Pillwein's Proof of Schöberl's Conjecture

Manuel Kauers

currently at INRIA-Rocquencourt
usually at RISC-Linz

Polynomial Inequalities

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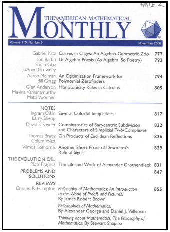

THE AMERICAN MATHEMATICAL MONTHLY

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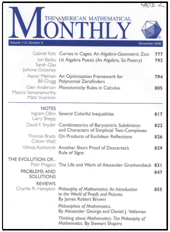
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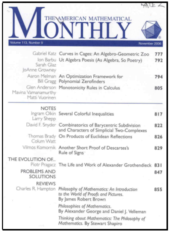
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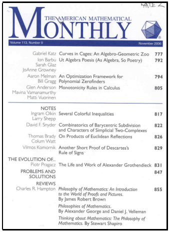

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- Problem 11251 (Marian Tetiva; vol. 113(10), 2006, p. 847): Let a, b, c be positive real numbers, two of which are ≤ 1, satisfying $ab + ac + bc = 3$. Show that

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} - \frac{3}{4} \geq \frac{3(a-1)(b-1)(c-1)}{2(a+b)(a+c)(b+c)}$$

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- Problem 11301 (Finbarr Holland; vol. 114(10), 2007, p. 547): Find the least number M such that for all a, b, c,

$$|ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2.$$

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- quantifiers $(\forall, \exists)$

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One such algorithm is due to Collins (Cylindrical Algebraic Decomposition, *CAD*, 1975).

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$$\xrightarrow{CAD} M \geq \tfrac{9}{32}\sqrt{2}$$

Polynomial Inequalities

Message:

Polynomial inequalities can be proven by CAD

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Polynomial inequalities can be proven by CAD without thinking.

The rest of this talk is about
inequalities that can be proven by CAD with thinking only.

Bernoulli's Inequality

$$\forall\ n \in \mathbb{N}\ \forall\ x \geq -1 : (x+1)^n \geq 1 + nx.$$

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- View $(x+1)^n - (1+nx)$ as a sequence of polynomials.
- View Bernoulli's inequality as a sequence of polynomial inequalities.

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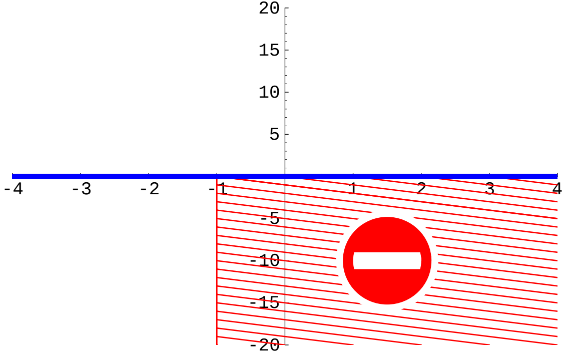

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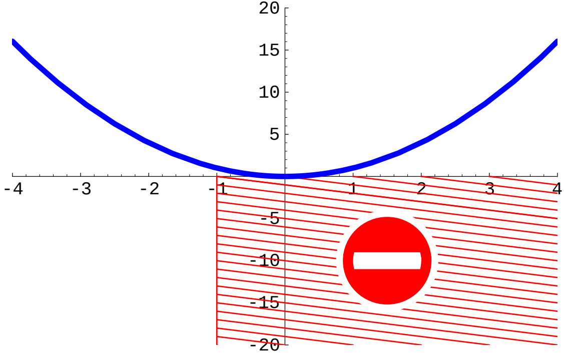

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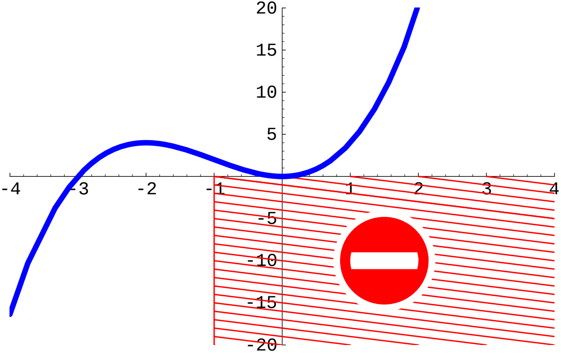

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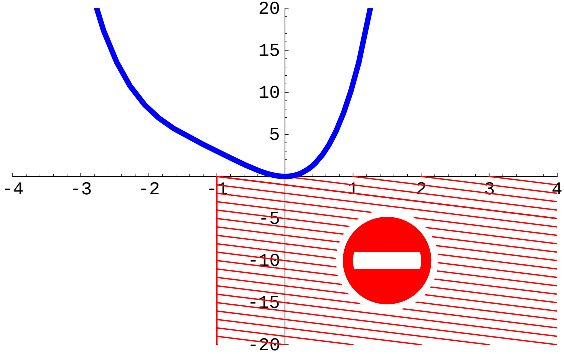

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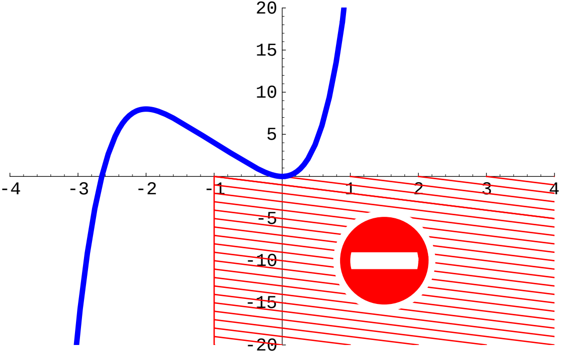

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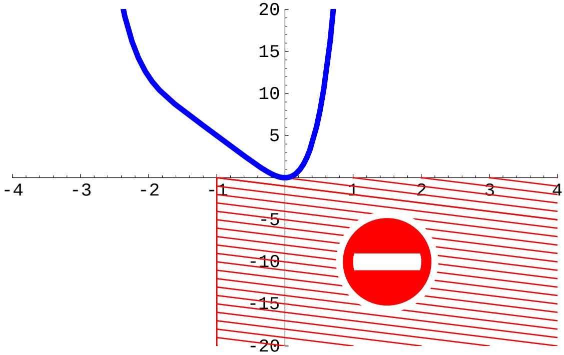

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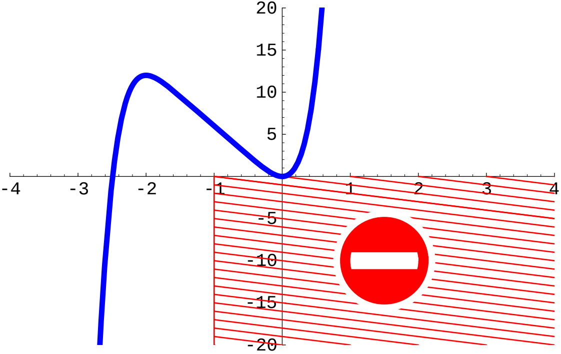

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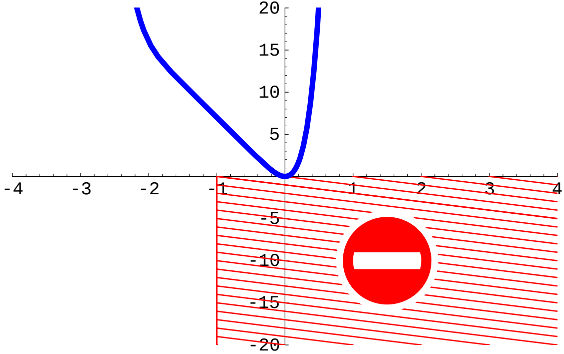

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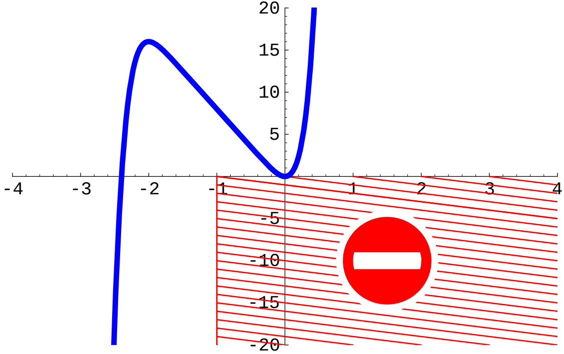

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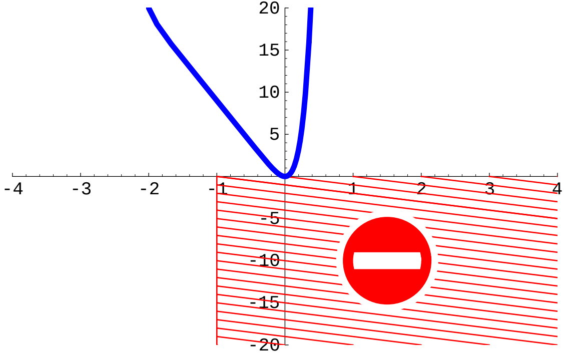

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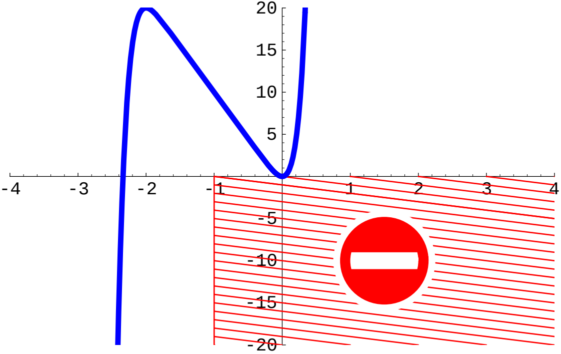

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- The resulting formula is indeed *true.*

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- This completes the proof.

