Computer Algebra for Special Function Inequalities

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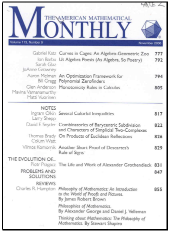
1. **Yakub's Inequality**

2. **Bernoulli's Inequality**

3. **Alzer's Inequality**

4. **Moll's Inequality**

Yakub's Inequality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Gabriel Katz	Curves in Cages: An Algebro-Geometric Zoo	777
Ion Barbu, Sarah Glaz, JoAnne Growney	Ut Algebra Poesis (As Algebra, So Poetry)	792
Aaron Melman, Bill Gragg	An Optimization Framework for Polynomial Zerofinders	794
Glen Anderson, Mavina Vamanamurthy, Matti Vuorinen	Monotonicity Rules in Calculus	805
NOTES		
Ingram Olkin, Larry Shepp	Several Colorful Inequalities	817
David F. Snyder	Combinatorics of Barycentric Subdivision and Characters of Simplicial Two-Complexes	822
Thomas Brady, Colum Watt	On Products of Euclidean Reflections	826
Vilmos Komornik	Another Short Proof of Descartes's Rule of Signs	829
THE EVOLUTION OF...		
Piotr Pragacz	The Life and Work of Alexander Grothendieck	831
PROBLEMS AND SOLUTIONS		847
REVIEWS		
Charles R. Hampton	*Philosophy of Mathematics: An Introduction to the World of Proofs and Pictures.* By James Robert Brown	855
	Philosophies of Mathematics. By Alexander George and Daniel J. Velleman	
	Thinking about Mathematics: The Philosophy of Mathematics. By Stewart Shapiro	

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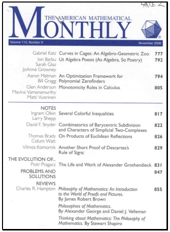

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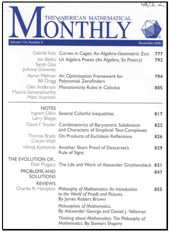
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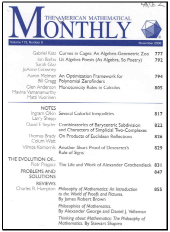
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- You should not need more than 30 seconds to come up with a *completely rigorous* solution to this problem
- ... because it can be done *by a computer!*
- Yakub's problem is therefore as *uninteresting* as asking for a proof that

$$317034851 \cdot 41539045 = 13169324942257295$$

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    where $a(x) = \text{Root}(16x^3 - 16x^4 + (729 - 1053x + 300x^2 + 8x^3)X - (216 + 132x + x^2)X^2 + 16X^3, 2))$

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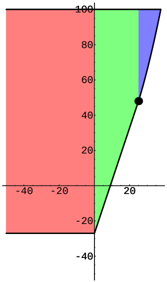

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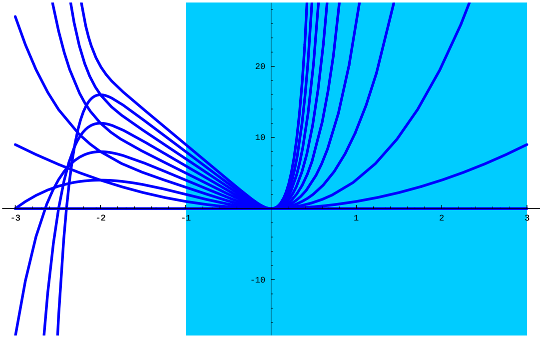
$$\forall\ x \geq -1\ \forall\ n \in \mathbb{N} : (x+1)^n \geq 1 + nx.$$



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This can be done with CAD.

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Replace the annoying term $(x+1)^n$ by a new variable y:

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The rest can be left to CAD. □

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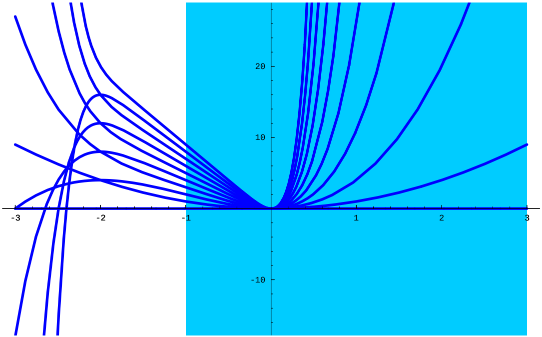

