Algorithms for Holonomic Functions

Manuel Kauers

Research Institute for Symbolic Computation
Johannes Kepler University
Austria

Context

Goal: Algorithms for dealing with functions:

Goal: Algorithms for dealing with functions:

- proving formulas

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions
- determining singularities

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions
- determining singularities
- approximating function values

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions
- determining singularities
- approximating function values
- plotting the graph

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions
- determining singularities
- approximating function values
- plotting the graph
- estimating the growth

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions
- determining singularities
- approximating function values
- plotting the graph
- estimating the growth

Want: Algorithms which take as input a function and produce answers to these questions as output.

Goal: Algorithms for dealing with functions:

- proving formulas
- evaluating sums and integrals
- computing series expansions
- determining singularities
- approximating function values
- plotting the graph
- estimating the growth

Want: Algorithms which take as input a function and produce answers to these questions as output.

Question: What does it mean to "take as input a function"?

Question: What does it mean to “take as input a function”?

Question: What does it mean to “take as input a function”?

Answer:

Question: What does it mean to “take as input a function”?

Answer: **NOTHING!**

Question: What does it mean to “take as input a function”?

Answer: **NOTHING!**

A real or complex function is an infinite object:

Question: What does it mean to “take as input a function”?

Answer: **NOTHING!**

A real or complex function is an infinite object:

- For each of the infinitely many x, we must specify $f(x)$

Question: What does it mean to "take as input a function"?

Answer: **NOTHING!**

A real or complex function is an infinite object:

- For each of the infinitely many x, we must specify $f(x)$
- For each value $f(x)$ we must specify infinitely many decimal digits

Question: What does it mean to “take as input a function”?

Answer: **NOTHING!**

A real or complex function is an infinite object:

- For each of the infinitely many x, we must specify $f(x)$
- For each value $f(x)$ we must specify infinitely many decimal digits

This is an infinite amount of information.

Question: What does it mean to "take as input a function"?

Answer: **NOTHING!**

A real or complex function is an infinite object:

- For each of the infinitely many x, we must specify $f(x)$
- For each value $f(x)$ we must specify infinitely many decimal digits

This is an infinite amount of information.

It is *impossible* to store this in a finite data structure.

Question: What does it mean to “take as input a function”?

Answer: **NOTHING!**

A real or complex function is an infinite object:

- For each of the infinitely many x, we must specify $f(x)$
- For each value $f(x)$ we must specify infinitely many decimal digits

This is an infinite amount of information.

It is *impossible* to store this in a finite data structure.

But algorithms can *only* operate on finite data structures.

Question: What does it mean to "take as input a function"?

Answer: **NOTHING!**

A real or complex function is an infinite object:

- For each of the infinitely many x, we must specify $f(x)$
- For each value $f(x)$ we must specify infinitely many decimal digits

This is an infinite amount of information.

It is *impossible* to store this in a finite data structure.

But algorithms can *only* operate on finite data structures.

Solution: Consider algorithms for suitably defined classes of functions.

Solution: Consider algorithms for suitably defined classes of functions.

A suitably defined class of functions should be

Solution: Consider algorithms for suitably defined classes of functions.

A suitably defined class of functions should be

- *not too big,* because we want to be able to write down each function in the class with a finite amount of data only, and we want to compute with these.

Solution: Consider algorithms for suitably defined classes of functions.

A suitably defined class of functions should be

- *not too big,* because we want to be able to write down each function in the class with a finite amount of data only, and we want to compute with these.
- *not too small,* because we want the class to contain as many functions as possible of those which appear in applications (e. g. in particle physics).

Solution: Consider algorithms for suitably defined classes of functions.

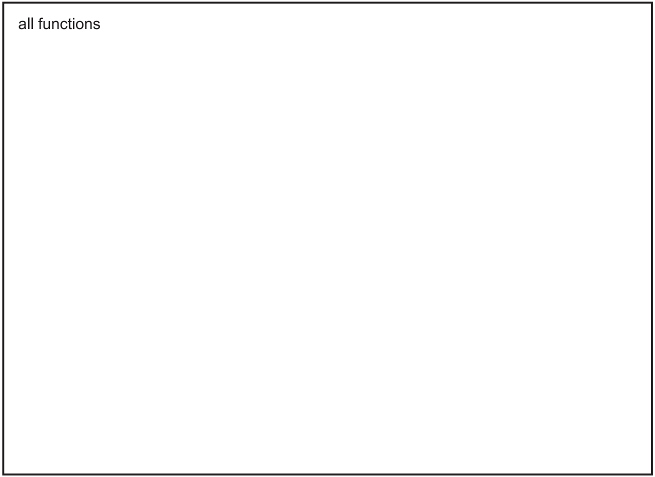
A suitably defined class of functions should be

- *not too big,* because we want to be able to write down each function in the class with a finite amount of data only, and we want to compute with these.
- *not too small,* because we want the class to contain as many functions as possible of those which appear in applications (e. g. in particle physics).

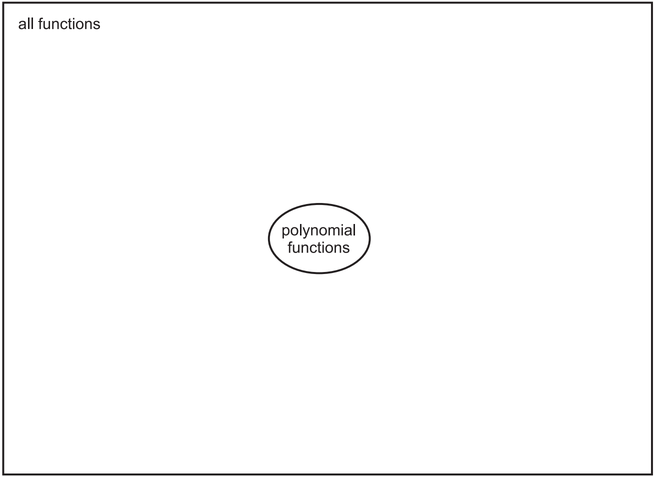
Deciding on the right function class is the first step in algorithmic problem solving.

Some common classes of functions:

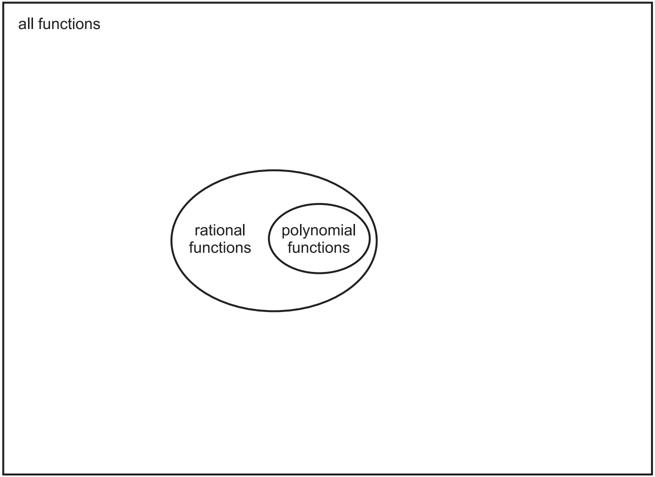
Some common classes of functions:



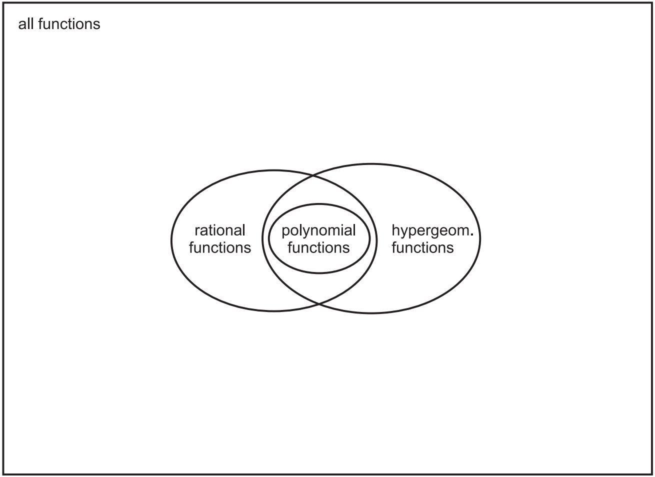
Some common classes of functions:



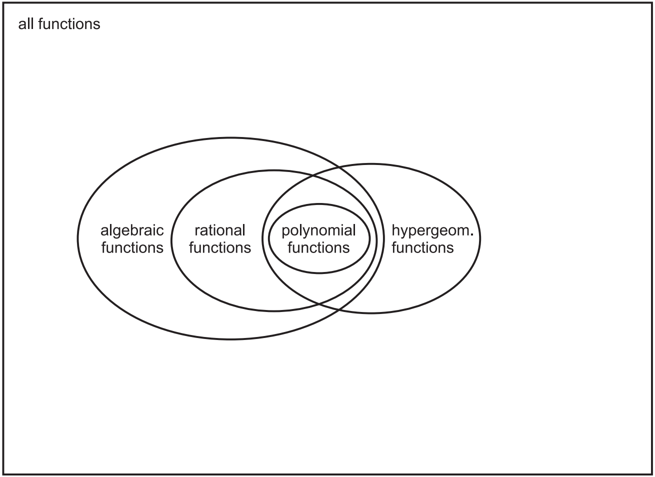
Some common classes of functions:



