n}{k}$, etc. while terms like $(-1)^n$ are excluded.

We illustrate our method with by example. In order to compute, for instance, $\lim\limits_{n\to\infty} \frac{\mathrm{H}_n \sum_{k=1}^{n} \mathrm{H}_k}{n\mathrm{H}_{2n}^2}$, where $\mathrm{H}_n := \sum\limits_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number, we first apply l'Hospital's rule obtaining

$$\lim_{n\to\infty} \frac{p_n}{q_n} \tag{1}$$

with

$$\begin{aligned}
p_n = {} & 4n^3\mathrm{H}_n\mathrm{H}_{2n} + 4n^2\sum_{k=1}^{n}\mathrm{H}_k + 10n^2\mathrm{H}_n\mathrm{H}_{2n} + 7n\sum_{k=1}^{n}\mathrm{H}_k + 4n^2\mathrm{H}_n \\
& + 4n^2\mathrm{H}_{2n} + 8n\mathrm{H}_n\mathrm{H}_{2n} + 7n\mathrm{H}_n + 3\sum_{k=1}^{n}\mathrm{H}_k + 4n + 6n\mathrm{H}_{2n} \\
& + 2\mathrm{H}_n\mathrm{H}_{2n} + 3\mathrm{H}_n + 2\mathrm{H}_{2n} + 3 \quad \text{and} \\
q_n = {} & 4n^3\mathrm{H}_n^2 + 8n^3\mathrm{H}_n + 10n^2\mathrm{H}_n^2 + 20n^2\mathrm{H}_n + 4n^2 + 8n\mathrm{H}_n^2 + 16n\mathrm{H}_n \\
& + 6n + 2\mathrm{H}_n^2 + 4\mathrm{H}_n + 2
\end{aligned}$$

after expanding products and clearing double denominators. Subsequent applications of l'Hospital's rule would lead to more and more complicated expressions and would most likely never come to a conclusion.

Instead, we determine in the next step a *dominant term* of both numerator and denominator and consider only the quotient of these "leading terms" — the other terms are negligible asymptotically. This is known from calculus exercises like

$$\lim_{n\to\infty} \frac{5n^3+3n^2+4}{3n^3+4n-5} = \tfrac{5}{3},$$

which we now extend from univariate to multivariate rational functions. Taking the point of view of $\Pi\Sigma$-fields [7], we may regard the expressions p_n and q_n of (1) as polynomials in the ring

$$R = \mathbb{Q}[n, \mathrm{H}_n, \mathrm{H}_{2n}, \sum_{k=1}^{n} \mathrm{H}_k].$$

This view regards n, H_n, etc. no longer as sequences but as pure algebraic variables. For monomials m_1, m_2 in R corresponding to admissible sequences $m_n^{(1)}, m_n^{(2)}$, respectively, let

$$m_1 \prec m_2 \quad :\Longleftrightarrow \quad \lim_{n\to\infty} \frac{m_n^{(1)}}{m_n^{(2)}} = 0.$$

This defines a strict partial ordering $\prec$ on the monomials of R.

Continuing the example of (1), we determine the leading term of both p_n and q_n with respect to $\prec$ by calling the algorithm recursively. In general, since $\prec$ is only a partial ordering, a unique greatest summand need not exists. This case is more complicated to deal with, but in the situation of (1) the leading terms are unique, and the problem reduces to

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{4n^3 \mathrm{H}_n \mathrm{H}_{2n}}{4n^3 \mathrm{H}_n^2} = \lim_{n\to\infty} \frac{\mathrm{H}_{2n}}{\mathrm{H}_n} \overset{\mathrm{l'H}}{=} \lim_{n\to\infty} \frac{4n+3}{4n+2} = 1.$$

Putting things together, we obtain

$$\lim_{n\to\infty} \frac{\mathrm{H}_n \sum_{k=1}^{n} \mathrm{H}_k}{n\mathrm{H}_{2n}^2} = 1.$$

4 Open Problems

Although a Maple implementation of the method outlined in the previous section works quite nice (see Section 5 for examples), some questions remain open and will be subject of further investigations.

