Fast Solvers for Dense Linear Systems

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Example

- Suppose you have given a sequence a_n of rational numbers, say

$$\frac{25}{24}, \frac{3898}{4213}, \frac{4774398}{5383247}, \frac{445394100}{509117429}, \frac{1875780301068}{2147400656503}, \frac{445092169340}{507340266747}, \ldots$$

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  for some rational function rat.
- How could you discover such a rational function?
- Make an *ansatz!*

Example

Find constants $c_i \in \mathbb{Q}$ such that

$$a_n = \frac{c_1 + c_2 n + c_3 H_n + c_4 H_n^{(2)} + c_5 H_n^{(3)}}{c_6 + c_7 n + c_8 H_n + c_9 H_n^{(2)} + c_{10} H_n^{(3)}},$$

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i.e.,

$$\begin{aligned} 0 = {} & c_1 + c_2 n + c_3 H_n + c_4 H_n^{(2)} + c_5 H_n^{(3)} \\ & - c_6 a_n - c_7 n a_n - c_8 H_n a_n - c_9 H_n^{(2)} a_n - c_{10} H_n^{(3)} a_n. \end{aligned}$$

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By plugging in $n = 1, \ldots, 10$ we get a *dense linear system:*

$$\begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

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$$a_n = \frac{c_1 + \cdots + c_{15} n H_n H_n^{(2)} + \cdots + c_{30} n^2 (H_n^{(3)})^2}{c_{31} + \cdots + c_{45} n H_n H_n^{(2)} + \cdots + c_{60} n^2 (H_n^{(3)})^2}.$$

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This leads to a system of size 60×60.
This system has a solution that corresponds to the closed form

$$\begin{aligned} a_n = \Big(&(n+3)H_n^2 + (2n+3)H_n + (3n-2)H_n^{(2)} H_n \\ &+ (2n-5)H_n^{(2)} + (n^2+n-3)H_n^{(3)} \\ &+ (2n+17)H_n^{(2)} H_n^{(3)} \Big) \Big/ \Big(3nH_n^2 + (5n-3)(H_n^{(2)})^2 \\ &+ (6n+5)(H_n^{(3)})^2 + (2n+3)H_n^{(2)} + (7n-5)H_n^{(3)} + 1 \Big). \end{aligned}$$

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The ugliest coefficient in this system would have been

908832599038694847038986851619916896699069828520278576734313218152228688617842975740915627396600
773096516860514938584475180035408435641902208677547085204403335118857901897921641508178647778278
950903964390545421753413156253428091388374361101038380706238279355922616786499296651605565677324
470873903641969510610033133866940362732235659419739168449043859859310108067614923918419572568852
463851315094097859434813883995756702579167128186328425670763241523886987083882016038071001636239
882720818524396979841994456391528090086739296315810673976687526368697214077911507428570965825294
889257827598342283599564261186266965141843600586071958087703197746205189825787434923775654359633
142865809525435636703214553432835616991039905573484634179460089512753393831372170001034464084815
860074912527360333164889060007697392681240306838092094762240357437235301741257767771407557323331
98776514572024833132166748245392570781813055455442682338791285775275321/608071561520469263771864
912900208340519341228462325866654070954648781382761160831047292475594970168876391229713333361460
617524426158506233015628532580104175799899603569619861748499212232349202704257338492766228143557
938393336466485636213537922123315123885938042342534943489837490551827553484761723686376518648743
365387695416861600852713536364490121065994222729396210947647475233184372489732847890966566597135
449686235059997946055799717491204008129578384888903681795059365804600893257023388718806123574709
883282534363429790748372716661107973838303728281458354476754486477224385836362983346375210030954
250430003579185696334806802111301940101874894701556977700464998889377408829983347785295119355949
072698840068582490079977153154387203675675429903671982942691774960800951099556416364355824981174
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And this was only a toy example...

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Given: a matrix $A \in \mathbb{Q}^{n\times n}$
Find: all $x \in \mathbb{Q}^n$ such that $A \cdot x = 0$.

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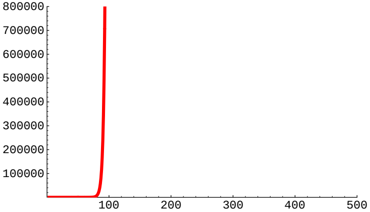

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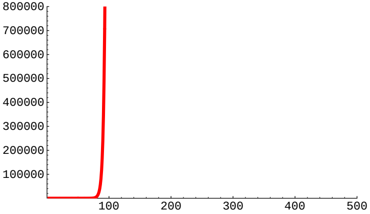

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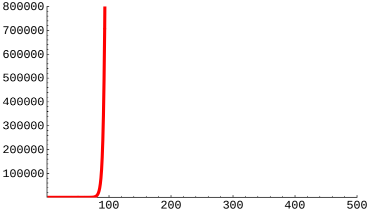

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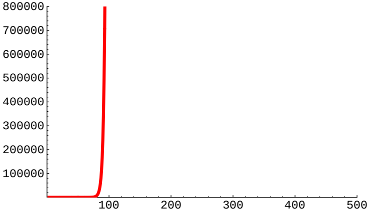

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Ex: expected runtime for solving a 300×300 system: 10^{33} years.
(If you are 100 000 times faster, you still have to wait 10^{27} years.)