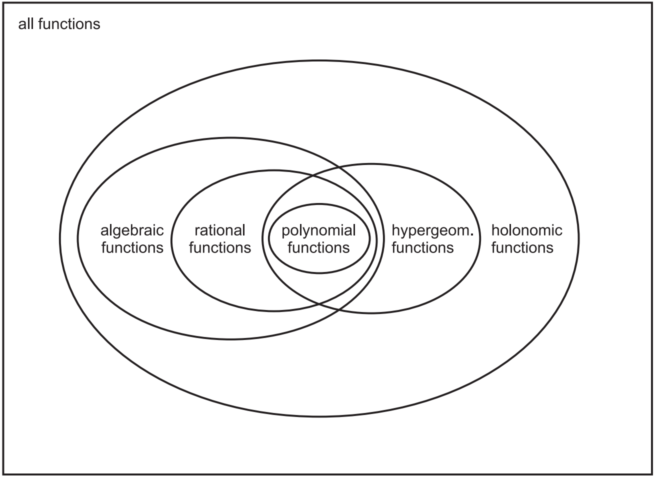
Some common classes of functions:



Some common classes of functions:

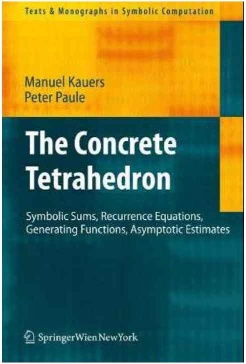


Some common classes of functions:



Commercial: A good reference for these classes of functions (and the corresponding algorithms) is

Commercial: A good reference for these classes of functions (and the corresponding algorithms) is



Holonomy: The Case of One Variable

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$:

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\qquad f'(x) - f(x) = 0$

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\qquad f'(x) - f(x) = 0$
- $\log(1-x)$:

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$:

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$
- Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$
- Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...
- Many functions which have no name and no closed form.

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

- $\exp(\exp(x) - 1)$.

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.
- Many functions which have no name and no closed form.

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

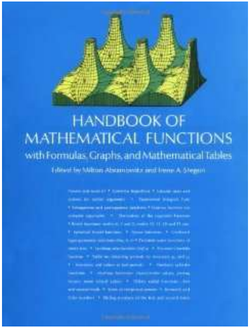
Not holonomic:

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.
- Many functions which have no name and no closed form.

This means that these functions can (provably) not be viewed as solutions of a linear differential equation with polynomial coefficients.

Definition (continuous case). A function f is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$



Approximately 60% of the functions in Abramowitz and Stegun's handbook fall into this category.

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequence: A holonomic function f is uniquely determined by

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequence: A holonomic function f is uniquely determined by

- The differential equation

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequence: A holonomic function f is uniquely determined by

- The differential equation
- A finite number of initial values $f(0), f'(0), f''(0), \ldots, f^{(k)}(0)$

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequence: A holonomic function f is uniquely determined by

- The differential equation
- A finite number of initial values $f(0), f'(0), f''(0), \dots, f^{(k)}(0)$ (Usually, $k = r$ suffices.)

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequence: A holonomic function f is uniquely determined by

- The differential equation
- A finite number of initial values $f(0), f'(0), f''(0), \dots, f^{(k)}(0)$ (Usually, $k = r$ suffices.)

Consequence: A holonomic function can be represented exactly by a finite amount of data.

Theorem. The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequence: A holonomic function f is uniquely determined by

- The differential equation
- A finite number of initial values $f(0), f'(0), f''(0), \dots, f^{(k)}(0)$ (Usually, $k = r$ suffices.)

Consequence: A holonomic function can be represented exactly by a finite amount of data.
(assuming that the constants appearing in equation and initial values belong to a suitable subfield of $\mathbb{C}$, e.g., to $\mathbb{Q}$.)

Examples.

Examples.

- $f(x) = \exp(x)$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x-1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1-x)$
  $\iff (x-1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x-1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$
- $f(x) =$ the fifth modified Bessel function of the first kind

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$
- $f(x) =$ the fifth modified Bessel function of the first kind
  $\iff x^2 f''(x) + xf'(x) - (x^2 + 25)f(x) = 0,$
  $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, f^{(5)}(0) = \frac{1}{32}$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x-1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$
- $f(x) =$ the fifth modified Bessel function of the first kind
  $\iff x^2 f''(x) + xf'(x) - (x^2 + 25)f(x) = 0,$
  $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, f^{(5)}(0) = \frac{1}{32}$
- $\ldots$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\qquad a_{n+1} - 2a_n = 0$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\qquad a_{n+1} - 2a_n = 0$
- $n!$:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $\quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $\quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $\quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...
- Many functions which have no name and no closed form.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

- 2^{2^n}.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

- 2^{2^n}.
- The sequence of prime numbers.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

- 2^{2^n}.
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

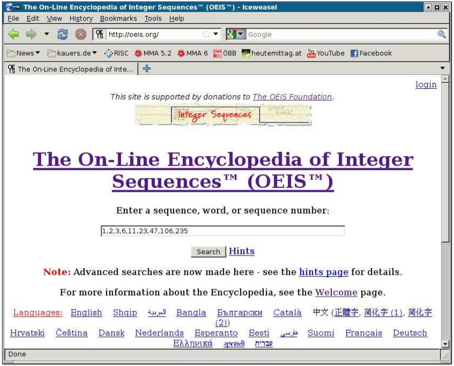
Not holonomic:

- 2^{2^n}.
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

This means that these sequences can (provably) not be viewed as solutions of a linear recurrence equation with polynomial coefficients.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$



Approximately 25% of the sequences in Sloane's Online Encyclopedia of Integer Sequences fall into this category.

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- The recurrence equation

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- The recurrence equation
- A finite number of initial values $a_0, a_1, a_2, \ldots, a_k$

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- The recurrence equation
- A finite number of initial values $a_0, a_1, a_2, \dots, a_k$ (Usually, $k = r$ suffices.)

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- The recurrence equation
- A finite number of initial values $a_0, a_1, a_2, \ldots, a_k$
  (Usually, $k = r$ suffices.)

Consequence: A holonomic sequence can be represented exactly by a finite amount of data.

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- The recurrence equation
- A finite number of initial values $a_0, a_1, a_2, \ldots, a_k$
  (Usually, $k = r$ suffices.)

Consequence: A holonomic sequence can be represented exactly by a finite amount of data.
(assuming that the constants appearing in equation and initial values belong to a suitable subfield of $\mathbb{C}$, e.g., to $\mathbb{Q}$.)

Examples.

Examples.

- $a_n = 2^n$

Examples.

- $a_n = 2^n$

  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$
  $\iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$
  $\iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$
  $\iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$
  $\iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$
  $\iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters
  $\iff a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0,$
  $a_0 = 1, a_1 = 1, a_2 = 2$

Examples.

- $a_n = 2^n$
  $\iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$
  $\iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters
  $\iff a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0,$
  $a_0 = 1, a_1 = 1, a_2 = 2$
- $\ldots$

Have:

Have:

- Finite data structure for representing holonomic objects

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Want:

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Want:

- Structural properties of the class of holonomic objects

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Want:

- Structural properties of the class of holonomic objects
- Algorithms for doing explicit computations with them

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$
\begin{aligned}
& a(x) \text{ is holonomic as function} \\
\iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.}
\end{aligned}
$$

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

The theorem is algorithmic:

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

The theorem is algorithmic:

- Given a differential equation for $a(x)$, we can compute a recurrence for $(a_n)_{n=0}^{\infty}$.

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

The theorem is algorithmic:

- Given a differential equation for $a(x)$, we can compute a recurrence for $(a_n)_{n=0}^{\infty}$.
- Given a recurrence for $(a_n)_{n=0}^{\infty}$, we can compute a differential equation for $a(x)$.

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

Examples.

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

Examples.

INPUT: $a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$
\begin{array}{ll}
 & a(x) \text{ is holonomic as function} \\
\iff & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.}
\end{array}
$$

Examples.

INPUT: $a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

$\Downarrow$

OUTPUT: $(n+1)a_{n+1} - a_n = 0, a_0 = 1$ (i.e., $a_n = \frac{1}{n!}$)

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

Examples.

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

Examples.

INPUT: $2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0$

Theorem (Conversion). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & a(x) \text{ is holonomic as function} \\ \iff \quad & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \end{aligned}$$

Examples.

INPUT: $2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0$

⬇

OUTPUT: $x^5 a^{(5)}(x) + (19x^2 + 3x - 1)x^2 a^{(4)}(x)$
$+ 2(55x^3 + 15x^2 - 2x - 1)a^{(3)}(x) + 6(37x + 12)xa''(x)$
$+ 12(11x + 3)a'(x) + 12a(x) = 0$

Theorem (Asymptotics).

Theorem (Asymptotics).

- If $a(x)$ is holonomic and has a singularity at ζ, then

$$a(x) \sim c\,\mathrm{e}^{P((\zeta-x)^{-1/r})}(\zeta - x)^{\alpha}\log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

Theorem (Asymptotics).

- If $a(x)$ is holonomic and has a singularity at ζ, then

$$a(x) \sim c\,\mathrm{e}^{P((\zeta-x)^{-1/r})}(\zeta-x)^{\alpha}\log(\zeta-x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})}n^{\gamma n}\phi^n n^{\alpha}\log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, ϕ, α, γ are constants, and $\beta \in \mathbb{N}$.

Algorithms.

Algorithms.

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.

Algorithms.

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.

Algorithms.

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Algorithms.

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Example.

Algorithms.

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Example.

INPUT:

$2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0, a_0 = a_1 = 1$

Algorithms.

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Example.

