## A Proof of George Andrews' and Dave Robbins' q-TSPP Conjecture[1]

(modulo a finite amount of routine calculations)

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*Pour Pierre Leroux, In Memoriam*

### Preface: Montréal, May 1985

In the historic conference *Combinatoire Énumérative* [LL] wonderfully organized by Gilbert Labelle and **Pierre Leroux** there were many stimulating lectures, including a very interesting one by Pierre Leroux himself, who talked about his joint work with Xavier Viennot [LV], on solving differential equations combinatorially! During the problem session of that very same *colloque*, chaired by Pierre Leroux, Richard Stanley raised some intriguing problems about the enumeration of plane partitions, that he later expanded into a fascinating article [Sta1]. Most of these problems concerned the enumeration of *symmetry classes* of *plane partitions*, that were discussed in more detail in another article of Stanley [Sta2]. *All* of the conjectures in the latter article have since been proved (see Dave Bressoud's modern classic [B]), *except* one, that, so far, *resisted* the efforts of the greatest minds in enumerative combinatorics. It concerns the proof of an explicit formula for the $q$-enumeration of *totally symmetric plane partitions*, conjectured independently by George Andrews and Dave Robbins ([Sta2], [Sta1] (conj. 7), [B] (conj. 13)). In this tribute to Pierre Leroux, we describe how to prove that last stronghold.

### $q$-TSPP: The Last Surviving Conjecture About Plane Partitions

Recall that a *plane partition* $\pi$ is an array $\pi = (\pi_{ij})$, $i, j \geq 1$, of positive integers $\pi_{ij}$ with finite sum $|\pi| = \sum \pi_{ij}$, which is weakly decreasing in rows and columns. By stacking $\pi_{ij}$ unit cubes on top of the $ij$ location, one gets the 3D Ferrers diagram, that can be identified with the plane-partition, and is a left-, up-, and bottom- justified structure of unit cubes, and we can refer to the locations $(i, j, k)$ of the individual unit cubes.

A plane partition is *totally symmetric* iff whenever $(i, j, k)$ is occupied (i.e. $\pi_{ij} \geq k$), it follows that all its (up to 5) permutations: $\{(i, k, j), (j, i, k), (j, k, i), (k, i, j), (k, j, i)\}$ are also occupied. In 1995, John Stembridge [Ste] proved Ian Macdonald's conjecture that the number of totally symmetric plane partitions (TSPPs) whose $3D$ Ferrers diagram is bounded inside the cube $[0, n]^3$ is given by

[1] Accompanied by Maple packages TSPP and qTSPP available from `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/qtspp.html`. The work of M. Kauers and C. Koutschan was supported in part by the Austrian FWF grants SFB F1305, P19462-N18, and P20162-N18.
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the nice product-formula

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2} \quad .$$

Ten years after Stembridge's completely human-generated proof, George Andrews, Peter Paule, and Carsten Schneider [APS] came up with a *computer-assisted* proof, that, however required lots of human ingenuity and ad hoc tricks, in addition to a considerable amount of computer time.

Way back in the early-to-mid eighties (ca. 1983), George Andrews and Dave Robbins independently conjectured a *q-analog* of this formula, namely that the *orbit-counting generating function* ([B], p. 200, [Sta1], p. 289) is given by

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \quad .$$

In this article we will show how to prove this conjecture (modulo a finite amount of routine computer calculations that may be already feasible today (with great technical effort), but that would most likely be routinely checkable on a standard desktop in twenty years).

**Soichi Okada's Crucial Insight**

Our starting point is an elegant reduction, by Soichi Okada [O], of the $q$-TSPP statement, to the problem of evaluating a certain "innocent-looking" determinant. This is also listed as Conjecture 46 (p. 42) in Christian Krattenthaler's celebrated essay [K] on the art of determinant evaluation.

