Restricted Lattice Walks and Creative Telescoping

Manuel Kauers
RISC

joint work with

A. Bostan, F. Chyzak, L. Pech, M. van Hoeij (part 1)
and S. Chen (part 2)

Symbolic Combinatorics

Symbolic Combinatorics

Symbolic

Combinatorics

Symbolic Combinatorics

Symbolic

Enumerative Combinatorics

Symbolic Combinatorics

Symbolic Computation

Enumerative Combinatorics

$$\begin{aligned} & \textit{Symbolic Combinatorics} \\ = {} & \textit{Symbolic Computation} \\ \cup {} & \textit{Enumerative Combinatorics} \end{aligned}$$

$$
\begin{aligned}
&\textit{Symbolic Combinatorics}\\
&= \textit{Symbolic Computation}\\
&\cap \textit{Enumerative Combinatorics}
\end{aligned}
$$

$$\begin{aligned} &\textit{Symbolic Combinatorics} \\ &= \textit{Symbolic Computation} \\ &+ \textit{Enumerative Combinatorics} \end{aligned}$$

Symbolic Combinatorics

$= $ *Symbolic Computation*

$+ $ *Enumerative Combinatorics*

In this talk:

Symbolic Combinatorics

= *Symbolic Computation*

\+ *Enumerative Combinatorics*

In this talk:

- Lattice Walk Counting

Symbolic Combinatorics

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- Creative Telescoping $\in$ Symbolic Computation
- And what one has to do with the other

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- Creative Telescoping $\in$ Symbolic Computation
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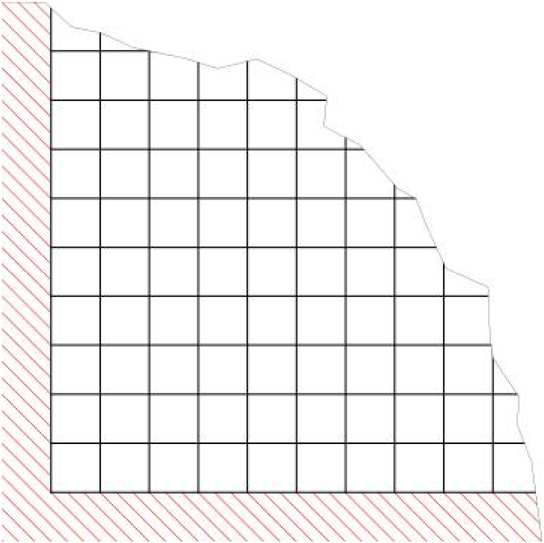
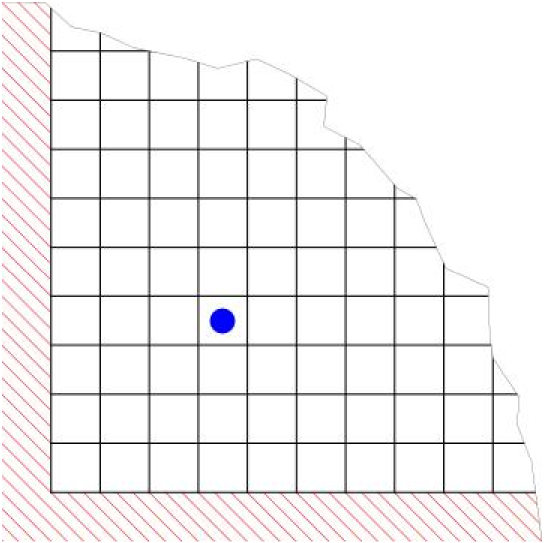
In this session:

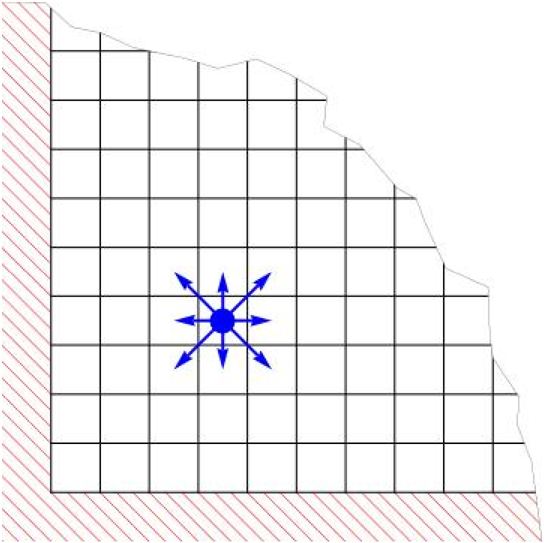
- Hopefully many other stories on how symbolic computation and enumerative combinatorics fertilize each other.