INPUT:

$$2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0, a_0 = a_1 = 1$$

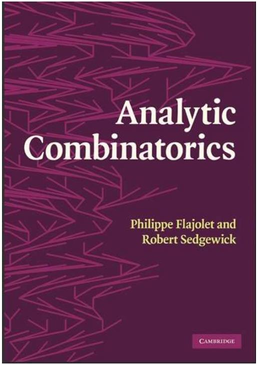
OUTPUT:

$$c\,\mathrm{e}^{\sqrt{n}-\frac{n}{2}} n^{n/2} \left(1 - \tfrac{119}{1152} n^{-1} + \tfrac{7}{24} n^{-1/2} + \tfrac{1967381}{39813120} n^{-2} + \mathrm{O}(n^{-3/2})\right)$$

with $c \approx 0.55069531490318374761598106274964784671382\ldots$

Commercial: A good reference for modern techniques for computing asymptotic expansions is:

Commercial: A good reference for modern techniques for computing asymptotic expansions is:



Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is algebraic and $b(0) = 0$, then $a(b(x))$ is holonomic.

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is algebraic and $b(0) = 0$, then $a(b(x))$ is holonomic.

The theorem is algorithmic:

Theorem (closure properties I). Let $a(x)$ and $b(x)$ be holonomic functions. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is algebraic and $b(0) = 0$, then $a(b(x))$ is holonomic.

The theorem is algorithmic:

- Differential equations for all these functions can be computed from given defining equations of $a(x)$ and $b(x)$.

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

The theorem is algorithmic:

Theorem (closure properties II). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

The theorem is algorithmic:

- Recurrence equations for all these sequences can be computed from given defining equations of $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$.

Examples.

Examples.

INPUT:

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

$(1-x)b''(x) - b'(x) = 0, b(0) = 0, b'(0) = -1$

(i.e., $b(x) = \log(1-x)$)

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

$(1 - x)b''(x) - b'(x) = 0, b(0) = 0, b'(0) = -1$

(i.e., $b(x) = \log(1 - x)$)

⬇ ($c(x) = a(x)b(x)$)

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

$(1-x)b''(x) - b'(x) = 0, b(0) = 0, b'(0) = -1$

(i.e., $b(x) = \log(1-x)$)

⬇ $(c(x) = a(x)b(x))$

OUTPUT:

$(x-1)c''(x) + (3-2x)c'(x) + (x-2)c(x),\ c(0) = 0, c'(0) = -1.$

Examples.

INPUT:

$(n+1)a_{n+1} - na_n, a_1 = 1$ (i.e., $a_n = \frac{1}{n}$)

Examples.

INPUT:

$(n+1)a_{n+1} - na_n, a_1 = 1$ (i.e., $a_n = \frac{1}{n}$)

⬇ $(c_n = \sum_{k=0}^{n} a_k)$

Examples.

INPUT:

$(n+1)a_{n+1} - na_n, a_1 = 1$ (i.e., $a_n = \frac{1}{n}$)

$\Downarrow$ $(c_n = \sum_{k=0}^{n} a_k)$

OUTPUT:

$(n+2)c_{n+2} - (2n+3)c_{n+1} + (n+1)c_n = 0, c_1 = 1, c_2 = \frac{3}{2}$

Examples.

INPUT:

$$(n+2)a_{n+2} - (2n+3)a_{n+1} + (n+1)a_n = 0, a_1 = 1, a_2 = \frac{3}{2}$$

(i.e., $a_n = \sum_{k=1}^{n} \frac{1}{k}$)

Examples.

INPUT:

$$(n+2)a_{n+2} - (2n+3)a_{n+1} + (n+1)a_n = 0, a_1 = 1, a_2 = \frac{3}{2}$$

(i.e., $a_n = \sum_{k=1}^{n} \frac{1}{k}$)

$\Downarrow$ $(c_n = \sum_{k=0}^{n} a_k)$

Examples.

INPUT:

$$(n+2)a_{n+2} - (2n+3)a_{n+1} + (n+1)a_n = 0, a_1 = 1, a_2 = \tfrac{3}{2}$$

(i.e., $a_n = \sum_{k=1}^{n} \frac{1}{k}$)

$\Downarrow$ $\quad (c_n = \sum_{k=0}^{n} a_k)$

OUTPUT:

$$(n^2+4n+4)c_{n+2} - (2n^2+9n+9)c_{n+1} + (n^2+5n+6)c_n = 0,$$

$$c_0 = 2, c_1 = \tfrac{9}{2}$$

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e. $a(x) = \exp(x)$)

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e. $a(x) = \exp(x)$)

$(1 - 4x)b(x)^2 - x^2 = 0$ (i.e. $b(x) = \frac{x}{\sqrt{1-4x}}$)

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e. $a(x) = \exp(x)$)

$(1-4x)b(x)^2 - x^2 = 0$ (i.e. $b(x) = \frac{x}{\sqrt{1-4x}}$)

⬇ ($c(x) = a(b(x))$)

Examples.

INPUT:

$a'(x) - a(x) = 0, a(0) = 1$ (i.e. $a(x) = \exp(x)$)

$(1 - 4x)b(x)^2 - x^2 = 0$ (i.e. $b(x) = \frac{x}{\sqrt{1-4x}}$)

$\Downarrow$ ($c(x) = a(b(x))$)

OUTPUT:

$(4x-1)^3(2x-1)c''(x)+4(x-1)(4x-1)^2c'(x)+(2x-1)^3c(x) = 0$,

$c(0) = 1, c'(0) = 1$

Implementations.

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

In[1]:= << **GeneratingFunctions.m**

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

In[1]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

In[1]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

In[2]:= $\mathbf{DEPlus}[a'[x] - a[x], a'[x] + 2a[x], a[x]]$

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

In[1]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

In[2]:= $\mathbf{DEPlus}[a'[x] - a[x], a'[x] + 2a[x], a[x]]$

Out[2]= $-2(-1 + x + 2x^2)a[x] + (4x^2 - 3)a'[x] + (2x + 1)a''[x] == 0$

Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

In[1]:= **<< GeneratingFunctions.m**

GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

In[2]:= $\mathbf{DEPlus}[a'[x] - a[x], a'[x] + 2a[x], a[x]]$

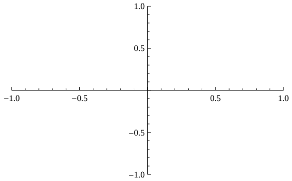
Out[2]= $-2(-1 + x + 2x^2)a[x] + (4x^2 - 3)a'[x] + (2x + 1)a''[x] == 0$

These packages are particularly useful for proving identities.

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

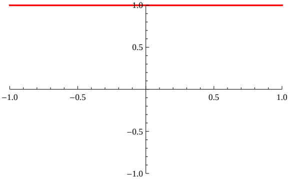
Legendre polynomials:



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

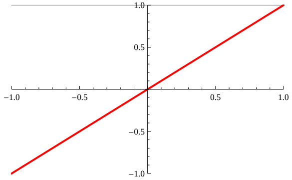
- $P_0(x) = 1$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

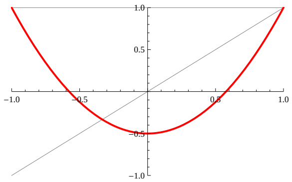
- $P_0(x) = 1$
- $P_1(x) = x$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

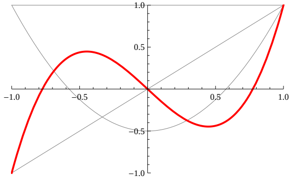
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

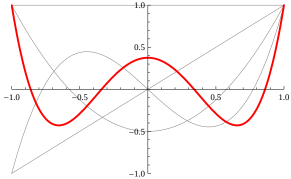
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

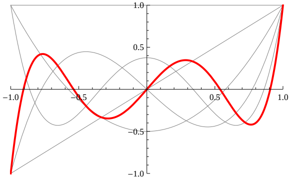
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
- $P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)$
- $\dots$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

$$P_{n+2}(x) = -\frac{n+1}{n+2}P_n(x) + \frac{2n+3}{n+2}xP_{n+1}(x)$$

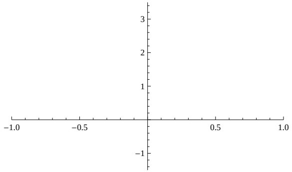
$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

$$\begin{aligned} P_{n+2}(x) &= -\frac{n+1}{n+2}P_n(x) + \frac{2n+3}{n+2}xP_{n+1}(x) \\ P_0(x) &= 1 \\ P_1(x) &= x \end{aligned}$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

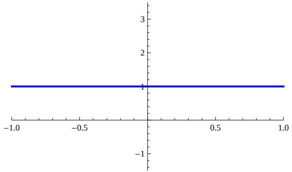
Jacobi polynomials:



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

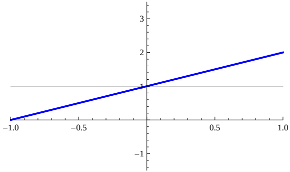
- $P_0^{(1,-1)}(x) = 1$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

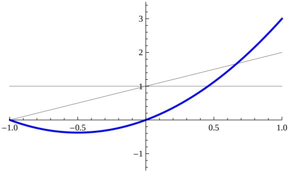
- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

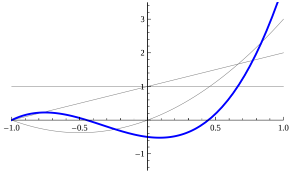
- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

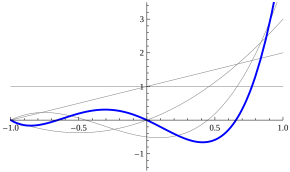
Jacobi polynomials:

- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

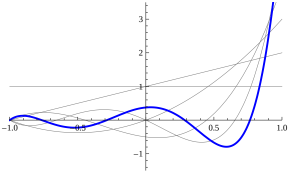
Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$
- $P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$
- $P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)$
- $P_5^{(1,-1)}(x) = \frac{3}{8}(1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5)$
- $\cdots$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

$$P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1} P_n^{(1,-1)}(x) + \frac{2n+3}{n+2} x P_{n+1}^{(1,-1)}(x)$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

$$P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1} P_n^{(1,-1)}(x) + \frac{2n+3}{n+2} x P_{n+1}^{(1,-1)}(x)$$
$$P_0^{(1,-1)}(x) = 1$$
$$P_1^{(1,-1)}(x) = 1 + x$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity?