**Okada's Determinant** (Shown by Okada to imply the $q$-TSPP conj.)

Let, as usual, $\delta(\alpha,\beta)$ be the Kronecker delta function ($\delta(\alpha,\beta)=1$ when $\alpha=\beta$ and $\delta(\alpha,\beta)=0$ when $\alpha \neq \beta$), and let, also as usual,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)(1-q^{b-1})\cdots(1-q)} \quad .$$

Define the discrete function $a(i,j)$ by:

$$a(i,j) = q^{i+j-1}\left(\begin{bmatrix} i+j-2 \\ i-1 \end{bmatrix} + q\begin{bmatrix} i+j-1 \\ i \end{bmatrix}\right) + (1+q^i)\delta(i,j) - \delta(i,j+1) \quad .$$

Soichi Okada ([O], see also [K], Conj. 46) proved that the following conjectured determinant evaluation would imply the $q$-TSPP conjecture.

For any nonnegative integer $n$, there holds:

$$\det(a(i,j))_{1 \leq i,j \leq n} = \prod_{1 \leq i \leq j \leq k \leq n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 \quad .$$

So in order to prove the $q$-TSPP conjecture, all we need is to prove Okada's conjectured determinant evaluation.

**The $q$-Holonomic Ansatz**

In [Z3], an empirical (yet fully rigorous!) approach is described to (symbolically!) evaluate determinants $A(n) := \det(a(i,j))_{1 \leq i,j \leq n}$, where $a(i,j)$ is a holonomic discrete function of $i$ and $j$. Note that this is an *approach*, not a method! It is not guaranteed to always work (and probably usually doesn't!).

Let's first describe this approach in more general terms, not just within the holonomic ansatz.

Suppose that $a(i,j)$ is given "explicitly" (as it sure is here), and we want to prove for *all* $n \geq 0$ that

$$\det(a(i,j))_{1 \leq i,j \leq n} = Nice(n) \quad ,$$

for some *explicit* expression $Nice(n)$ (as it sure is here).

The approach is to *pull out of the hat* another "explicit" (possibly in a much broader sense of the word *explicit*) discrete function $B(n,j)$, and then verify the identities

$$\sum_{j=1}^{n} B(n,j)a(i,j) = 0 \quad , \quad (1 \leq i < n < \infty) \quad , \qquad (Soichi)$$

$$B(n,n) = 1 \quad , \quad (1 \leq n < \infty) \quad . \qquad (Normalization)$$

If we could do that, then by *uniqueness*, it would follow that $B(n,j)$ equals the co-factor of the $(n,j)$ entry of the $n \times n$ determinant divided by the $(n-1) \times (n-1)$ determinant (that is the co-factor of the $(n,n)$ entry in the $n \times n$ determinant). Finally one has to prove the identity

$$\sum_{j=1}^{n} B(n,j)a(n,j) = Nice(n)/Nice(n-1) \quad . \qquad (Okada)$$

While [Z3] was concerned with the holonomic ansatz ([Z1]), everything goes *verbatim* to the $q$-holonomic ansatz. Recall that the nice features of both the holonomic and $q$-holonomic ansatzes are

(i) They are closed under addition and multiplication .

(ii) It is always algorithmically decidable whether A=B .

(iii) If $F(i_1, i_2, \dots, i_r)$ is a $q$-holonomic function of its variables, then

$$G(i_1, \dots, i_{r-1}) := \sum_{i_r=-\infty}^{\infty} F(i_1, \dots, i_{r-1}, i_r)$$

is $q$-holonomic in the surviving variables $i_1, \dots, i_{r-1}$. Furthermore, if one has a $q$-holonomic description of $F$, one can get a $q$-holonomic description of $G$.

**What is a $q$-Holonomic Description?**

A $q$-holonomic discrete function of **one** variable $f(n)$ is determined in terms of a *linear recurrence equation*

$$f(n) = \sum_{r=1}^{R} a_r(q, q^n) f(n-r) \quad ,$$

where $a_r(q, q^n)$ are certain rational functions of their arguments, together with the **initial values**

$$f(0), ..., f(R-1) \quad .$$

Analogously, A $q$-holonomic discrete function $f(m,n)$ of **two** variables $m$ and $n$ is determined in terms of a *linear recurrence equation*

$$f(m,n) = \sum_{r=1}^{R} a_r(q, q^m, q^n) f(m, n-r) \quad ,$$

where $a_r(q, q^m, q^n)$ are rational functions in their arguments, together with the **initial values** $f(m,0), ..., f(m, R-1)$ that should be given holonomic descriptions on their own right as holonomic functions of a single variable (as described above). This can be continued to discrete functions of any number of variables.