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Message:

We may use CAD to construct an induction proof for the positivity of a quantity satisfying a recurrence.

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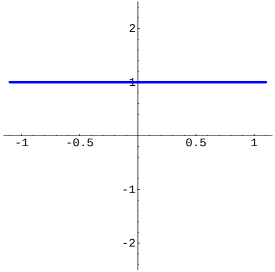

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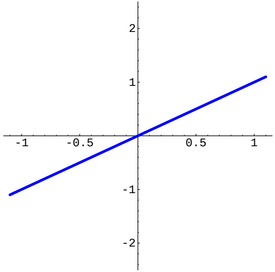

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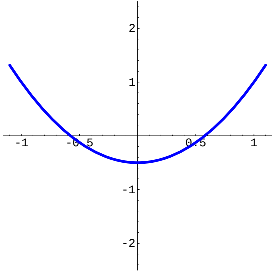

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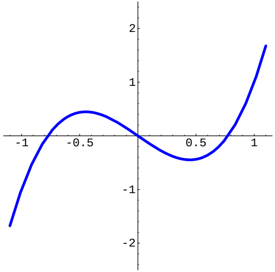

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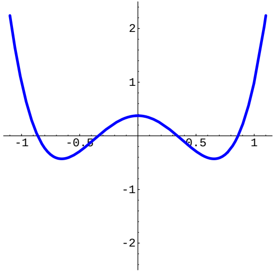

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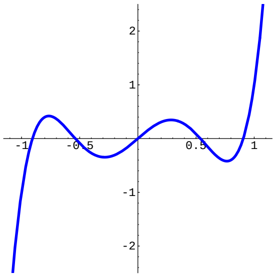

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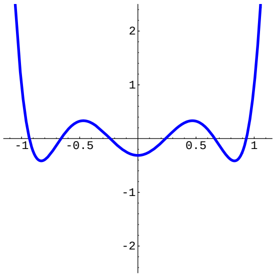

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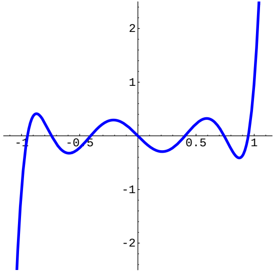

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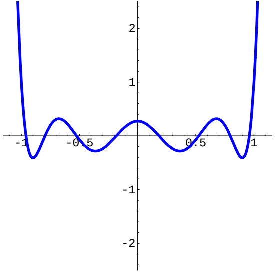

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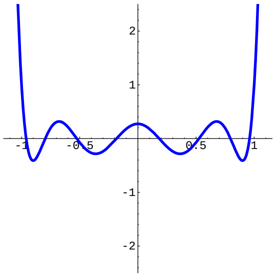

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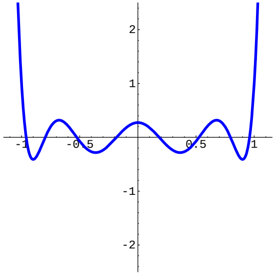

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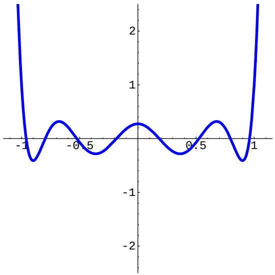

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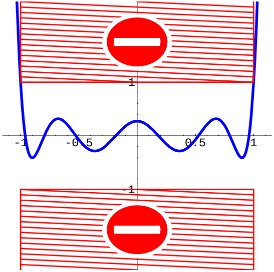

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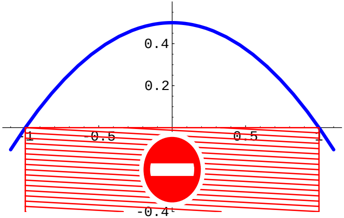

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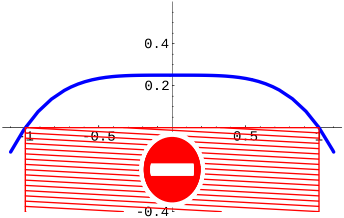

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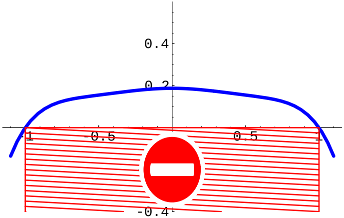

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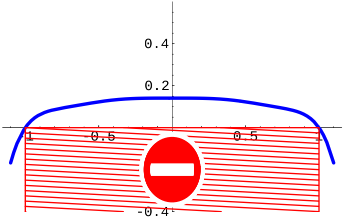

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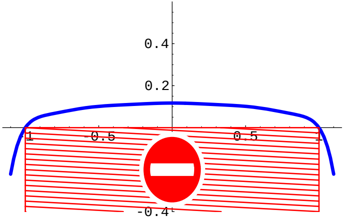

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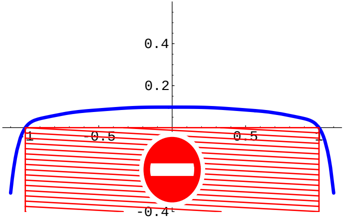

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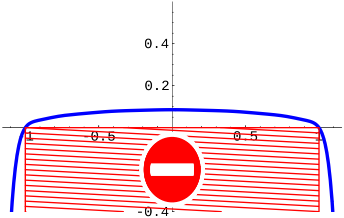

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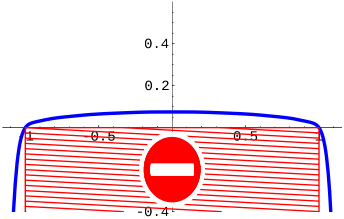

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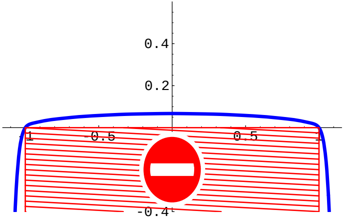

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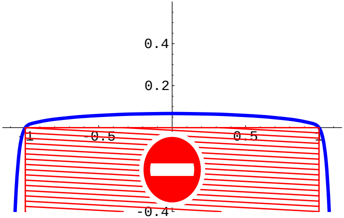

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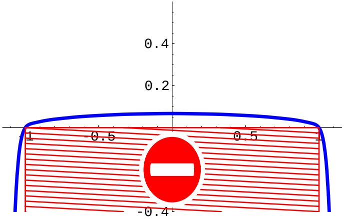

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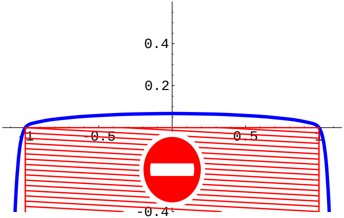

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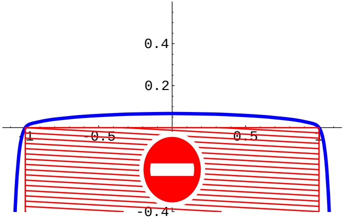

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A proof for general n can be obtained in the same way as for Bernoulli's inequality.

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Use the recurrence for $P_n(x)$ to obtain

$$\begin{aligned}\Delta_n(x) &= \tfrac{(n+1)}{n+2}P_n(x)^2 - \tfrac{2n+3}{n+2}xP_{n+1}(x)P_n(x) + P_{n+1}(x)^2\\ \Delta_{n+1}(x) &= \tfrac{(n+1)^2}{(n+2)^2}P_n(x)^2 - \tfrac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)}P_{n+1}(x)P_n(x)\\ &\quad + \tfrac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}P_{n+1}(x)^2\end{aligned}$$

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The induction base $\Delta_0(x) \geq 0$ is trivial.

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Here is an inequality about $P_n(x)$ that can be shown with CAD:

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Relaxing $P_n(x)$ to y, and $P_{n+1}(x)$ to z, and $n \in \mathbb{N}$ to $n \geq 0$ leads to the formula

$$\forall n\, \forall x\, \forall y\, \forall z : \big(n \geq 0 \wedge x^2 \leq 1 \wedge \tfrac{n+1}{n+2}y^2 - \tfrac{2n+3}{n+2}xyz + z^2 \geq 0\big)$$
$$\Rightarrow \big(\tfrac{(n+1)^2}{(n+2)^2}y^2 - \tfrac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)}yz + \tfrac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}z^2 \geq 0\big),$$

which is indeed *true.* This proves the induction step.