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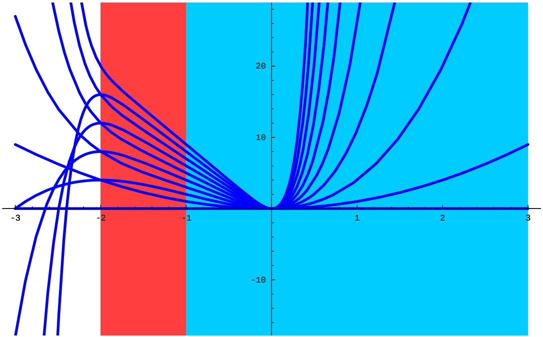

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- Another trick is needed here, because

  $$n \geq 1 \wedge x \geq -2 \wedge y \geq 1 + nx \Rightarrow (x+1)y \geq 1 + (n+1)x$$

  is *false.* (CAD can be used also for constructing counterexamples.)

Bernoulli's Inequality

$$\forall\ x \geq -2\ \forall\ n \in \mathbb{N} : (x+1)^n \geq 1 + nx.$$

- Extending the induction step helps:

$$(1+x)^n \geq 1 + nx \wedge (1+x)^{n+1} \geq 1 + (n+1)x$$
$$\Rightarrow (1+x)^{n+2} \geq 1 + (n+2)x$$

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follows from

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CAD does the rest. $\Box$

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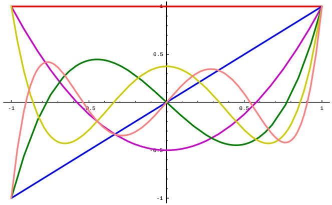
$$P_n(x) := \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$



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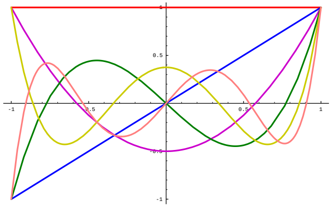

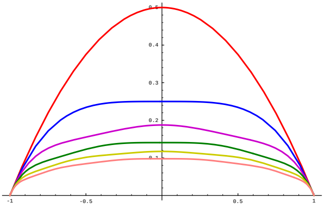

Turan's inequality says

$$P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0.$$

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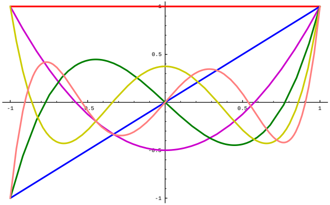

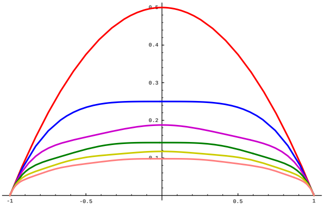

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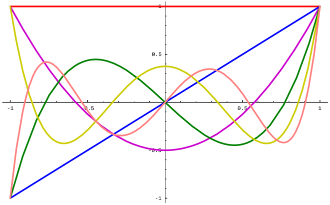

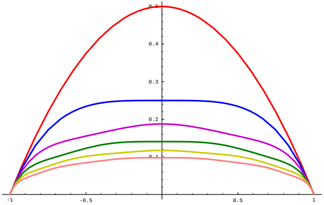

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Alzer has conjectured the sharper variant

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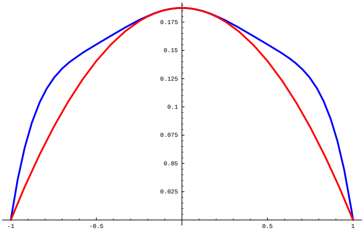

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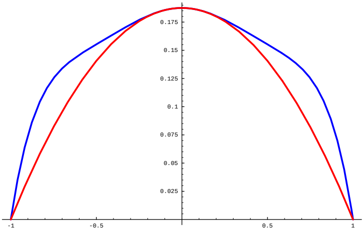

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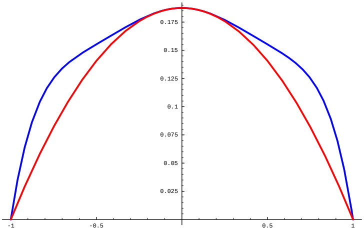

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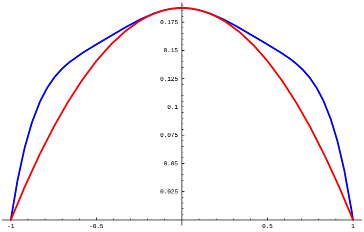

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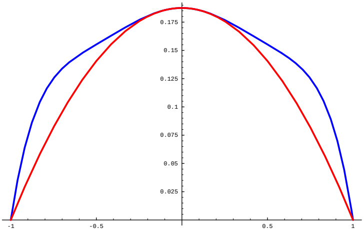

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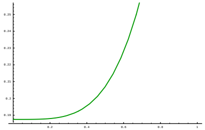
is increasing on $(0, 1)$.