Note that while every holonomic discrete function can be described as above, not every function that is described as above, with *arbitrary* $a_r$'s is necessarily holonomic (usually it isn't!). However there are efficient algorithms for deciding whether a candidate discrete function given as above is holonomic or not. One empirical way of doing this is to use the description to crank many values, and then "guess" a pure recurrence with polynomial coefficients in the other variable, $m$, that can be routinely proved *a posteriori*.

**The Computational Challenge**

*A priori* we don't have a clue whether the normalized cofactors, let's call them $A'(n,j)$, are $q$-holonomic. One has to *hope*. By cranking out enough values, one tries to guess a "$q$-holonomic" description as above. Without this guess, there is nothing one can do!

Using several clever tools of the trade in the art of efficient guessing -a straightforward brute-force approach is hopeless!-we were able to come up with a conjectured $q$-holonomic description of these normalized cofactors. It is contained in the Maple package `qTSPP` accompanying this article. The much easier $q = 1$ case (that would give a new proof to the already proved Stembridge theorem) is contained in the Maple package `TSPP`.

Note that *once* we have a *conjectured* $q$-holonomic description of the normalized cofactors, proving it rigorously is completely routine. Identity (*Soichi*) is a decidable $q$-holonomic identity and there exist algorithms of Chyzak, Salvy, and Takayama ([CS], [T]) to prove it. Also (*Normalization*) is automatically satisfied, it being part of the definition of $B(n, j)$. Finally (*Okada*) is also of the form $A = B$ where both sides are $q$-holonomic. The left side is holonomic because of the closure under multiplication and definite-summation, and the right side, $Nice(n)/Nice(n-1)$ is not just $q$-holonomic (a solution of *some* linear recurrence with polynomial coefficients (in $q, q^n$) ) but in fact *closed-form* (the defining recurrence is first-order).

**The Semi-Rigorous Shortcut**

We believe that even today, performing the above steps is feasible, but it would require a huge technical effort. But why bother? First, if we wait for twenty more years, Moore's Law will probably enable us to finish up these finitely-many routine calculations with no sweat. Besides, since now we **know for sure** that a *fully rigorous* proof **exists**, do we really want to see it? It won't give us any new *insight*. The beauty of the present approach is in the *meta-insight*, reducing the statement of the conjecture to a *finite* calculation. Furthermore, we know *a priori* that there exists an operator $P(q, q^n, N)$ (where $N$ is the shift operator in $n$: Nf(n):=f(n+1)) that annihilates the difference of the left and right sides of (*Okada*). If that operator has order $L$, say, then a completely rigorous proof would be to check (*Okada*) for $0 \leq n \leq L-1$. At present, we are unable to find $P$, and hence do not know the value of $L$. But it is very reasonable that $L$ would be less than, say, 400, and checking the first 400 cases of (*Okada*) (and analogously for (*Soichi*)) is certainly doable (we did it for $L = 100$, and $L = 400$ for TSPP, but you are welcome to go further). These are done in procedures `CheckqTSPP` in the Maple package `qTSPP`, and `CheckTSPP` in the Maple package `TSPP`, respectively. The corresponding input and output files can be found in the webpage of this article mentioned above. As a technical aside, let's confess that Maple running on our computer was only able to check (Soichi) and (Okada) for $L \leq 30$, for *symbolic* $q$, but for random numerical choices of $q$ it went up to $L = 100$, and it is easy to see that with sufficiently many choices of numerical $q$ for a given $L$, one can prove it for symbolic $q$.

In 1993, Zeilberger [Z2] proposed the notion of *semi-rigorous* proof. At the time he didn't have any *natural* examples. The present determinant evaluation, that was shown by Okada to imply a long-standing open problem in enumerative combinatorics, is an *excellent* example of a semi-rigorous proof that is (at least) as good as a rigorous proof. Let us conclude by promising that if any one is willing to pay us $\$10^7$ (ten million US dollars), we will be more than glad to fill-in-the details.

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