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity? $\quad\longrightarrow\quad$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\longrightarrow$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\longrightarrow$ By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.

$$\underbrace{\sum_{k=0}^{n} \frac{2k+1}{k+1}}_{\text{recurrence of order 1}} P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence} \\ \text{of order 1}}} \; \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence} \\ \text{of order 2}}}}_{\text{recurrence of order 2}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Big)}_{\text{recurrence of order 3}} = 0$$

$$\underbrace{\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Big)}_{\text{recurrence of order 3}}}_{\text{recurrence of order 7}} = 0$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\begin{aligned} \mathrm{lhs}_{n+7} = {} & (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+6} \\ & + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+5} \\ & + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+4} \\ & + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+3} \\ & + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+2} \\ & + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+1} \\ & + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n} \end{aligned}$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

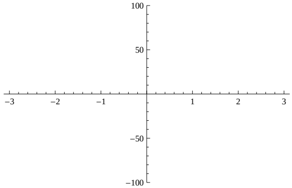
$$\begin{aligned} \mathrm{lhs}_{n+7} = (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n+6} \\ + (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n+5} \\ + (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n+4} \\ + (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n+3} \\ + (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n+2} \\ + (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n+1} \\ + (\cdots \textsf{messy} \cdots)\, &\mathrm{lhs}_{n} \end{aligned}$$

Therefore the identity holds *for all* $n \in \mathbb{N}$
if and only if it holds *for* $n = 0, 1, 2, \ldots, 6$.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \ \frac{1}{n!} \ t^n \ = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

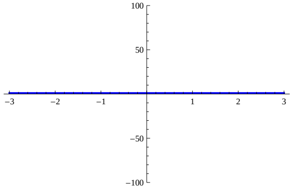
Hermite polynomials:



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

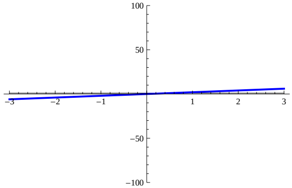
- $H_0(x) = 1$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

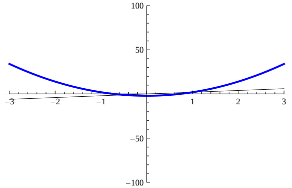
- $H_0(x) = 1$
- $H_1(x) = 2x$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

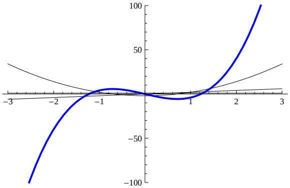
- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

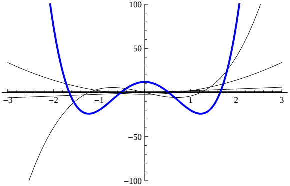
Hermite polynomials:

- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

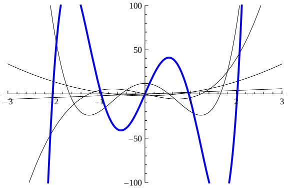
Hermite polynomials:



- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:



- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$
- $H_5(x) = 32x^5 - 160x^3 + 120x$
- $\dots$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

$$\begin{aligned} H_{n+2}(x) &= 2xH_{n+1}(x) - 2(n+1)H_n(x) \\ H_0(x) &= 1 \\ H_1(x) &= 2x \end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity among analytic functions.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity among analytic functions.

Consider x and y as fixed parameters.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity among analytic functions.

Consider x and y as fixed parameters.

Then both sides are functions in t.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\Big(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\Big)$$

This is an identity among analytic functions.

Consider x and y as fixed parameters.

Then both sides are functions in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\Big(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\Big) = 0$$

This is an identity among analytic functions.

Consider x and y as fixed parameters.

Then both sides are functions in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity among analytic functions.

Consider x and y as fixed parameters.

Then both sides are functions in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity among analytic functions.

Consider x and y as fixed parameters.

Then both sides are functions in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

Then the function is identically zero.

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \ \frac{1}{n!} \ t^n \ - \ \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)H_n(y)}_{\substack{\text{rec. of} \\ \text{ord. 2}}} \ \frac{1}{n!} \ t^n \ - \ \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}} \; \frac{1}{n!} \; t^n \; - \; \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}} \underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}} \; \frac{1}{n!} \; t^n \; - \; \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}} \underbrace{\frac{1}{n!}\; t^n}_{\text{rec. of ord. 1}} - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\;\underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}}\; t^n \;-\; \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \ \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}} \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}} \; t^n}_{\text{differential equation of order 5}} \; - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty}\underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\exp\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{diff.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{diff.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}} t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\left(\underbrace{\frac{4t(xy-t(x^2+y^2))}{1-4t^2}}_{\text{diff.eq. of order 1}}\right)}_{\text{differential equation of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\ \underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{diff.eq. of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}} = 0$$

$$\underbrace{\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}} \underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{diff.eq. of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 5}} = 0$$

$$\underbrace{\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}\, t^n}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}}_{\text{differential equation of order 5}} - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{diff.eq. of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 5}} = 0$$

$\rightsquigarrow$ recurrence equation of order 4

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\,t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} = \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3} + \tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2} \\ + \tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1} - \tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\, t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} = {} & \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3} + \tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2} \\ & + \tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1} - \tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

Because of $\mathrm{lhs}_0=\mathrm{lhs}_1=\mathrm{lhs}_2=\mathrm{lhs}_3=0$, we have $\mathrm{lhs}_n=0$ for all n.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\,t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} = {} & \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3} + \tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2} \\ & + \tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1} - \tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

Because of $\mathrm{lhs}_0=\mathrm{lhs}_1=\mathrm{lhs}_2=\mathrm{lhs}_3=0$, we have $\mathrm{lhs}_n=0$ for all n.

This completes the proof.

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\binom{n}{k} = 4^{-n}\binom{2n}{n}$$

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\binom{n}{k} \qquad = \qquad 4^{-n}\binom{2n}{n}$$

Problem: $\binom{n}{k}$ depends on two variables.

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\binom{n}{k} \qquad = \qquad 4^{-n}\binom{2n}{n}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\binom{n}{k} \qquad = \qquad 4^{-n}\binom{2n}{n}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{n!}{k!(n-k)!} \qquad = \qquad 4^{-n}\binom{2n}{n}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$n! \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} \qquad = \qquad 4^{-n} \binom{2n}{n}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!} \qquad = \qquad 4^{-n}\binom{2n}{n}\frac{1}{n!}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!} \qquad = \qquad 4^{-n}\binom{2n}{n}\frac{1}{n!}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{k=0}^{n} a_k b_{n-k}$$

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!} \qquad = \qquad 4^{-n}\binom{2n}{n}\frac{1}{n!}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{n=0}^{\infty}\Bigl(\sum_{k=0}^{n}a_k b_{n-k}\Bigr)x^n$$

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!} \qquad = \qquad 4^{-n}\binom{2n}{n}\frac{1}{n!}$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)x^n = \left(\sum_{n=0}^{\infty}a_n x^n\right)\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n = \sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n$$

Problem: $\binom{n}{k}$ depends on two variables.

The summation closure closure property is not directly applicable.

Trick: Switch to the function level!

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)x^n = \left(\sum_{n=0}^{\infty}a_n x^n\right)\left(\sum_{n=0}^{\infty}b_n x^n\right)$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n = \sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n-\sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n=0$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n}_{} - \sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n = 0$$

$$= \left(\sum_{n=0}^{\infty}\frac{(-4)^{-n}}{n!}\binom{2n}{n}x^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n}- \sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n = 0$$

$$= \left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n}_{} - \sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n = 0$$

$$\underbrace{= \left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n}-\sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n=0$$

$$=\underbrace{\left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n} - \sum_{n=0}^{\infty}4^{-n}\binom{2n}{n}\frac{1}{n!}x^n = 0$$

$$= \underbrace{\underbrace{\left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}}_{\text{differential equation of order 3}}$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n} - \sum_{n=0}^{\infty}\underbrace{4^{-n}\binom{2n}{n}\frac{1}{n!}}_{\text{rec. of order 1}}x^n = 0$$

$$= \underbrace{\underbrace{\left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}}_{\text{differential equation of order 3}}$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n}_{} - \underbrace{\sum_{n=0}^{\infty}\underbrace{4^{-n}\binom{2n}{n}\frac{1}{n!}}_{\text{rec. of order 1}}x^n}_{\text{diff.eq. of order 3}} = 0$$

$$= \underbrace{\underbrace{\left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}}_{\text{differential equation of order 3}}$$

$$
\underbrace{
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n - \underbrace{\sum_{n=0}^{\infty}\underbrace{4^{-n}\binom{2n}{n}\frac{1}{n!}}_{\text{rec. of order 1}}x^n}_{\text{diff.eq. of order 3}} = 0 \\
&= \underbrace{\underbrace{\left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}}_{\text{differential equation of order 3}}
\end{aligned}
}_{\text{differential equation of order 5}}
$$