Currently, we deal with monotonicity by restricting the admissible sequences to a class where everything is asymptotically monotonous, and we employ a set of ad-hoc criterions for checking that numerator and denominator go to infinity. Naturally, future work will focus on enlarging the class of admissible terms and on a more systematic treatment for deciding unboundedness.

The method described in Section 3 is a semi-decision procedure in the sense that every computed limit is correct but the method may fail or it may run forever. Refined versions should focus on the treatment of sequences which currently cause a failure of the method, and an a priory criteria for termination would be interesting.

5 Some Examples

For giving a flavor of the problems that can be treated by our method, we provide some examples, none of which can be found by the standard general purpose CA-systems.

- $\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left(k^2 \sum_{i=1}^{k} \frac{2^i}{i}\right)}{2^n n} = 4,$
- $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \binom{3k}{k}\binom{5k}{k}}{\binom{3n}{n}\binom{5n}{n}} = \frac{84375}{83351},$
- $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{2^k}{k^2}\binom{2k}{k}}{\left(\sum_{k=1}^{n} \frac{2^k}{k^2}\right)\sum_{k=1}^{n}\binom{2k}{k}} = \frac{3}{7},$
- $\lim_{n\to\infty} \frac{(\mathrm{H}_n)^3 \sum_{k=1}^{n} \frac{1}{\mathrm{H}_k}}{n \sum_{k=1}^{n} \frac{\mathrm{H}_k}{k}} = 2$
- $\lim_{n\to\infty} \frac{n^3\left(\sum_{k=1}^{n} \frac{2^k}{k^2}\right)^2}{2^n \sum_{k=1}^{n} \frac{2^k}{k}} = 2,$
- $\lim_{n\to\infty} \frac{\mathrm{H}_n \sum_{k=1}^{n} \frac{2^k}{k}}{n \sum_{k=1}^{n} \frac{2^k \mathrm{H}_k}{k^2}} = 1,$
- $\lim_{n\to\infty} \frac{\sum_{k=1}^{2n} \frac{2^k k!}{k^2}}{\left(\sum_{k=1}^{n} \frac{4^k}{k^2}\right)\sum_{k=1}^{2n} k!} = \frac{3}{16},$
- $\lim_{n\to\infty} \frac{\sum_{k=1}^{2n} \frac{(\mathrm{H}_k)^2}{k}}{(\mathrm{H}_n)^3} = \frac{1}{3}.$

References

[1] Michael Boshernitzan. Orders of infinity generated by difference equations. In *American Journal of Mathematics,* 106:1067–1089, 1984.

[2] Michael Boshernitzan. Discrete Orders of Infinity. In *American Journal of Mathematics,* 106:1147–1198, 1984.

[3] Keith O. Geddes, Gaston H. Gonnet. A New Algorithm for Computing Symbolic Limits Using Generalized Hierarchical Series. In *Proceedings of ISSAC'88,* 1988

[4] Ronald L. Graham, Donald E. Knuth, Oren Patashnik. *Concrete Mathematics.* Addison-Wesley, second edition, 1994.

[5] Dominik Gruntz. *On Computing Limits in a Symbolic Manipulation System.* ETH Diss 11432, 1996.

[6] Maxwell Rosenlicht. Hardy fields. In *Journal of Mathematical Analysis and Applications,* 93:286–311, 1983.

[7] Michael Karr. Summation in Finite Terms. In *Journal of the ACM,* 28:305–350, 1981

[8] Konrad Knopp. *Theorie und Anwendung der unendlichen Reihen.* Springer, fifth edition, 1964.

[9] George Pólya, Gabor Szegő. *Problems and Theorems in Analysis I.* Springer, 1978.

[10] Bruno Salvy, John Shackell. Symbolic Asymptotics: Functions of Two Variables, Implicit Functions. In *Journal of Symbolic Computation,* 25:329–349, 1998.

[11] Carsten Schneider. *Symbolic Summation in Difference Fields.* PhD Thesis RISC, 2001.

[12] John Shackell. Growth Estimate of exp-log Functions. In *Journal of Symbolic Computation,* 10:611–632, 1990.

[13] John Shackell, Bruno Salvy. Symbolic Asymptotics: Multiseries of Inverse Functions. In *Journal of Symbolic Computation,* 27:543–563, 1999.