The induction base $\Delta_0(x) \geq 0$ is trivial. This completes the proof.

Legendre Polynomials: Turan's Inequality

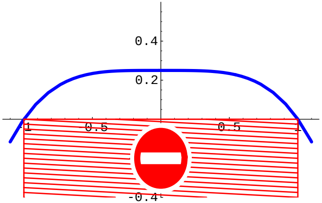
Message:

A “deep” special function inequality may be just an immediate consequence of a polynomial inequality.

Legendre Polynomials: Turan's Inequality

Turan's inequality

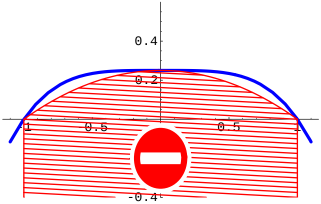
$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0$$



Legendre Polynomials: Turan's Inequality

Turan's inequality can be improved to

$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq \alpha_n(1 - x^2)$$



where $\alpha_n = \Delta_n(0)$.

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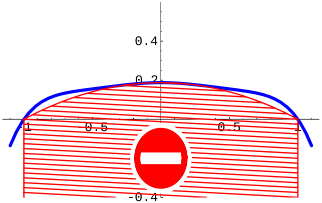

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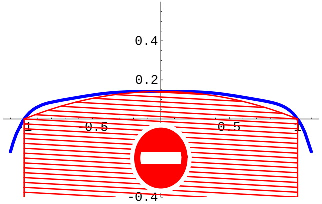

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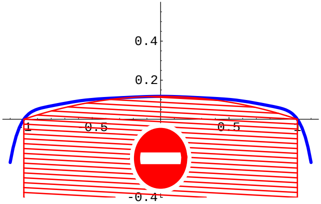

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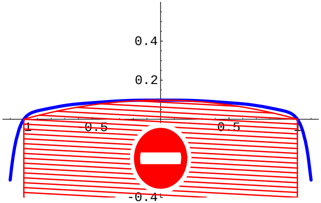

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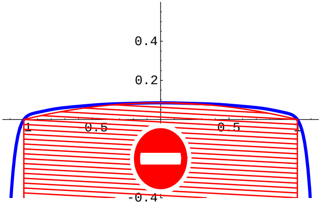

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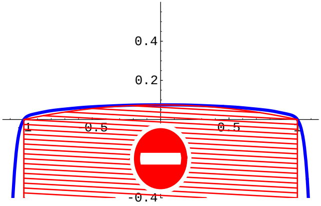

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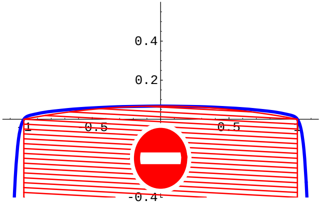

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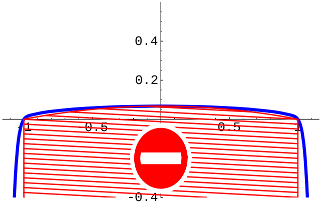

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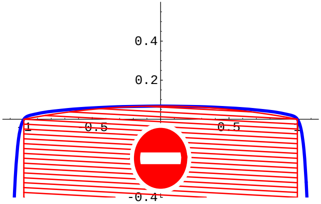

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We have the recurrence

$$(n+3)(n+4)\alpha_{n+2}$$
$$= (2n+5)\alpha_{n+1} + (n+1)(n+2)\alpha_n.$$

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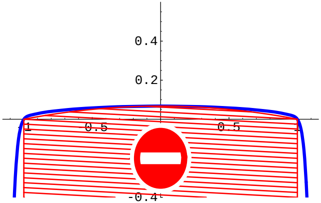

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A Tarski formula encoding the induction step would be...

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Turan's inequality can be improved to

$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq \alpha_n(1-x^2)$$

$$\forall n, x, y, z, a, b : \big(n \geq 0 \wedge x^2 \leq 1 \wedge \tfrac{n+1}{n+2}y^2 - \tfrac{2n+3}{n+2}xyz + z^2 \geq a(1-x^2)$$
$$\wedge \tfrac{(n+1)^2}{(n+2)^2}y^2 - \tfrac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)}yz + \tfrac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}z^2 \geq b(1-x^2)\big)$$
$$\Rightarrow \big(\tfrac{(n+1)^2((n+3)^3-(2n+5)x^2)}{(n+4)(n+3)^2(n+2)^2}y^2$$
$$+ \tfrac{(n+1)(2(2n+3)(2n+5)x^2-(2n^4+21n^3+83n^2+142n+86))}{(n+4)(n+3)^2(n+2)^2}xyz$$
$$+ \tfrac{((n+4)(n+2)^4-(2n+3)^2(2n+5)x^4+(n+1)(2n+3)(2n+5)x^2)}{(n+4)(n+3)(n+2)}z^2$$
$$\geq \tfrac{(n+1)(n+2)}{(n+3)(n+4)}a + \tfrac{(2n+5)}{(n+3)(n+4)}b\big).$$

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\end{aligned}$$

Unfortunately, this is *false.*

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Unfortunately, this is *false.* We must be more careful.

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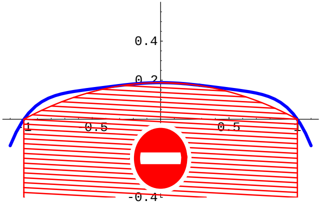
$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq \alpha_n(1-x^2)$$



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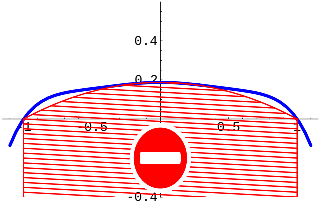

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- By symmetry, it suffices to consider $x \geq 0$.

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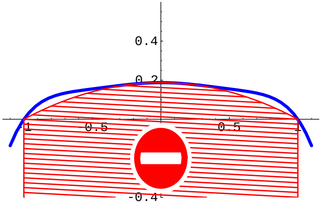

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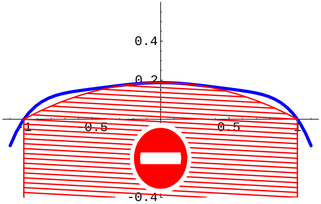

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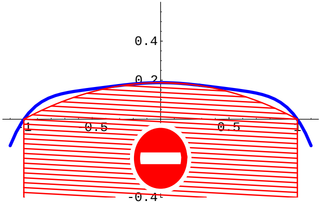

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New idea: Show that $\dfrac{d}{dx}\dfrac{\Delta_n(x)}{1-x^2} \geq 0$

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We have

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A positivity proof for the latter expression by CAD and induction on n succeeds.

Legendre Polynomials: Turan's Inequality

Message:

A special function inequality may require some non-obvious manipulation before an induction proof via CAD succeeds.

Schöberl's Conjecture

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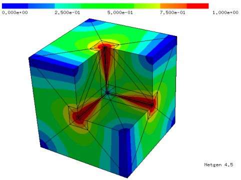
- In the higher order finite element method (FEM), solutions of PDEs are locally approximated by polynomials.



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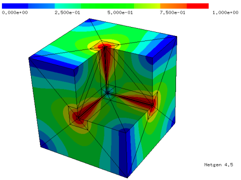
- In the higher order finite element method (FEM), solutions of PDEs are locally approximated by polynomials.
- Some basis polynomials lead to better numerical performance than the standard basis $1, x, x^2, x^3, \ldots$.



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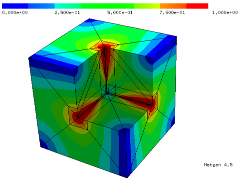

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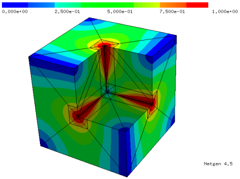

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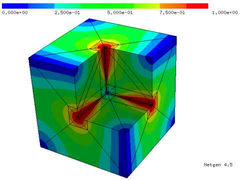

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  - $\displaystyle\int_{-1}^{1} f_n(x)q(x)dx = q(0)$ for all q with $\deg q \le n$.
  - $\displaystyle\int_{-1}^{1} |f_n(x)| \le C$ for some constant C.

Schöberl's Conjecture

- The *Legendre kernel polynomials*

$$k_n(x,y) := \frac{n+1}{2(x-y)}\left(P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)\right)$$

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- But not the second.

Schöberl's Conjecture

- Schöberl next considered the "gliding averages"

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$$\boxed{\sum_{k=0}^{n} (4k+1)(2n-2k+1) P_{2k}(0) P_{2k}(x) \geq 0}$$

for all $x \in [-1,1]$ and all $n \in \mathbb{N}$.

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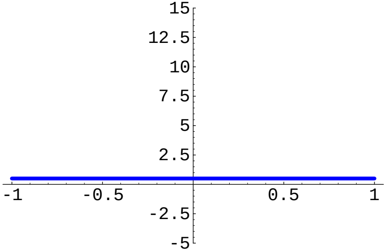
- Hence was born the *Schöberl conjecture.*

Schöberl's Conjecture

Consider

$$S_n(x) := \sum_{k=0}^{n} (4k+1)(2n-2k+1) P_{2k}(0) P_{2k}(x)$$

for $n = 0, 1, \ldots, 20$.