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\,\frac{1}{(n-k)!}\right)x^n}\;-\;\underbrace{\sum_{n=0}^{\infty}\underbrace{4^{-n}\binom{2n}{n}\frac{1}{n!}}_{\text{rec. of order 1}}x^n}_{\text{diff.eq. of order 3}}=0$$

$$=\underbrace{\underbrace{\left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)}_{\text{differential equation of order 3}}\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}}_{\text{differential equation of order 3}}$$

differential equation of order 5

$\rightsquigarrow$ recurrence equation of order 7

$$\underbrace{\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\frac{1}{k!}\frac{1}{(n-k)!}\right)x^n}_{} - \underbrace{\sum_{n=0}^{\infty}\underbrace{4^{-n}\binom{2n}{n}\frac{1}{n!}}_{\text{rec. of order 1}}x^n}_{\text{diff.eq. of order 3}} = 0$$

$$= \left(\sum_{n=0}^{\infty}\underbrace{\frac{(-4)^{-n}}{n!}\binom{2n}{n}}_{\text{rec. of order 1}}x^n\right)\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right)}_{\text{diff.eq. of ord. 1}}$$

differential equation of order 3

differential equation of order 3

differential equation of order 5

$\rightsquigarrow$ recurrence equation of order 7

The identity is proved as soon as it is checked for the first 7 terms.

$$\sum_{k=0}^{n}(-4)^{-k}\binom{2k}{k}\binom{n}{k} = 4^{-n}\binom{2n}{n}$$

$$\sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \binom{n}{k} = 4^{-n} \binom{2n}{n}$$

- *Of course,* this particular example can be done easily with Zeilberger's algorithm.

$$\sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \binom{n}{k} = 4^{-n} \binom{2n}{n}$$

- *Of course,* this particular example can be done easily with Zeilberger's algorithm.
- *Of course,* the the holonomic machinary is more general than the hypergeometric one.

$$\sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \binom{n}{k} = 4^{-n} \binom{2n}{n}$$

- *Of course,* this particular example can be done easily with Zeilberger's algorithm.
- *Of course,* the the holonomic machinary is more general than the hypergeometric one.
- *Of course,* a good implementation will do the whole computation in one stroke.

Algorithms for executing closure properties are *rigorous.*

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

In practice, *experimental* results are as reliable as *rigorous* ones.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

In practice, *experimental* results are as reliable as *rigorous* ones.

Idea: In order to find a recurrence for $(a_n)_{n=0}^{\infty}$,

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

In practice, *experimental* results are as reliable as *rigorous* ones.

Idea: In order to find a recurrence for $(a_n)_{n=0}^{\infty}$,

- Compute a finite (but large) number N of sequence terms.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

In practice, *experimental* results are as reliable as *rigorous* ones.

Idea: In order to find a recurrence for $(a_n)_{n=0}^{\infty}$,

- Compute a finite (but large) number N of sequence terms.
- Compute recurrences which match $a_0, a_1, \ldots, a_N$.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

In practice, *experimental* results are as reliable as *rigorous* ones.

Idea: In order to find a recurrence for $(a_n)_{n=0}^{\infty}$,

- Compute a finite (but large) number N of sequence terms.
- Compute recurrences which match $a_0, a_1, \ldots, a_N$.
- Guess that these recurrences continue to hold for $n \geq N$.

Algorithms for executing closure properties are *rigorous.*

Their output constitutes a formal mathematical proof.

The prize is that the computations sometimes take long.

It can be faster to compute only *experimental* results.

Or to combine *experimental* computations with *rigorous* ones.

In practice, *experimental* results are as reliable as *rigorous* ones.

Idea: In order to find a recurrence for $(a_n)_{n=0}^{\infty}$,

- Compute a finite (but large) number N of sequence terms.
- Compute recurrences which match $a_0, a_1, \dots, a_N$.
- Guess that these recurrences continue to hold for $n \geq N$.
- If desired, prove this by an independent argument.

Example: What's next?

Example: What's next?

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \textbf{???}$$

Example: What's next?

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \textbf{???}$$

We have $(2+n)a_{n+1} - (4n+2)a_n = 0$ for $n = 0, \ldots, 7$

Example: What's next?

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \textbf{???}$$

We have $(2+n)a_{n+1} - (4n+2)a_n = 0$ for $n = 0, \ldots, 7$

A program can find this.

Example: What's next?

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \textbf{???}$$

We have $(2+n)a_{n+1} - (4n+2)a_n = 0$ for $n = 0, \ldots, 7$

A program can find this.

If the recurrence is also true for $n > 7$, then the next terms should be $4862, 16796, \ldots$

Example: What's next?

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \textbf{???}$$

We have $(2+n)a_{n+1} - (4n+2)a_n = 0$ for $n = 0, \ldots, 7$

A program can find this.

If the recurrence is also true for $n > 7$, then the next terms should be $4862, 16796, \ldots$

Whether the recurrence is also true for $n > 7$, this cannot be judged by looking at any finite amount of data.

Example: What's next?

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \textbf{???}$$

We have $(2+n)a_{n+1} - (4n+2)a_n = 0$ for $n = 0, \ldots, 7$

A program can find this.

If the recurrence is also true for $n > 7$, then the next terms should be $4862, 16796, \ldots$

Whether the recurrence is also true for $n > 7$, this cannot be judged by looking at any finite amount of data.

But the more data we check, the more "likely" it becomes.

Example: What's the recurrence for

$$\sum_{k=0}^{n}\left(\binom{3k}{k}\sum_{i=0}^{k}\binom{k}{i}^{10}\sum_{i=0}^{k} i^{10}\binom{k}{i}\right) \quad ?$$

Example: What's the recurrence for

$$\sum_{k=0}^{n}\left(\binom{3k}{k}\sum_{i=0}^{k}\binom{k}{i}^{10}\sum_{i=0}^{k}i^{10}\binom{k}{i}\right) \quad ?$$

- It is *clear* by closure properties that a recurrence exist.

Example: What's the recurrence for

$$\sum_{k=0}^{n}\left(\binom{3k}{k}\sum_{i=0}^{k}\binom{k}{i}^{10}\sum_{i=0}^{k}i^{10}\binom{k}{i}\right) \quad ?$$

- It is *clear* by closure properties that a recurrence exist.
- It might still be hard to actually compute it.

Example: What's the recurrence for

$$\sum_{k=0}^{n}\left(\binom{3k}{k}\sum_{i=0}^{k}\binom{k}{i}^{10}\sum_{i=0}^{k}i^{10}\binom{k}{i}\right) \quad ?$$

- It is *clear* by closure properties that a recurrence exist.
- It might still be hard to actually compute it.
- *Efficient shortcut:* Evaluate the sum for $n = 0, \ldots, 500$, say, and compute a recurrence from this data.

Example: What's the recurrence for

$$\sum_{k=0}^{n}\left(\binom{3k}{k}\sum_{i=0}^{k}\binom{k}{i}^{10}\sum_{i=0}^{k}i^{10}\binom{k}{i}\right) \quad ?$$

- It is *clear* by closure properties that a recurrence exist.
- It might still be hard to actually compute it.
- *Efficient shortcut:* Evaluate the sum for $n = 0, \ldots, 500$, say, and compute a recurrence from this data.
- Result (with high probability): A recurrence of order 6 with polynomial coefficients of degree 94.

Summary

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.
- Many questions about holonomic functions can be answered computationally (rigorously or not).

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.
- Many questions about holonomic functions can be answered computationally (rigorously or not).
- Software packages for Maple and Mathematical are available for these tasks.

Algorithms for Holonomic Functions

Manuel Kauers

Research Institute for Symbolic Computation
Johannes Kepler University
Austria

Recall: The Case of One Variable

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.
- Many questions about holonomic functions can be answered computationally (rigorously or not).

- Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
- Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.
- Many questions about holonomic functions can be answered computationally (rigorously or not).
- Software packages for Maple and Mathematical are available for these tasks.

Holonomy: The Case of Several Variables

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

- $\exp(x - y)$: 2 continuous and 0 discrete variables.

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

- $\exp(x - y)$: 2 continuous and 0 discrete variables.
- $\binom{n}{k}$: 0 continuous and 2 discrete variables.

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

- $\exp(x - y)$: 2 continuous and 0 discrete variables.
- $\binom{n}{k}$: 0 continuous and 2 discrete variables.
- $P_n(x)$ 1 continuous and 1 discrete variable.

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

We want to *differentiate* the x_i and to *shift* the n_j:

$$\frac{\partial^5}{\partial x^5} \frac{\partial^3}{\partial y^3} f(x, y, n + 4, k + 23)$$

We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are continuous variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

We want to *differentiate* the x_i and to *shift* the n_j:

$$\frac{\partial^5}{\partial x^5} \frac{\partial^3}{\partial y^3} f(x, y, n+4, k+23)$$

Compact notation:

$$D_x^5 D_y^3 S_n^4 S_k^{23} f$$

Definition. A function $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called holonomic, if

Definition. A function $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called holonomic, if

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

$$p_0 f + p_1 D_{x_k} f + p_2 D_{x_k}^2 f + \cdots + p_r D_{x_k}^r f = 0.$$

Definition. A function $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called holonomic, if

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

$$p_0 f + p_1 D_{x_k} f + p_2 D_{x_k}^2 f + \cdots + p_r D_{x_k}^r f = 0.$$

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

$$p_0 f + p_1 S_{n_k} f + p_2 S_{n_k}^2 f + \cdots + p_r S_{n_k}^r f = 0.$$

Definition. A function $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called holonomic, if

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

$$p_0 f + p_1 D_{x_k} f + p_2 D_{x_k}^2 f + \cdots + p_r D_{x_k}^r f = 0.$$

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

$$p_0 f + p_1 S_{n_k} f + p_2 S_{n_k}^2 f + \cdots + p_r S_{n_k}^r f = 0.$$

Warning! This is just a somewhat oversimplified approximation to the official definition

Examples.

Examples.

- $f(x, y) = \exp(x - y)$ is holonomic because

$$D_x f - f = 0 \quad \text{and} \quad D_y f + f = 0.$$

Examples.