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$$\begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \end{pmatrix}$$

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$$\begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 0 & \frac{1}{168} & \frac{8}{945} & \frac{1}{105} \\ 0 & \frac{1}{198} & \frac{16}{2145} & \frac{2}{231} \end{pmatrix}$$

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Ugliest intermediate coefficient: $\frac{1}{186376544704350}$

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What to do? *Goal:* Find ways to avoid expression swell.

Technique I: Gauss-Bareiss Elimination

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$$\begin{pmatrix} a_{1,1} & a_{1,2} & * & * & * \\ a_{2,1} & a_{2,2} & * & * & * \\ a_{3,1} & a_{3,2} & * & * & * \\ a_{4,1} & a_{4,2} & * & * & * \\ a_{5,1} & a_{5,2} & * & * & * \end{pmatrix}$$

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$$\begin{pmatrix} a_{1,1} & a_{1,2} & ** & ** & ** \\ 0 & a_{1,1}a_{2,2} - a_{1,2}a_{2,1} & ** & ** & ** \\ 0 & a_{1,1}a_{3,2} - a_{1,2}a_{3,1} & ** & ** & ** \\ 0 & a_{1,1}a_{4,2} - a_{1,2}a_{4,1} & ** & ** & ** \\ 0 & a_{1,1}a_{5,2} - a_{1,2}a_{5,1} & ** & ** & ** \end{pmatrix}$$

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Thm. All elements in the remaining matrix are divisible by $a_{1,1}$.

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Ex. $\color{red}{***} = a_{1,1}(-a_{1,4}a_{2,2}a_{4,1} + a_{1,2}a_{2,4}a_{4,1} + a_{1,4}a_{2,1}a_{4,2}$
$- a_{1,1}a_{2,4}a_{4,2} - a_{1,2}a_{2,1}a_{4,4} + a_{1,1}a_{2,2}a_{4,4})$

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In fact, the resulting algorithm as only polynomial *bit complexity.*

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We need another idea here.

Technique II: Homomorphic Images

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Let $\mathbb{Z}_p := \{0, 1, 2, 3, \ldots, p - 1\}$.

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Let $\mathbb{Z}_p := \{0, 1, 2, 3, \ldots, p-1\}$.

Define $+$ and $\cdot$ on $\mathbb{Z}_p$ via

$$a + b := (a + b) \bmod p \quad a \cdot b := (a \cdot b) \bmod p \qquad (a, b \in \mathbb{Z}_p)$$

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The algebraic domain $\mathbb{Z}_p$ is called a *finite field of characteristic p*.

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The map m is called a *homomorphism.*

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Let $m\colon \mathbb{Z} \to \mathbb{Z}_p$ be the map $a \mapsto a \bmod p$.

Then

$$m(a+b) = m(a) + m(b), \quad m(a \cdot b) = m(a) \cdot m(b) \qquad (a, b \in \mathbb{Z}).$$

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Example: $m(4/3) = 6$ in $\mathbb{Z}_7$, because $3 \cdot 6 = 4$ in $\mathbb{Z}_7$.

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Global strategy:

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- This is called *rational reconstruction.*

Homomorphic Images

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This might be too large to be efficient. We prefer to compute with small primes.

Homomorphic Images

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Example: If $a = 3$ in $\mathbb{Z}_7$ and $b = 4$ in $\mathbb{Z}_{11}$,
then $(-3) \cdot 7 + 2 \cdot 11 = 1$
and $c = 3 + (4 - 3)(-3)7 = -18 = 59$ in $\mathbb{Z}_{77}$.

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Cool: The images $m_1(x), \ldots, m_k(x)$ can be computed independently *in parallel,* each prime on a seperate processor.

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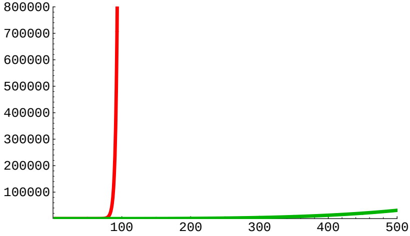
This allows to crack much larger systems in a reasonable time, even on a single processor machine.



Homomorphic Images

Feature: This technique extends to linear systems with *polynomial* coefficients:

$$\begin{array}{ccc}
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\downarrow & & \uparrow \\
m(A) \in \mathbb{Z}_p[t]^{n\times n} & & m(x) \in \mathbb{Z}_p[t]^{n} \\
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- Linear systems can be solved in polynomial time. Seriously.
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- The algorithms presented in this talk are known since long.
- Modern algorithms are even faster than this. (But also more difficult.)
- In applications, special knowledge about a matrix should always be taken into account (sparsity, structure, ...) before a general purpose algorithm is applied.