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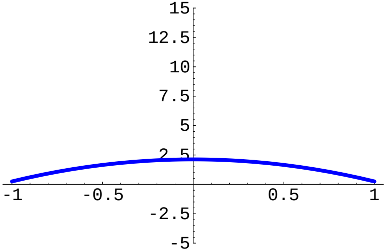

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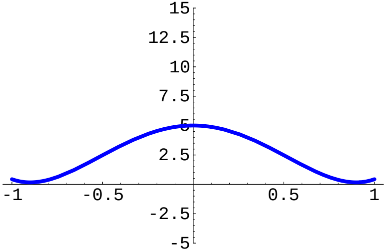

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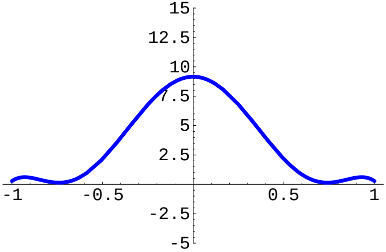

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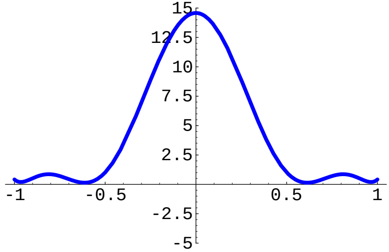

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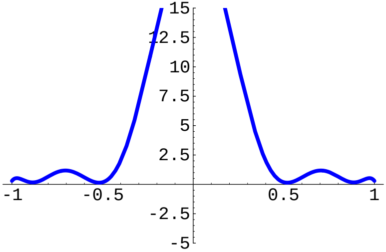

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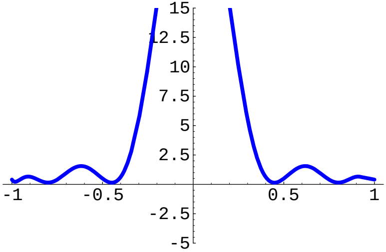

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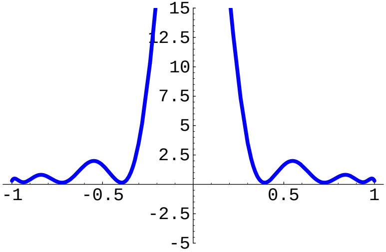

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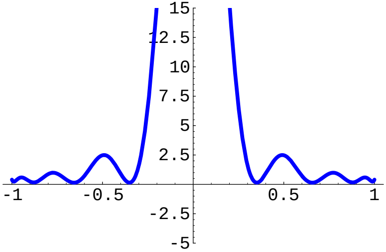

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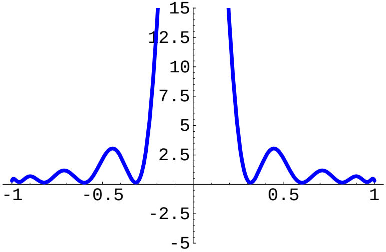

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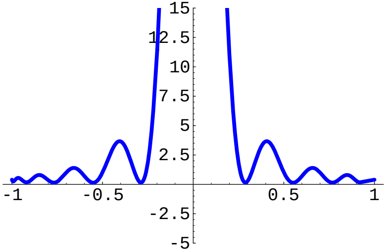

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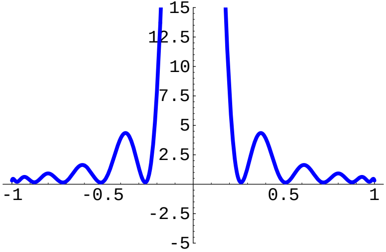

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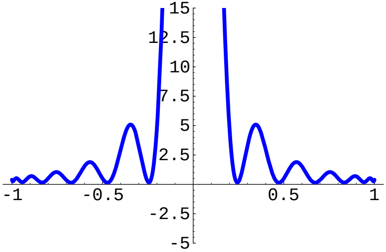

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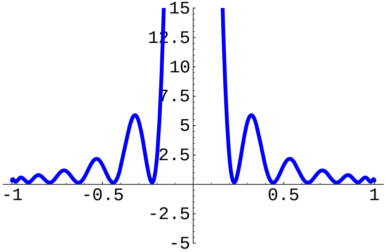

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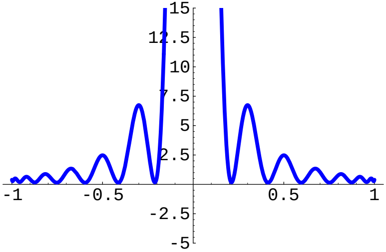

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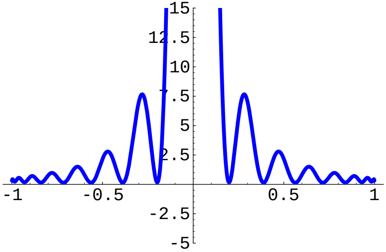

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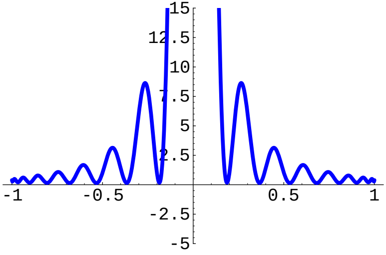

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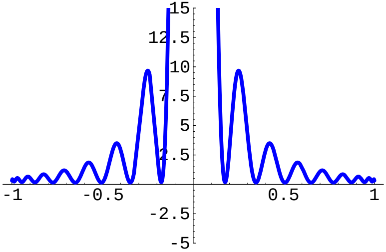

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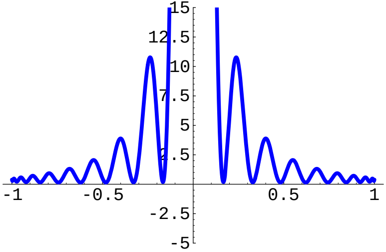

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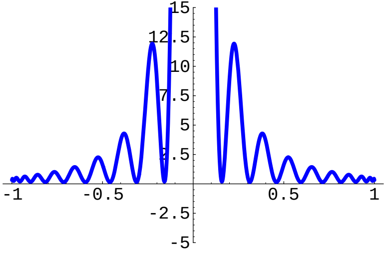

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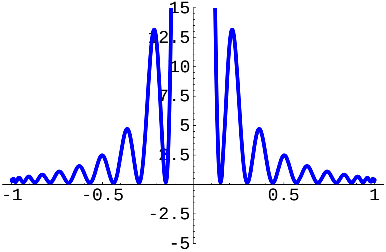

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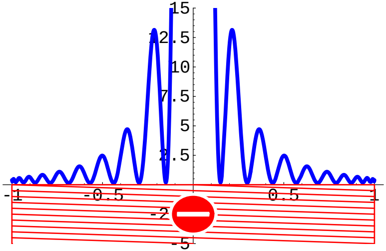

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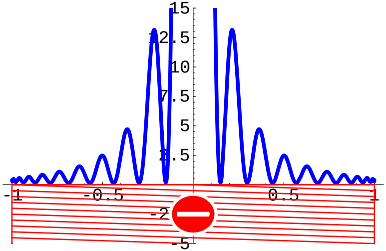

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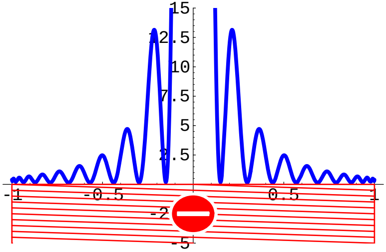

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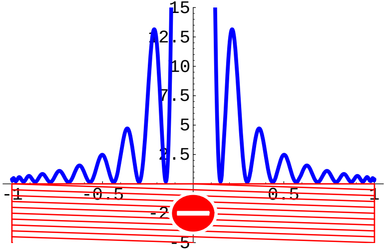

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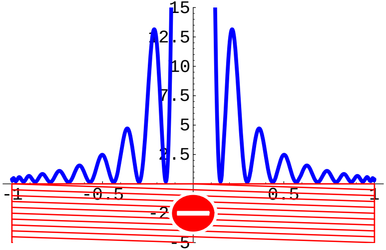

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- For specific $n \in \mathbb{N}$, it can be shown without thinking.
- It can be also be shown for $x = -1$, $x = 0$, $x = +1$.
- But a proof for general x, n could not be found for some years.

Schöberl's Conjecture

Message:

Special function inequalities arise
in real world applications.