- $f(x,y) = \exp(x-y)$ is holonomic because

$$D_x f - f = 0 \quad \text{and} \quad D_y f + f = 0.$$

- $f(n,k) = \binom{n}{k}$ is holonomic because

$$(1-k+n)S_n f - (n+1)f = 0 \quad \text{and} \quad (k+1)S_k f + (k-n)f = 0.$$

Examples.

- $f(x,y) = \exp(x-y)$ is holonomic because

$$D_x f - f = 0 \quad \text{and} \quad D_y f + f = 0.$$

- $f(n,k) = \binom{n}{k}$ is holonomic because

$$(1-k+n)S_n f - (n+1)f = 0 \quad \text{and} \quad (k+1)S_k f + (k-n)f = 0.$$

- $f(x,n) = P_n(x)$ is holonomic because

$$(x^2-1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and}$$
$$(n+2)S_n^2 f - (2nx-3x)S_n f + (n+1)f = 0$$

Counterexamples.

Counterexamples.

- $f(x, n) = \sqrt{x+n}$ is not holonomic.

Counterexamples.

- $f(x, n) = \sqrt{x + n}$ is not holonomic.

  It satisfies a differential equation in x, but no recurrence in n.

Counterexamples.

- $f(x,n) = \sqrt{x+n}$ is not holonomic.

  It satisfies a differential equation in x, but no recurrence in n.

- $f(x,n) = \binom{x}{n}$ is not holonomic.

Counterexamples.

- $f(x, n) = \sqrt{x + n}$ is not holonomic.

  It satisfies a differential equation in x, but no recurrence in n.
- $f(x, n) = \binom{x}{n}$ is not holonomic.

  It satisfies a recurrence in n, but no differential equation in x.

Counterexamples.

- $f(x, n) = \sqrt{x + n}$ is not holonomic.

  It satisfies a differential equation in x, but no recurrence in n.
- $f(x, n) = \binom{x}{n}$ is not holonomic.

  It satisfies a recurrence in n, but no differential equation in x.
- $f(n, k) = S_1(n, k)$ [Stirling numbers] is not holonomic.

Counterexamples.

- $f(x,n) = \sqrt{x+n}$ is not holonomic.

  It satisfies a differential equation in x, but no recurrence in n.

- $f(x,n) = \binom{x}{n}$ is not holonomic.

  It satisfies a recurrence in n, but no differential equation in x.

- $f(n,k) = S_1(n,k)$ [Stirling numbers] is not holonomic.

  It satisfies the recurrence

$$S_n S_k f + n S_n f - f = 0,$$

  but no "pure" recurrence in S_k or S_n.

Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations

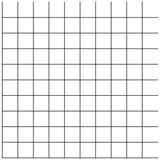
$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations

$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

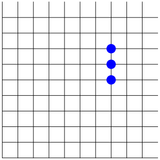


Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations

$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

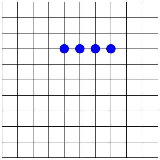


Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations

$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

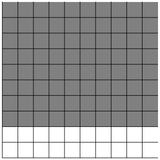


Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations

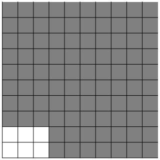
$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$



Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations

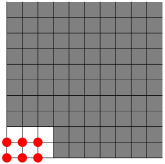


$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations



$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

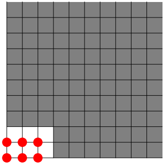
The solution is uniquely determined by

$$f(0,0), f(1,0), f(2,0), f(1,0), f(1,1), f(2,1).$$

Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.

Example.

- Consider the equations



$$(\dots)S_n^2 f + (\dots)S_n f + (\dots)f = 0$$
$$(\dots)S_k^3 f + (\dots)S_k^2 f + (\dots)S_k f + (\dots)f = 0$$

  The solution is uniquely determined by

$$f(0,0), f(1,0), f(2,0), f(1,0), f(1,1), f(2,1).$$

Simiarly for differential equations and for systems containing mixed equations.

Holonomy requires for every variable a pure equation.

Holonomy requires for every variable a pure equation.

But if there are mixed equations *in addition,* they are welcome.

Holonomy requires for every variable a pure equation.

But if there are mixed equations *in addition,* they are welcome.

Example.

- $f(x, n) = P_n(x)$ satisfies

$$(x^2 - 1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and}$$
$$(n+2)S_n^2 f - (2nx - 3x)S_n f + (n+1)f = 0$$

Holonomy requires for every variable a pure equation.

But if there are mixed equations *in addition,* they are welcome.

Example.

- $f(x, n) = P_n(x)$ satisfies

$$(x^2 - 1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and}$$
$$(n+2)S_n^2 f - (2nx - 3x)S_n f + (n+1)f = 0 \quad \text{and}$$
$$(x^2 - 1)D_x f - (n+1)S_n f + (n+1)xf = 0.$$

Holonomy requires for every variable a pure equation.

But if there are mixed equations *in addition,* they are welcome.

Example.

- $f(x,n) = P_n(x)$ satisfies

$$(x^2-1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and}$$
$$(n+2)S_n^2 f - (2nx-3x)S_n f + (n+1)f = 0 \quad \text{and}$$
$$(x^2-1)D_x f - (n+1)S_n f + (n+1)xf = 0.$$

In this case, any two equations imply the other.

Holonomy requires for every variable a pure equation.

But if there are mixed equations *in addition,* they are welcome.

Example.

- $f(x,n) = P_n(x)$ satisfies

$$(x^2-1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and}$$
$$(n+2)S_n^2 f - (2nx-3x)S_n f + (n+1)f = 0 \quad \text{and}$$
$$(x^2-1)D_x f - (n+1)S_n f + (n+1)xf = 0.$$

  In this case, any two equations imply the other.

In general, mixed equations may contain additional information.

Holonomy requires for every variable a pure equation.

But if there are mixed equations *in addition,* they are welcome.

Example.

- $f(x, n) = P_n(x)$ satisfies

$$(x^2 - 1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and}$$
$$(n+2)S_n^2 f - (2nx - 3x)S_n f + (n+1)f = 0 \quad \text{and}$$
$$(x^2 - 1)D_x f - (n+1)S_n f + (n+1)xf = 0.$$

  In this case, any two equations imply the other.

In general, mixed equations may contain additional information.

A system of equations is called *holonomic* if it implies for every variable a pure equation.

Have:

Have:

- Finite data structure for representing holonomic objects

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Want:

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Want:

- Structural properties of the class of holonomic objects

Have:

- Finite data structure for representing holonomic objects
- Coverage of many important examples

Want:

- Structural properties of the class of holonomic objects
- Algorithms for doing explicit computations with them

Theorem (closure properties). Let f and g be holonomic functions. Then:

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic
- $D_x f$ is holonomic for every continuous variable x

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic
- $D_x f$ is holonomic for every continuous variable x
- $\int_x f$ is holonomic for every continuous variable x

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic
- $D_x f$ is holonomic for every continuous variable x
- $\int_x f$ is holonomic for every continuous variable x
- $S_n f$ is holonomic for every discrete variable n

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic
- $D_x f$ is holonomic for every continuous variable x
- $\int_x f$ is holonomic for every continuous variable x
- $S_n f$ is holonomic for every discrete variable n
- $\sum_{k=0}^{n} f(\dots, k, \dots)$ is holonomic for every discrete variable n

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic
- $D_x f$ is holonomic for every continuous variable x
- $\int_x f$ is holonomic for every continuous variable x
- $S_n f$ is holonomic for every discrete variable n
- $\sum_{k=0}^{n} f(\ldots, k, \ldots)$ is holonomic for every discrete variable n
- If $h_1, \ldots, h_p$ are algebraic functions in $x_1, \ldots, x_p$, free of $n_1, \ldots, n_q$, then $f(h_1, \ldots, h_p, n_1, \ldots, n_q)$ is holonomic.

Theorem (closure properties). Let f and g be holonomic functions. Then:

- $f + g$ is holonomic
- fg is holonomic
- $D_x f$ is holonomic for every continuous variable x
- $\int_x f$ is holonomic for every continuous variable x
- $S_n f$ is holonomic for every discrete variable n
- $\sum_{k=0}^{n} f(\dots, k, \dots)$ is holonomic for every discrete variable n
- If $h_1, \dots, h_p$ are algebraic functions in $x_1, \dots, x_p$, free of $n_1, \dots, n_q$, then $f(h_1, \dots, h_p, n_1, \dots, n_q)$ is holonomic.
- If $h_1, \dots, h_q$ are integer-linear functions in $n_1, \dots, n_q$, free of $x_1, \dots, x_p$, then $f(x_1, \dots, x_p, h_1, \dots, h_q)$ is holonomic.

The theorem is algorithmic:

The theorem is algorithmic:

- Holonomic systems for all these functions can be computed from given holonomic systems of f and g.

The theorem is algorithmic:

- Holonomic systems for all these functions can be computed from given holonomic systems of f and g.

There is software available for this.

The theorem is algorithmic:

- Holonomic systems for all these functions can be computed from given holonomic systems of f and g.

There is software available for this.

- For Maple: `mgfun` by Chyzak, distributed together with Maple.

The theorem is algorithmic:

- Holonomic systems for all these functions can be computed from given holonomic systems of f and g.

There is software available for this.

- For Maple: `mgfun` by Chyzak, distributed together with Maple.
- For Mathematica: `HolonomicFunctions.m` by Koutschan, available from the RISC combinatorics software website.

Example.

Example.