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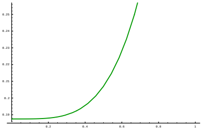

- f_n is increasing iff $\frac{d}{dx}f_n(x) \geq 0$

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- Observe

$$\begin{aligned} \frac{d}{dx} f_n(x) = \Big(& (n-1)nP_n(x)^2 \\ & - (2nx^2 + x^2 - 1)P_n(x)P_{n+1}(x) \\ & + (n+1)xP_{n+1}(x)^2 \Big) \Big/ \Big(n(1-x^2)^2 \Big) \end{aligned}$$

and leave the rest to CAD and induction. □

1. Yakub's Inequality
2. Bernoulli's Inequality
3. Alzer's Inequality
4. **Moll's Inequality**

Moll's Inequality

For $0 \leq l \leq m \in \mathbb{Z}$, let

$$d_l(m) = \sum_{j=0}^{l} \sum_{s=0}^{m-j} \sum_{k=s+l}^{m} \frac{(-1)^{k-l-s}}{2^{3k}} \binom{2k}{k} \binom{2m+1}{2s+2j} \times \binom{m-s-j}{m-k} \binom{s+j}{j} \binom{k-s-j}{l-j}.$$

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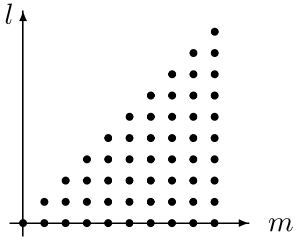
These numbers appear in the closed form of

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}} dx \quad (a > -1, m \in \mathbb{N})$$

Moll's Inequality



Theorem (Moll) $d_l(m) > 0$

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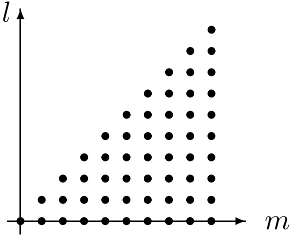

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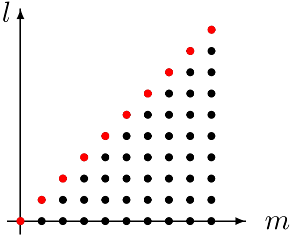

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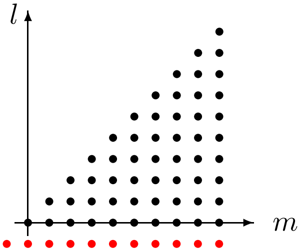

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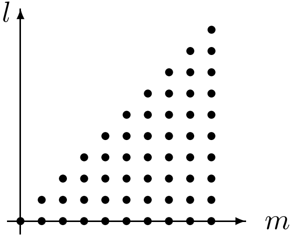

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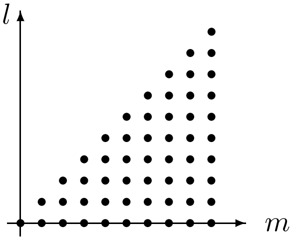

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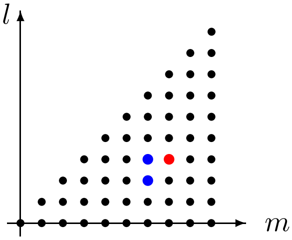

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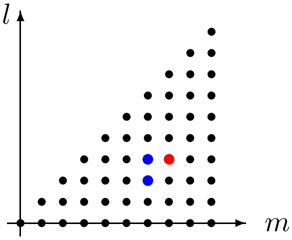

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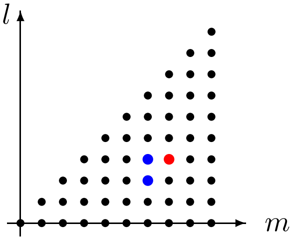

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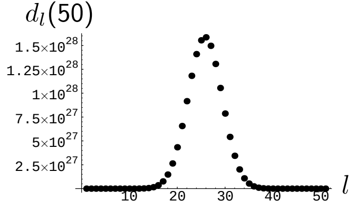