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$$\sum_{k=0}^{n} P_k(x) \geq 0 \quad (x \in [-1, 1], n \in \mathbb{N})$$

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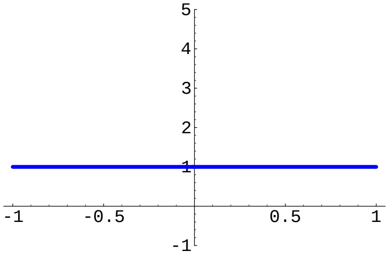

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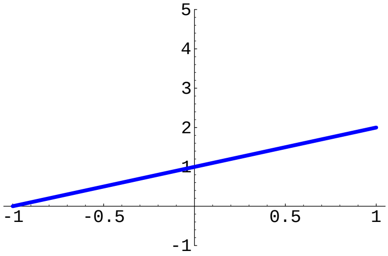

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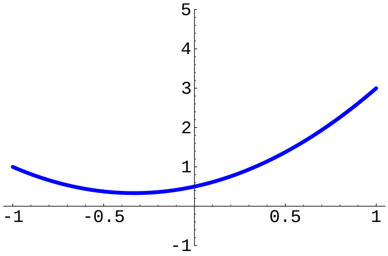

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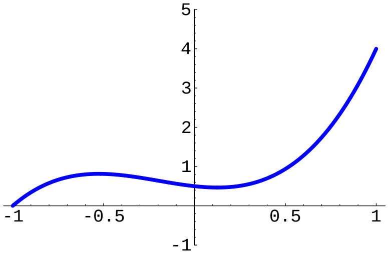

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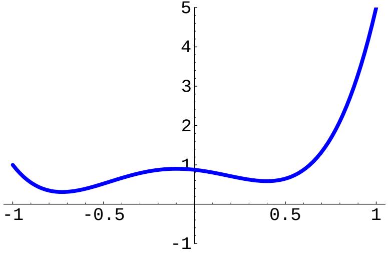

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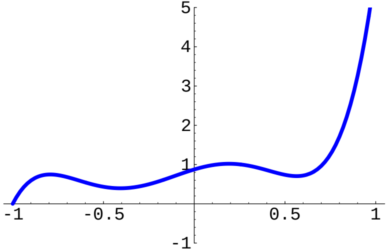

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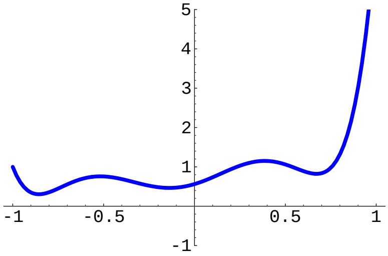

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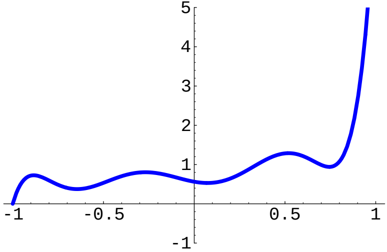

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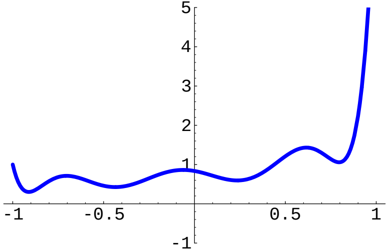

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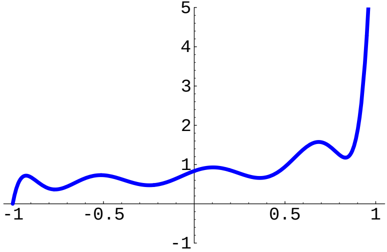

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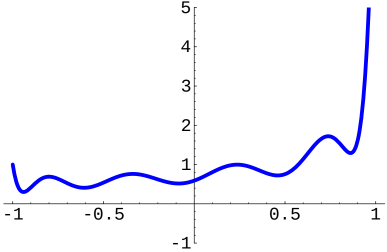

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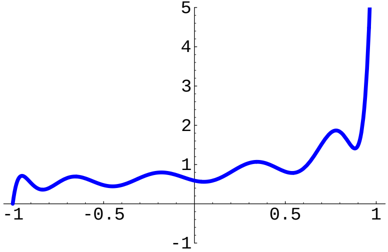

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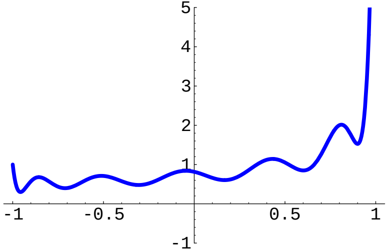

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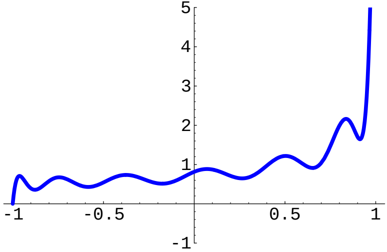

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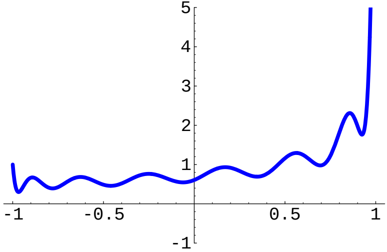

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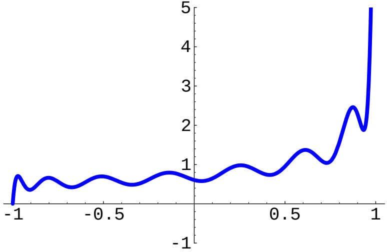

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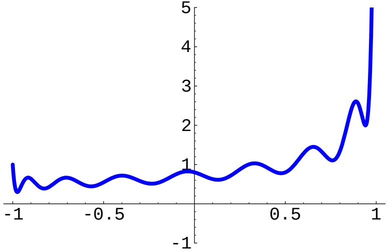

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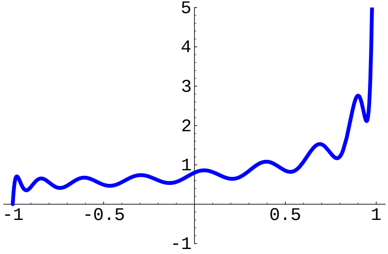

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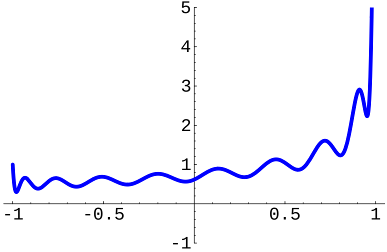

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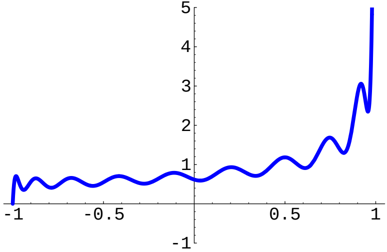

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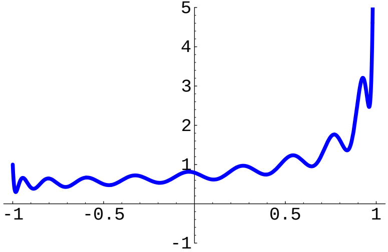

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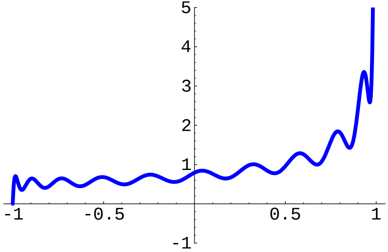

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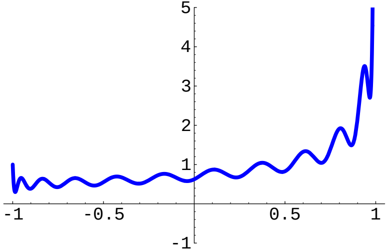

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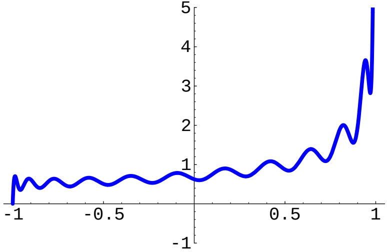

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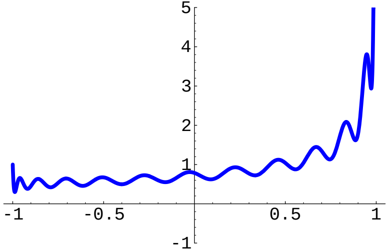

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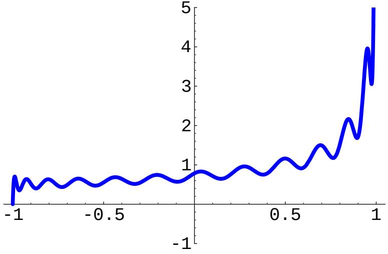

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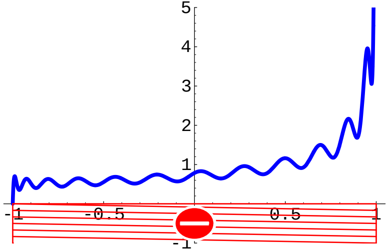

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As $P_k(x) = P_k^{(0,0)}(x)$, it includes Fejer's inequality.

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- This is work in progress.
- Now back to Schöberl's conjecture...