- $f(x, n) = n! x^n \exp(x) P_{2n+3}(\sqrt{1 - x^2})$

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= **<< HolonomicFunctions.m**

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= **<< HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= $<<$ **HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz,
Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

In[2]:= $\mathbf{Annihilator[\boldsymbol{n!x^n}Exp[\boldsymbol{x}]LegendreP[2\boldsymbol{n}+3, Sqrt[1-\boldsymbol{x}^2]],}$
$\mathbf{\{Der[\boldsymbol{x}], \boldsymbol{S}[\boldsymbol{n}]\}]}$

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= **<< HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

In[2]:= **Annihilator[$n!x^n$Exp[x]LegendreP[$2n+3$, Sqrt[$1-x^2$]], {Der[x], S[n]}]**

Out[2]= $$\Big\{(-9x^2 - \ldots)D_x + (4n^2 + \ldots)S_n + (13nx^4 + \ldots),$$
$$(16n^3 + \cdots)S_n^2 + (64n^4x^3 + \ldots)S_n + (16n^5x^2 + \cdots)\Big\}$$

Example.

- $f(n,k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}$

Example.

- $f(n,k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}$

In[3]:= $\mathbf{Annihilator[Binomial[}\boldsymbol{n, k}\mathbf{] + Sum[1/}\boldsymbol{k}\mathbf{!, \{}\boldsymbol{k}\mathbf{, 0, }\boldsymbol{n}\mathbf{\}], \{}\boldsymbol{S}\mathbf{[}\boldsymbol{n}\mathbf{], }\boldsymbol{S}\mathbf{[}\boldsymbol{k}\mathbf{]\}]}$

Example.

- $f(n,k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}$

In[3]:= **Annihilator[Binomial[n, k] + Sum[1/k!, {k, 0, n}], {S[n], S[k]}]**

Out[3]= $\Big\{(2k^2 + \dots)S_k^2 + (n^2 + \cdots)S_k + (3kn + \cdots),$
$(n^2 + \cdots)S_n S_k + (3kn + \cdots)S_n + (2kn + \cdots)S_k + (n^2 + \cdots),$
$(4kn^3 + \cdots)S_n^2 + (n^4 + \cdots)S_n + (k^2 n^2 + \cdots)S_k - (n^3 + \cdots)\Big\}$

Example.

- $f(x,n) = \int_0^x P_n(t)dt$

Example.

- $f(x,n) = \int_0^x P_n(t)dt$

In[4]:= **Annihilator[Integrate[LegendreP[*n*, *t*], {*t*, 0, *x*}], {Der[*x*], *S*[*n*]}]**

Example.

- $f(x,n) = \int_0^x P_n(t)dt$

In[4]:= **Annihilator[Integrate[LegendreP[*n*, *t*], {*t*, 0, *x*}], {Der[*x*], *S*[*n*]}]**

Out[4]= $\Big\{(2n^5x^2 + \cdots)S_n^3 + \cdots\cdots, \quad (2n^3x^2 + \cdots)D_x S_n + \cdots\cdots,$
$(2n^2x^5 + \cdots)D_x^2 S_n + \cdots\cdots, \quad (nx^7 + \cdots)D_x^3 + \cdots\cdots\Big\}$

Low-level commands for executing closure properties "by hand":

Low-level commands for executing closure properties “by hand”:

- **DFinitePlus**

Low-level commands for executing closure properties “by hand”:

- **DFinitePlus**
- **DFiniteTimes**

Low-level commands for executing closure properties “by hand”:

- **DFinitePlus**
- **DFiniteTimes**
- **DFiniteSubstitute**

Low-level commands for executing closure properties “by hand”:

- **DFinitePlus**
- **DFiniteTimes**
- **DFiniteSubstitute**
- **DFiniteOreAction**

Low-level commands for executing closure properties “by hand”:

- **DFinitePlus**
- **DFiniteTimes**
- **DFiniteSubstitute**
- **DFiniteOreAction**
- **DFiniteDE2RE**

Low-level commands for executing closure properties "by hand":

- **DFinitePlus**
- **DFiniteTimes**
- **DFiniteSubstitute**
- **DFiniteOreAction**
- **DFiniteDE2RE**
- **DFiniteRE2DE**

Low-level commands for executing closure properties “by hand”:

- **DFinitePlus**
- **DFiniteTimes**
- **DFiniteSubstitute**
- **DFiniteOreAction**
- **DFiniteDE2RE**
- **DFiniteRE2DE**

Use this commands for functions whose definition is not known to **Annihilator** or for expressions where the **Annihilator** command takes a long time.

Example.

- $P_n(x) + x^n \exp(x)$

Example.

- $P_n(x) + x^n \exp(x)$

In[5]:= $\boldsymbol{annP = \text{OreGroebnerBasis}[\{(x^2-1)\text{Der}[x] - (n+1)S[n] + (x+nx), (n+2)S[n]^2 - (2nx+3x)S[n] + (n+1)\}, \text{OreAlgebra}[\text{Der}[x], S[n]]];}$

Example.

- $P_n(x) + x^n \exp(x)$

In[5]:= $\boldsymbol{annP = \mathbf{OreGroebnerBasis}[\{(x^2-1)\mathbf{Der}[x] - (n+1)S[n] + (x+nx), (n+2)S[n]^2 - (2nx+3x)S[n] + (n+1)\}, \mathbf{OreAlgebra}[\mathbf{Der}[x], S[n]]];}$

In[6]:= $\boldsymbol{annE = \mathbf{OreGroebnerBasis}[\{x\mathbf{Der}[x] - (n+x), S[n] - x\}, \mathbf{OreAlgebra}[\mathbf{Der}[x], S[n]]];}$

Example.

- $P_n(x) + x^n \exp(x)$

In[5]:= $\boldsymbol{annP} = \mathbf{OreGroebnerBasis}[\{(\boldsymbol{x}^2 - 1)\mathbf{Der}[\boldsymbol{x}] - (\boldsymbol{n} + 1)\boldsymbol{S}[\boldsymbol{n}]$
$\quad + (\boldsymbol{x} + \boldsymbol{nx}), (\boldsymbol{n} + 2)\boldsymbol{S}[\boldsymbol{n}]^2 - (2\boldsymbol{nx} + 3\boldsymbol{x})\boldsymbol{S}[\boldsymbol{n}] + (\boldsymbol{n} + 1)\},$
$\quad \mathbf{OreAlgebra}[\mathbf{Der}[\boldsymbol{x}], \boldsymbol{S}[\boldsymbol{n}]]];$

In[6]:= $\boldsymbol{annE} = \mathbf{OreGroebnerBasis}[\{\boldsymbol{x}\mathbf{Der}[\boldsymbol{x}] - (\boldsymbol{n} + \boldsymbol{x}),$
$\quad \boldsymbol{S}[\boldsymbol{n}] - \boldsymbol{x}\}, \mathbf{OreAlgebra}[\mathbf{Der}[\boldsymbol{x}], \boldsymbol{S}[\boldsymbol{n}]]];$

In[7]:= $\mathbf{DFinitePlus}[\boldsymbol{annP}, \boldsymbol{annE}]$

Example.

- $P_n(x) + x^n \exp(x)$

In[5]:= $\boldsymbol{annP = \mathbf{OreGroebnerBasis}[\{(x^2-1)\mathbf{Der}[x]-(n+1)S[n] + (x+nx), (n+2)S[n]^2-(2nx+3x)S[n]+(n+1)\}, \mathbf{OreAlgebra}[\mathbf{Der}[x], S[n]]];}$

In[6]:= $\boldsymbol{annE = \mathbf{OreGroebnerBasis}[\{x\mathbf{Der}[x]-(n+x), S[n]-x\}, \mathbf{OreAlgebra}[\mathbf{Der}[x], S[n]]];}$

In[7]:= $\boldsymbol{\mathbf{DFinitePlus}[annP, annE]}$

Out[7]= $\{D_x(nx^3-nx+x^3-x)+S_n(-3n^2x-2nx^2-5nx-3x^2-x)+S_n^2(n^2+nx+2n+2x)+n^2x^2+n^2+2nx^2+nx+n+x^2+x, D_xS_n(nx^2-n+x^3-x)+(x^2-x^4)D_x+S_n(n^2(-x)-nx)+n^2-nx^3+nx+n-x^3+x, D_x(n^2x^2-n^2-2nx^5+2nx^4+4nx^3-3nx^2-2nx+n-x^6+2x^4-x^2)+D_x^2(nx^5-2nx^3+nx+x^6-2x^4+x^2)-n^3x^3+2n^3x-3n^2x^4-n^2x^3+3n^2x^2+n^2x+S_n(-n^3+2n^2x^3-2n^2x+nx^4+4nx^3-nx^2-2nx+n+x^4+2x^3-x^2)-nx^5-5nx^4+nx^3+3nx^2-nx-x^5-2x^4+x^3\}$

Theorem (Summation/Integration).

Theorem (Summation/Integration).

- If f is holonomic, then so is

$$\int_{-\infty}^{\infty} f(t, x_2, \ldots, x_p, n_1, \ldots, n_q)dt,$$

provided that this integral exists.

Theorem (Summation/Integration).

- If f is holonomic, then so is

$$\int_{-\infty}^{\infty} f(t, x_2, \ldots, x_p, n_1, \ldots, n_q)dt,$$

provided that this integral exists.

- If f is holonomic, then so is

$$\sum_{k=-\infty}^{\infty} f(x_1, \ldots, x_p, k, n_2, \ldots, n_q),$$

provided that this sum exists.

Theorem (Summation/Integration).

- If f is holonomic, then so is

$$\int_{-\infty}^{\infty} f(t, x_2, \ldots, x_p, n_1, \ldots, n_q)dt,$$

provided that this integral exists.

- If f is holonomic, then so is

$$\sum_{k=-\infty}^{\infty} f(x_1, \ldots, x_p, k, n_2, \ldots, n_q),$$

provided that this sum exists.