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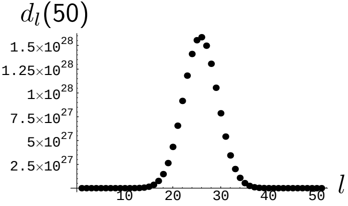

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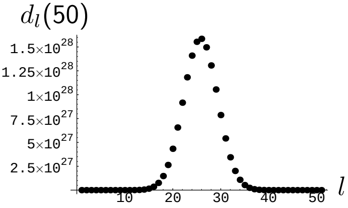

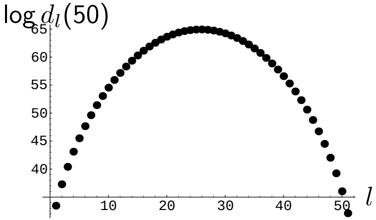

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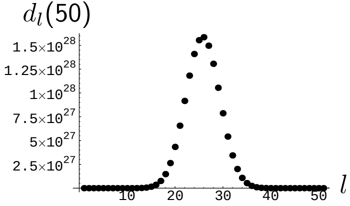

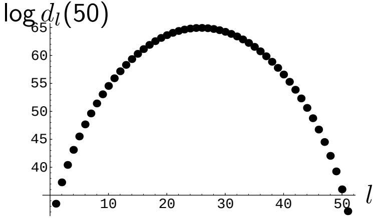

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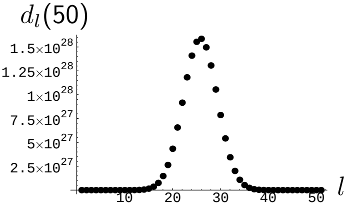

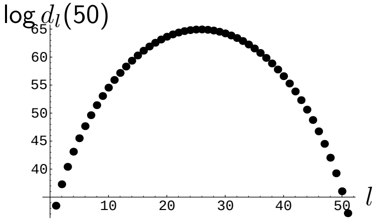

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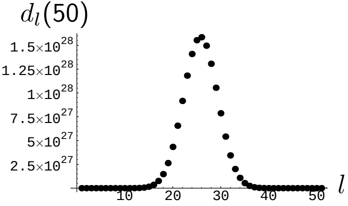

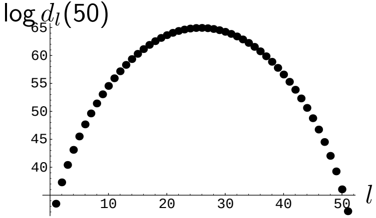

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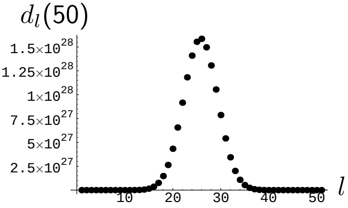

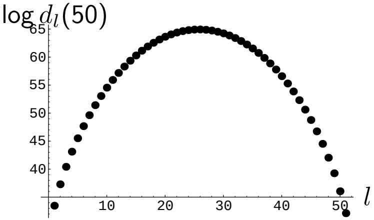

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Using CAD and some recurrence equations, it can be *found* that

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- This is *better* because the $d_l(m)$ occur only linearly.
- It is *worse* because of the root expression

Moll's Inequality

How to show $d_{l-1}(m)d_{l+1}(m) \le d_l(m)^2$?

Observation: It suffices to show the stronger condition

$$d_l(m+1) \ge \frac{-2l^2+(m+1)(4m+3)+\sqrt{l(4l^3-3l-4m(m+1))+u(l,m)}}{2(m+1)(m-l+1)} d_l(m)$$

for some appropriate $u(l,m) \ge 0$.

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$$d_l(m+1) \geq \frac{-2l^2+(m+1)(4m+3)+\sqrt{l(4l^3-3l-4m(m+1))+u(l,m)}}{2(m+1)(m-l+1)} d_l(m)$$

for some appropriate $u(l,m) \geq 0$.
Choosing $u(l,m) = 4l^2 + 4l^3 + 4lm(m+1)$ turns the radicand into a square and we are left with

$$d_l(m+1) \geq \frac{4m^2+7m+l+1}{2(m+1-l)(m+1)} d_l(m).$$

Moll's Inequality

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This can be done with CAD and induction. $\square$.