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- Proving estimates for each part by hand
- Combining the estimates for both components with CAD and induction

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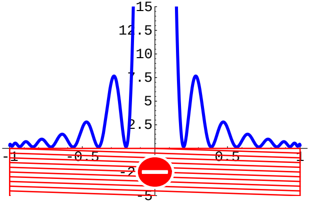
Consider the graph of

$$S_n(x) := \sum_{k=0}^{n} (4k+1)(2n-2k+1)P_{2k}(0)P_{2k}(x)$$

for $n = 15$



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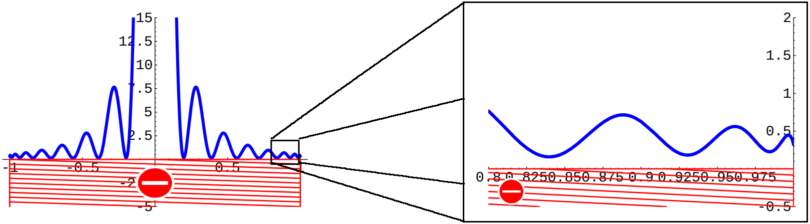

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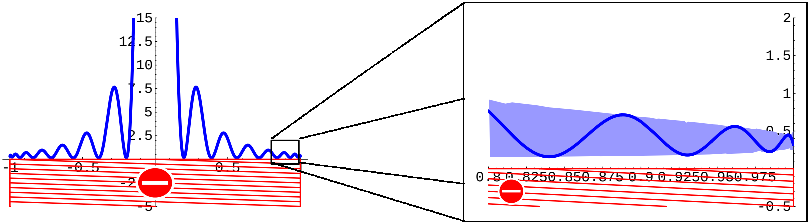

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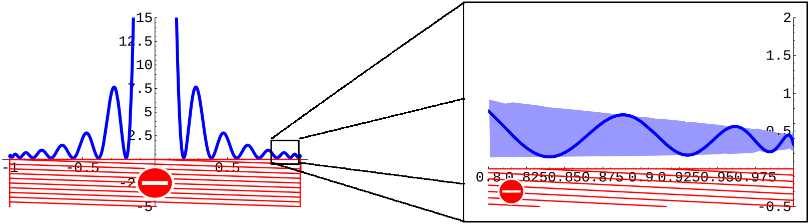

What is the reason for this gap?

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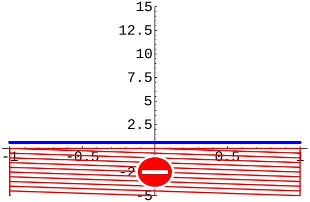

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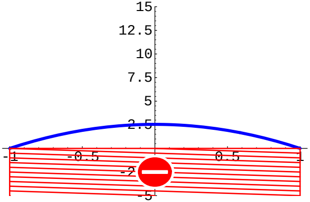

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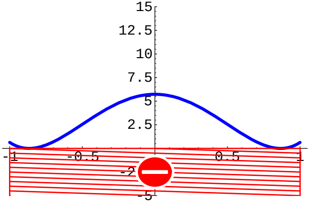

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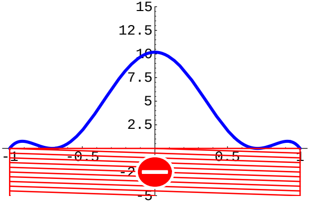

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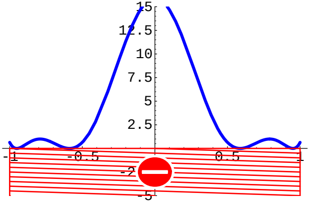

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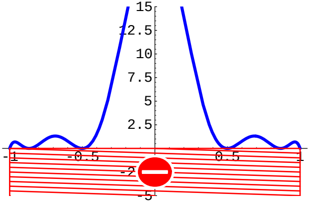

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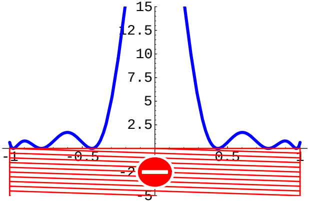

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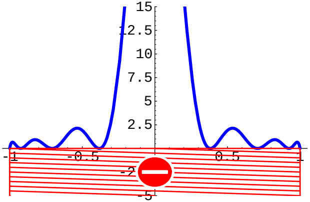

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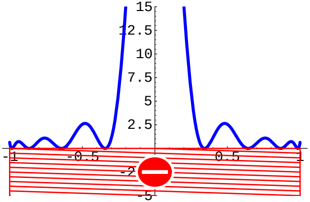

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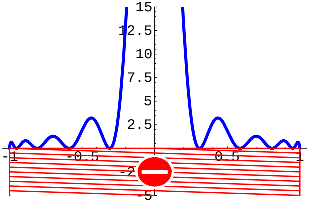

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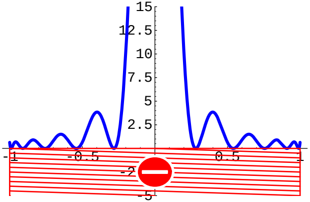

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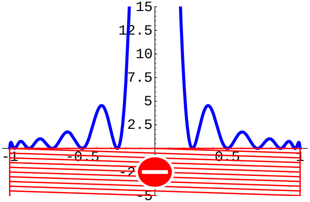

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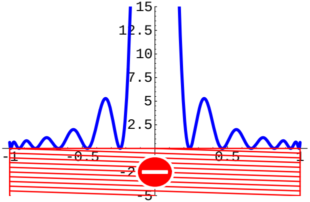

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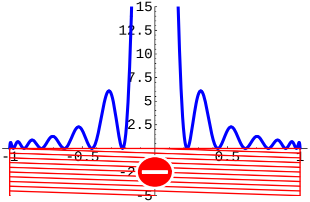

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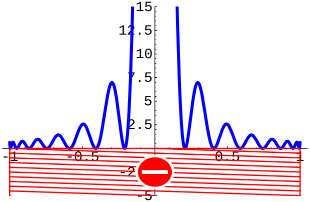

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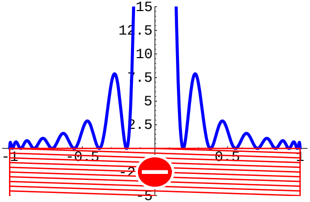

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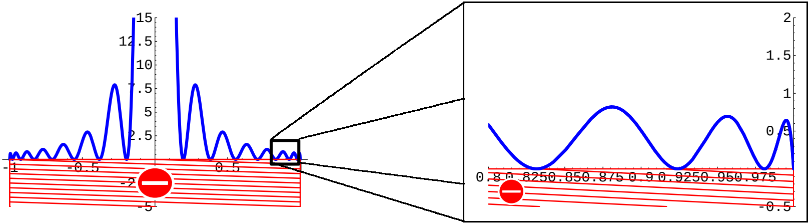

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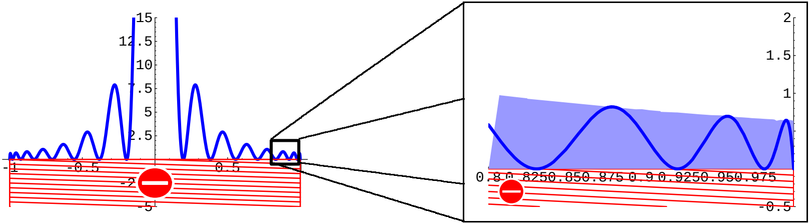

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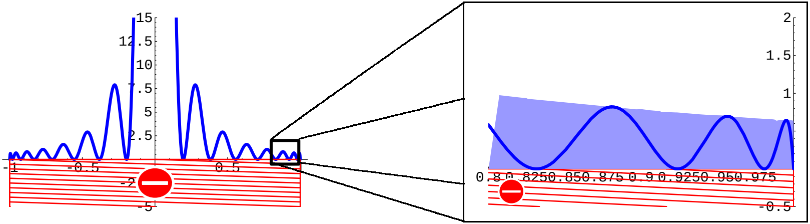

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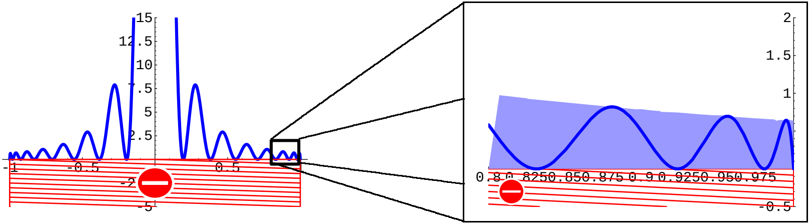

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Note: $S_n(x) = S_n^0(x)$.