Warning! Strictly speaking, this item only holds for the official definition of holonomic.

Note the difference between indefinite and definite summation and integration:

Note the difference between indefinite and definite summation and integration:

Indefinite:

Definite:

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Definite:

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k, m).$$

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Sum and summand have the same number of variables.

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k, m).$$

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Sum and summand have the same number of variables.

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k, m).$$

The sum has one variable less than the summand.

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Sum and summand have the same number of variables.

$$\Downarrow$$

easy

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k, m).$$

The sum has one variable less than the summand.

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Sum and summand have the same number of variables.

$$\Downarrow$$

easy

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k, m).$$

The sum has one variable less than the summand.

$$\Downarrow$$

hard

Note the difference between indefinite and definite summation and integration:

Indefinite:

$$g(x, y) = \int_0^x f(t, y)\, dt.$$

Sum and summand have the same number of variables.

$$\Downarrow$$

easy

Definite:

$$g(y) = \int_{-\infty}^{\infty} f(t, y)\, dt.$$

The sum has one variable less than the summand.

$$\Downarrow$$

hard

The situation for integration is fully analogous.

Examples.

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies

$$(n+2)S_n^2 f - (10n+15)S_n f + (9n+9)f = 0.$$

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies

$$(n+2)S_n^2 f - (10n+15)S_n f + (9n+9)f = 0.$$

- $f(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2) dt$ satisfies

$$16x^2 D_x^3 f + (16x^2 + 96x) D_x^2 f + (72x + 99) D_x f + 48 f = 0.$$

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies

$$(n+2)S_n^2 f - (10n+15)S_n f + (9n+9)f = 0.$$

- $f(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2) dt$ satisfies

$$16x^2 D_x^3 f + (16x^2 + 96x) D_x^2 f + (72x + 99) D_x f + 48 f = 0.$$

- $f(x,t) = \sum_{n=0}^{\infty} P_n(t) x^n$ satisfies

$(x^2 - 2tx + 1) D_t f - x f = 0$ and $(x^2 - 2tx + 1) D_x f + (x - t) f = 0.$

Examples.

- $f(n) = \int_0^1 \int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{\epsilon/2} z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}} (1-w^{n+1}-(1-w)^{n+1}) dw\, dz$ satisfies

$$\begin{aligned}(8\epsilon n^7 + \cdots)S_n^3 f - (24\epsilon n^7 + \cdots)S_n^2 f \\ - (24\epsilon n^7 + \cdots)S_n f + (8\epsilon n^7 + \cdots)f = 0.\end{aligned}$$

Examples.

- $f(n) = \int_0^1 \int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{\epsilon/2} z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}} (1-w^{n+1}-(1-w)^{n+1}) dw\, dz$ satisfies

$$(8\epsilon n^7 + \cdots)S_n^3 f - (24\epsilon n^7 + \cdots)S_n^2 f \\ - (24\epsilon n^7 + \cdots)S_n f + (8\epsilon n^7 + \cdots) f = 0.$$

- $f(x) = \displaystyle\int_0^1 t^2(1-t)^2 {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| xt\right) dt$ satisfies

$$x^2(x-1)D_x^3 f + (\dots)D_x^2 f + (\dots)D_x f + 3abf = 0.$$

How does this work?

How does this work?

Basic principle: Assume we have $f(x, 0) = f(x, 1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x, y)dy$.

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

Suppose f satisfies an equation of the form

$$a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y\big(h(x,y)f\big)$$

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

Suppose f satisfies an equation of the form

$$a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y\big(h(x,y)f\big)$$

Then integrating both sides gives

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

Suppose f satisfies an equation of the form

$$a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y\big(h(x,y)f\big)$$

Then integrating both sides gives

$$a(x)F(x) + b(x)D_x F(x) + c(x)D_x^2 F(x) = 0$$

Example.

- $f(t, x) = t^2\sqrt{t+1}\exp(-xt^2)$.

Example.

- $f(t, x) = t^2 \sqrt{t+1} \exp(-xt^2)$.

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\,\exp(-xt^2)$.

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$\Longrightarrow$

Example.

- $f(t,x) = t^2\sqrt{t+1}\,\exp(-xt^2)$.

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$$\Longrightarrow 16x^2 D_x^3 f + (16x^2 + 96x)D_x^2 f + (72x + 99)D_x f + 48f$$
$$= D_t\big(-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$.

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$$\Longrightarrow \overbrace{16x^2D_x^3 f + (16x^2 + 96x)D_x^2 f + (72x + 99)D_x f + 48f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\big(-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\,\exp(-xt^2)$.

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$$\Longrightarrow \overbrace{16x^2 D_x^3 f + (16x^2 + 96x)D_x^2 f + (72x + 99)D_x f + 48f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{“Certificate”}}f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\,\exp(-xt^2)$. $F(x) = \int_0^\infty f(x,t)dt$

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$$\Longrightarrow \overbrace{16x^2D_x^3f + (16x^2+96x)D_x^2f + (72x+99)D_xf + 48f}^{\text{"Telescoper": free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{"Certificate"}}f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$. $F(x) = \int_0^\infty f(x,t)dt$

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$$\implies \overbrace{16x^2 D_x^3 f + (16x^2 + 96x)D_x^2 f + (72x + 99)D_x f + 48f}^{\text{"Telescoper": free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{"Certificate"}}f\big)$$

$\implies$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$. $F(x) = \int_0^\infty f(x,t)dt$

$$2t(t+1)D_t f + (4t^3x + 4t^2x - 5t - 4)f = 0,$$
$$D_x f + t^2 f = 0.$$

$$\Longrightarrow \overbrace{16x^2 D_x^3 f + (16x^2 + 96x)D_x^2 f + (72x+99)D_x f + 48f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\Big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{“Certificate”}} f\Big)$$

$$\Longrightarrow 16x^2 D_x^3 F + (16x^2 + 96x)D_x^2 F + (72x+99)D_x F + 48F = 0$$

How to construct a creative telescoping relation?

How to construct a creative telescoping relation?

There are algorithms for this task.

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra
- Chyzak's algorithm (generalizing Zeilberger's algorithm)

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra
- Chyzak's algorithm (generalizing Zeilberger's algorithm)
- Takayama's algorithm

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra
- Chyzak's algorithm (generalizing Zeilberger's algorithm)
- Takayama's algorithm

Depending on the problem at hand, any of these algorithms may be much more efficient than the others.

Koutschan's package provides the command **FindCreativeTelescoping**.

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2)$

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2)$

```
In[1]:= << HolonomicFunctions.m
```

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2)$

In[1]:= << **HolonomicFunctions.m**
HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2)$

In[1]:= **<< HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

In[2]:= $\mathbf{FindCreativeTelescoping}[\boldsymbol{t}^{\mathbf{2}}\mathbf{Sqrt}[\boldsymbol{t}+\mathbf{1}]\mathbf{Exp}[-\boldsymbol{x}\boldsymbol{t}^{\mathbf{2}}], \{\mathbf{Der}[\boldsymbol{t}]\}, \{\mathbf{Der}[\boldsymbol{x}]\}]$

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2)$

In[1]:= $<<$ **HolonomicFunctions.m**
HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

In[2]:= **FindCreativeTelescoping[t^2Sqrt[$t+1$]Exp[$-xt^2$], {Der[t]}, {Der[x]}]**

Out[2]= $$\Big\{\{16x^2 D_x^3 + (16x^2 + 96x) D_x^2 + (72x + 99) D_x f + 48\}, \{\{-2(4t^5 x - 4t^3 x - 9t^3 - t^2 + 8t)\}\}\Big\}$$

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$

In[3]:= $\mathbf{FindCreativeTelescoping[LegendreP[}\boldsymbol{n, x}\mathbf{]}\boldsymbol{t^n}\mathbf{, \{}\boldsymbol{S}\mathbf{[}\boldsymbol{n}\mathbf{] - 1\},}$
$\mathbf{\{Der[}\boldsymbol{x}\mathbf{], Der[}\boldsymbol{t}\mathbf{]\}]}$

Koutschan's package provides the command **FindCreativeTelescoping**.

Examples

- $F(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$

In[3]:= $\mathbf{FindCreativeTelescoping[LegendreP}[n,x]t^n, \{S[n]-1\}, \{\mathbf{Der}[x], \mathbf{Der}[t]\}\mathbf{]}$

Out[3]= $\Big\{\{(1+t^2-2tx)D_t+(t-x),\ (-1-t^2+2tx)D_x+t\},$
$\{\{(-1+x^2)D_x-\frac{n(tx-1)}{t}\},\{(-1+tx)D_x-nt\}\}\Big\}$

Summary

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.
- A holonomic system is one which implies a pure relation for every variable.

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.
- A holonomic system is one which implies a pure relation for every variable.
- Holonomic system plus initial values characterize a holonomic function uniquely by a finite amount of data.

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.
- A holonomic system is one which implies a pure relation for every variable.
- Holonomic system plus initial values characterize a holonomic function uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.
- A holonomic system is one which implies a pure relation for every variable.
- Holonomic system plus initial values characterize a holonomic function uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.
- A holonomic system is one which implies a pure relation for every variable.
- Holonomic system plus initial values characterize a holonomic function uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.
- In particular, summation and integration preserves holonomy.

- Holonomic means to satisfy a holonomic system of linear differential/recurrence equations with polynomial coefficients.
- A holonomic system is one which implies a pure relation for every variable.
- Holonomic system plus initial values characterize a holonomic function uniquely by a finite amount of data.
- Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.
- Many more can be composed out of known ones by applying holonomic closure properties.
- In particular, summation and integration preserves holonomy.
- Software packages for Maple and Mathematical are available for computing with holonomic functions.