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$$= \frac{4}{\pi x^2}\begin{cases} (U_{n+1}(x) - xU_n(x))^2 & \text{if } n \text{ is even} \\ \end{cases}$$

Situation at the boundary

For $\alpha = 1/2$, the sum $S_n^\alpha(x)$ can be written in *closed form.*

With $U_k(x) := \frac{\pi}{4}\binom{k+1}{1/2}P_k^{(1/2,1/2)}(x)$, we have

$$\begin{aligned} S_n^{1/2}(x) &= \frac{4}{\pi}\sum_{k=0}^{n}(2n-2k+1)U_{2k}(0)U_{2k}(x) \\ &= \frac{2}{\pi x^2}\bigl(1+(-1)^n-2(-1)^n(1-x^2)U_n(x)^2\bigr) \end{aligned}$$

This identity was found with *symbolic summation.*

$$= \frac{4}{\pi x^2}\begin{cases} (U_{n+1}(x)-xU_n(x))^2 & \text{if } n \text{ is even} \\ (1-x^2)U_n(x)^2 & \text{if } n \text{ is odd} \end{cases}$$

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$$S_n^\alpha(x) = g_n^\alpha(x) - f_n^\alpha(x)$$

where $f_n^\alpha(x)$ is a sum expression that vanishes for $\alpha = \pm 1/2$, and $g_n^\alpha(x)$ is a closed form expression.

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This can be done in many ways.

Split

A good choice turned out to be

$$f_n^\alpha(x) = \sum_{k=0}^{2n} \frac{4^{-\alpha}(1-4\alpha^2)}{(2\alpha+2k-1)(2\alpha+2k+3)} \frac{\binom{2\alpha+k}{\alpha}}{\binom{\alpha+k}{\alpha}} P_k^{(\alpha,\alpha)}(0) P_k^{(\alpha,\alpha)}(x)$$

$$g_n^\alpha(x) = 2^{-2\alpha-1}(2n+1) \frac{\binom{2\alpha+2n+1}{\alpha}}{\binom{\alpha+2n}{\alpha}} P_{2n}^{(\alpha,\alpha)}(0) \times \left(x P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right)$$

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Indeed, for this choice we have

$$S_n^\alpha(x) = \tfrac{1}{x^2}\left(g_n^\alpha(x) - f_n^\alpha(x)\right) \quad \text{and} \quad f_n^{\pm 1/2}(x) = 0.$$

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This can be verified (but not discovered!) by symbolic summation.

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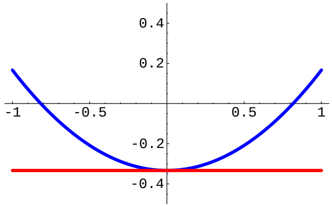

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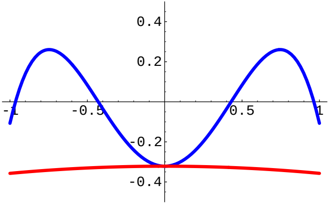

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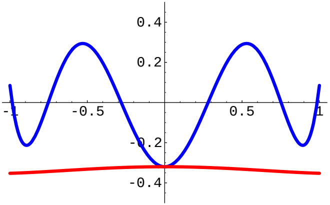

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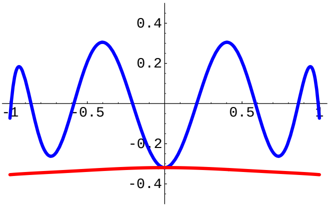

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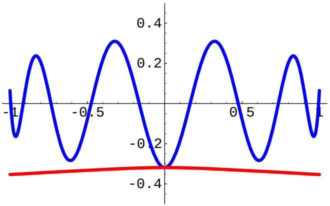

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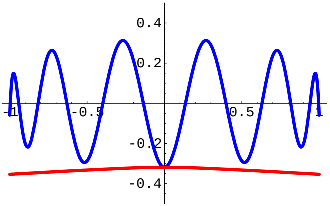

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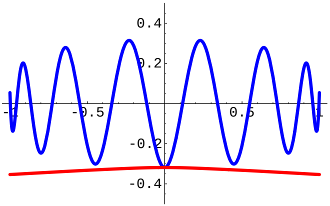

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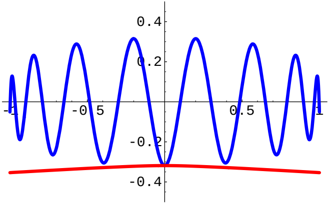

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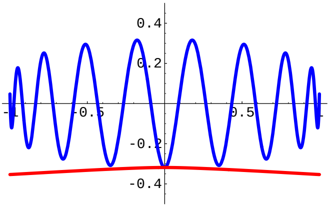

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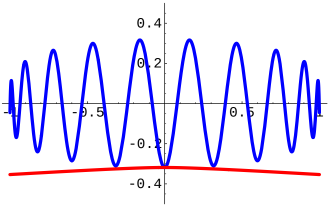

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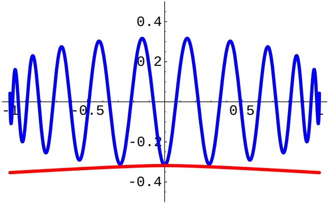

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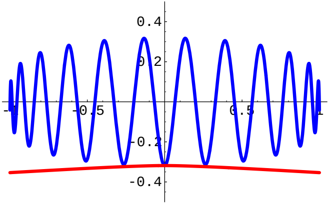

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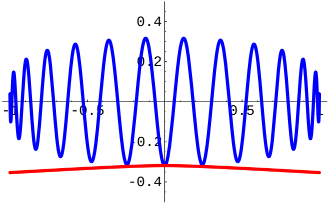

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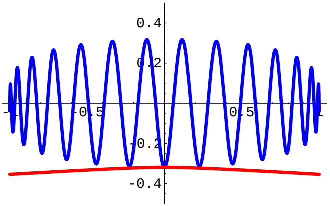

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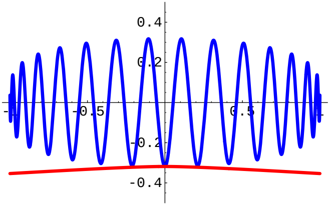

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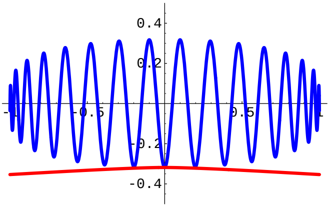

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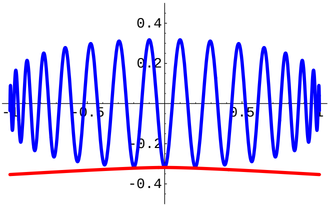
The closed form part $g_n^\alpha(x)$ contains the main oscillations.



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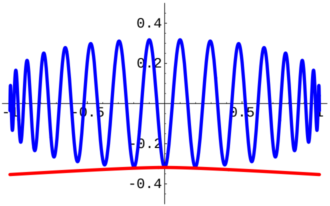

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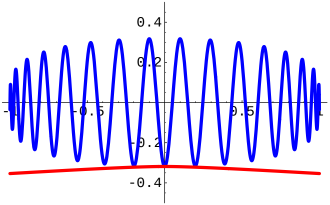

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Next goal: Find $e_n^\alpha(x)$ in closed form such that

$$f_n^\alpha(x) \leq e_n^\alpha(x).$$

Bound the sum

Consider

$$f_n^{\alpha}(x,y) = \sum_{k=0}^{2n} \frac{4^{-\alpha}(1-4\alpha^2)}{(2\alpha+2k-1)(2\alpha+2k+3)} \frac{\binom{2\alpha+k}{\alpha}}{\binom{\alpha+k}{\alpha}} P_k^{(\alpha,\alpha)}(x) P_k^{(\alpha,\alpha)}(y)$$

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Then $f_n^\alpha(x) = f_n^\alpha(x,0)$.

It can be shown without too much effort that

$$f_n^\alpha(x,y) \le \tfrac{1}{2}\big(f_n^\alpha(x,x) + f_n^\alpha(y,y)\big) \quad (\alpha \in [-\tfrac{1}{2}, \tfrac{1}{2}])$$

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So we may set

$$e_n^\alpha(x) := \tfrac{1}{2}\big(f_n^\alpha(x,x) + f_n^\alpha(0,0)\big).$$

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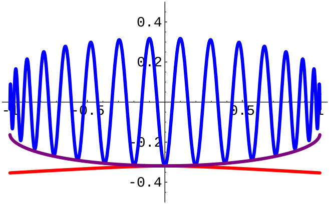

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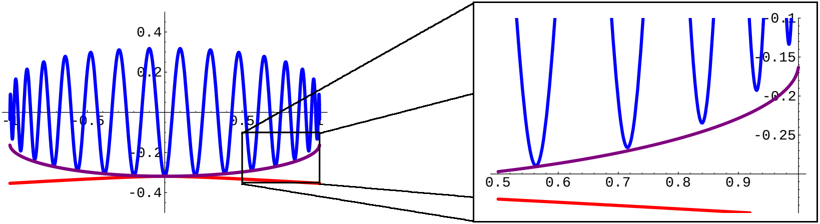

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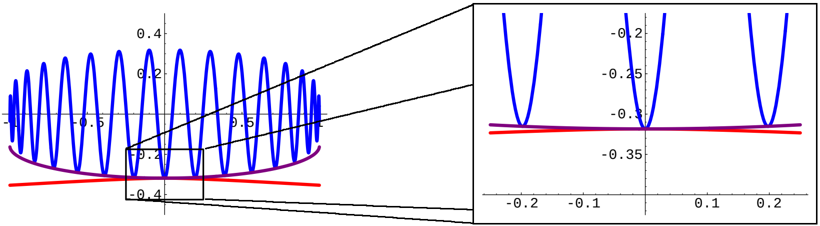

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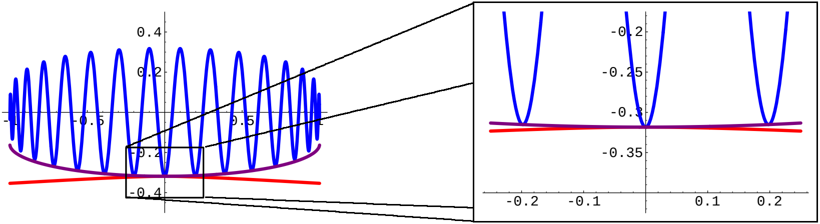

Looks promising...

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We have

$$g_n^\alpha(x) = 2^{-2\alpha-1}(2n+1)\frac{\binom{2\alpha+2n+1}{\alpha}}{\binom{\alpha+2n}{\alpha}} P_{2n}^{(\alpha,\alpha)}(0) \times \left(x P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right)$$

$$e_n^\alpha(x) = 2^{-2\alpha-1}(2n+1)\frac{\binom{2\alpha+2n+1}{\alpha}}{\binom{\alpha+2n}{\alpha}} \times \left(x P_{2n}^{(\alpha,\alpha)}(x) P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{\alpha+2n+1}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x)^2 - \frac{\alpha+2n+1}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(0)^2 - \frac{(2n+1)(2\alpha+2n+1)}{(\alpha+2n+1)(2\alpha+4n+1)} P_{2n+1}^{(\alpha,\alpha)}(x)^2 \right)$$

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It remains to show $g_n^\alpha(x) \geq e_n^\alpha(x)$.

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After some simplifications, it remains to show

$$\begin{aligned}&(\alpha+2n+1)^2(2\alpha+4n+1)(P_{2n}^{(\alpha,\alpha)}(0)^2+P_{2n}^{(\alpha,\alpha)}(x)^2)\\ &\quad+(2n+1)(2\alpha+2n+1)(2\alpha+4n+3)P_{2n+1}^{(\alpha,\alpha)}(x)^2\\ &\quad-(\alpha+2n+1)(2\alpha+4n+1)\\ &\qquad\times(2\alpha+4n+3)xP_{2n}^{(\alpha,\alpha)}(x)P_{2n+1}^{(\alpha,\alpha)}(x)\geq 0\end{aligned}$$

for $-1 \leq x \leq 1$ and $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$.

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This would still be a hard thing to do by hand.

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for $-1 \leq x \leq 1$ and $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$.

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A Tarski formula for the induction step is…

... and leaving the rest to the computer

$$\forall n,\alpha,x,y,z,w\big((n \geq 0 \wedge -1 \leq x \leq 1 \wedge -1 \leq 2\alpha \leq 1 \wedge (2\alpha+4n+1)(y^2+z^2)(\alpha+2n+1)^2 - (2\alpha+4n+1)(2\alpha+4n+3)wxz(\alpha+2n+1) + (2n+1)(2\alpha+2n+1)(2\alpha+4n+3)w^2 \geq 0) \Rightarrow (2n+3)(\alpha+2n+1)^2(\alpha+2n+3)^2(2\alpha+2n+3)(2\alpha+4n+5)y^2(\alpha+2n+2)^2 + (\alpha+2n+1)^2(64n^5 - 256x^2n^4 + 160\alpha n^4 + 464n^4 + 144\alpha^2n^3 - 512\alpha x^2n^3 - 1184x^2n^3 + 928\alpha n^3 + 1344n^3 + 56\alpha^3n^2 + 628\alpha^2n^2 - 384\alpha^2x^2n^2 - 1776\alpha x^2n^2 - 1984x^2n^2 + 2016\alpha n^2 + 1944n^2 + 8\alpha^4n + 164\alpha^3n + 912\alpha^2n - 128\alpha^3x^2n - 888\alpha^2x^2n - 1984\alpha x^2n - 1434x^2n + 1944\alpha n + 1404n + 12\alpha^4 + 120\alpha^3 + 441\alpha^2 - 16\alpha^4x^2 - 148\alpha^3x^2 - 496\alpha^2x^2 - 717\alpha x^2 - 378x^2 + 702\alpha + 405)z^2(\alpha+2n+2)^2 - w^2(-256n^7 + 4096x^4n^6 - 3072x^2n^6 - 896\alpha n^6 - 1728n^6 + 12288\alpha x^4n^5 + 25088x^4n^5 - 1216\alpha^2n^5 - 9216\alpha x^2n^5 - 19968x^2n^5 - 5184\alpha n^5 - 4864n^5 + 15360\alpha^2x^4n^4 + 62720\alpha x^4n^4 + 62464x^4n^4 - 800\alpha^3n^4 - 5872\alpha^2n^4 - 11008\alpha^2x^2n^4 - 49920\alpha x^2n^4 - 53120x^2n^4 - 12160\alpha n^4 - 7408n^4 - 256\alpha^4n^3 + 10240\alpha^3x^4n^3 + 62720\alpha^2x^4n^3 + 124928\alpha x^4n^3 + 81216x^4n^3 - 3104\alpha^3n^3 - 11072\alpha^2n^3 - 6656\alpha^3x^2n^3 - 47744\alpha^2x^2n^3 - 106240\alpha x^2n^3 - 74176x^2n^3 - 14816\alpha n^3 - 6592n^3 - 32\alpha^5n^2 - 752\alpha^4n^2 + 3840\alpha^4x^4n^2 + 31360\alpha^3x^4n^2 + 93696\alpha^2x^4n^2 + 121824\alpha x^4n^2 + 58320x^4n^2 - 4448\alpha^3n^2 - 10192\alpha^2n^2 - 2112\alpha^4x^2n^2 - 21696\alpha^3x^2n^2 - 76416\alpha^2x^2n^2 - 111264\alpha x^2n^2 - 57396x^2n^2 - 9888\alpha n^2 - 3424n^2 - 64\alpha^5n - 736\alpha^4n + 768\alpha^5x^4n + 7840\alpha^4x^4n + 31232\alpha^3x^4n + 60912\alpha^2x^4n + 58320\alpha x^4n + 21978x^4n - 2784\alpha^3n - 4576\alpha^2n - 320\alpha^5x^2n - 4608\alpha^4x^2n - 23296\alpha^3x^2n - 53568\alpha^2x^2n - 57396\alpha x^2n - 23340x^2n - 3424\alpha n - 960n - 32\alpha^5 - 240\alpha^4 + 64\alpha^6x^4 + 784\alpha^5x^4 + 3904\alpha^4x^4 + 10152\alpha^3x^4 + 14580\alpha^2x^4 + 10989\alpha x^4 + 3402x^4 - 640\alpha^3 - 800\alpha^2 - 16\alpha^6x^2 - 352\alpha^5x^2 - 2504\alpha^4x^2 - 8240\alpha^3x^2 - 13881\alpha^2x^2 - 11670\alpha x^2 - 3897x^2 - 480\alpha - 112)(\alpha+2n+2)^2 - 2(\alpha+2n+1)wx(128n^6 - 1024x^2n^5 + 384\alpha n^5 + 1408n^5 + 448\alpha^2n^4 - 2560\alpha x^2n^4 - 5504x^2n^4 + 3520\alpha n^4 + 5592n^4 + 256\alpha^3n^3 + 3296\alpha^2n^3 - 2560\alpha^2x^2n^3 - 11008\alpha x^2n^3 - 11488x^2n^3 + 11184\alpha n^3 + 10888n^3 + 72\alpha^4n^2 + 1424\alpha^3n^2 + 7870\alpha^2n^2 - 1280\alpha^3x^2n^2 - 8256\alpha^2x^2n^2 - 17232\alpha x^2n^2 - 11688x^2n^2 + 16332\alpha n^2 + 11258n^2 + 8\alpha^5n + 272\alpha^4n + 2278\alpha^3n + 7692\alpha^2n - 320\alpha^4x^2n - 2752\alpha^3x^2n - 8616\alpha^2x^2n - 11688\alpha x^2n - 5814x^2n + 11258\alpha n + 5940n + 16\alpha^5 + 220\alpha^4 + 1124\alpha^3 + 2669\alpha^2 - 32\alpha^5x^2 - 344\alpha^4x^2 - 1436\alpha^3x^2 - 2922\alpha^2x^2 - 2907\alpha x^2 - 1134x^2 + 2970\alpha + 1257)z(\alpha+2n+2)^2 \geq 0\big)$$

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This completes the proof of Schöberl's conjecture.

Pillwein's Proof

Message:

A special function inequality may require some *very* non-obvious manipulation before an induction proof via CAD succeeds.

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- Stronger computer algebra methods for proving special function inequalities would be highly appreciated...
- ...because these inequalities are *soo* difficult.