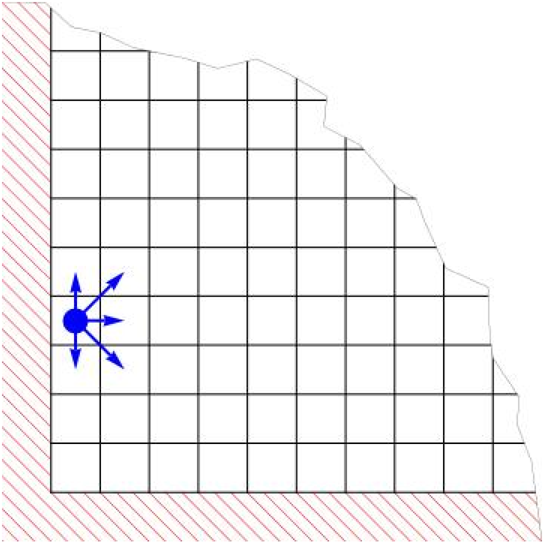
1. The Combinatorics Part.

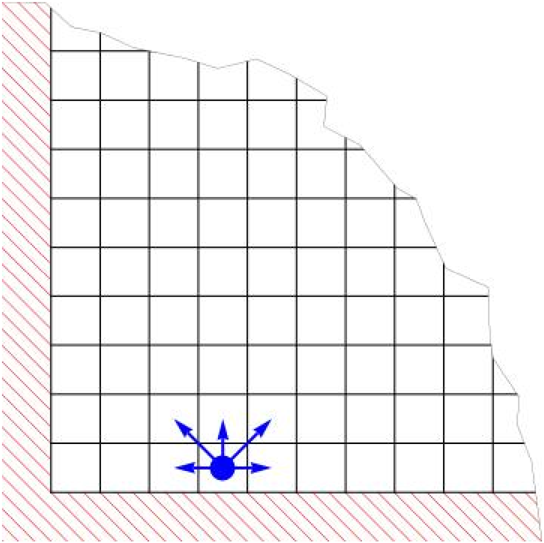
Enumeration of Restricted Lattice Walks

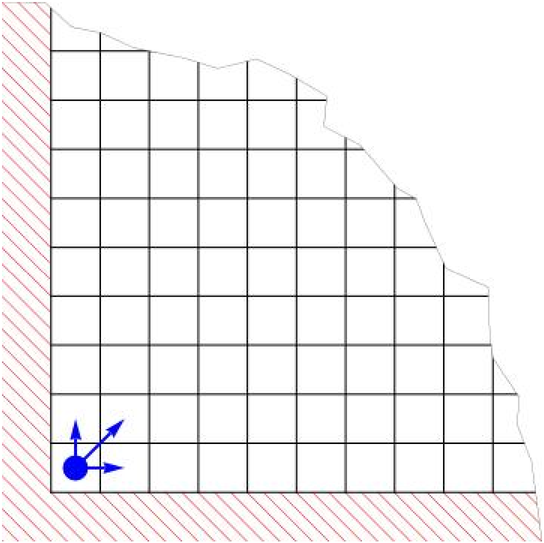


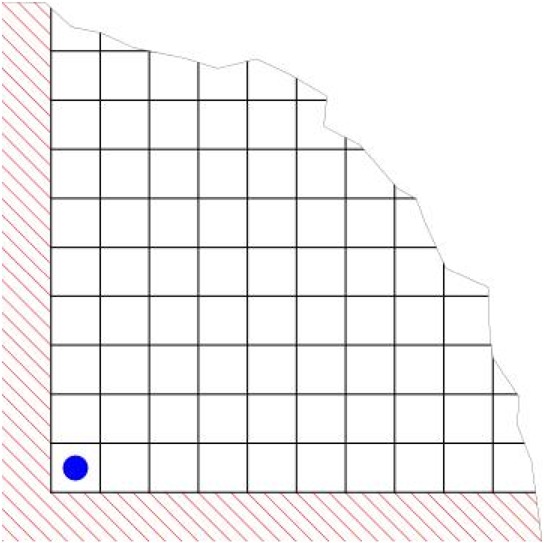


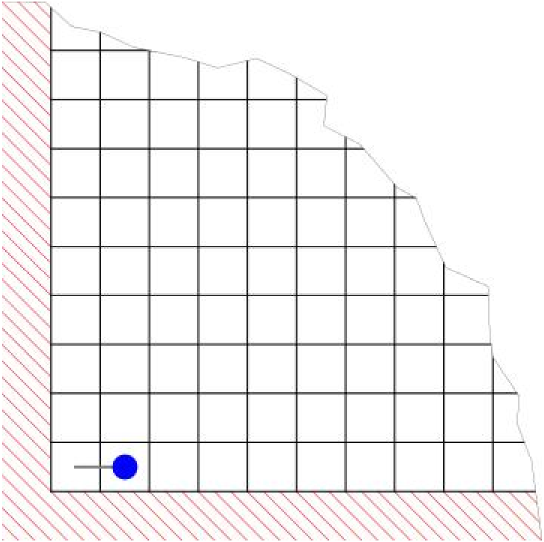


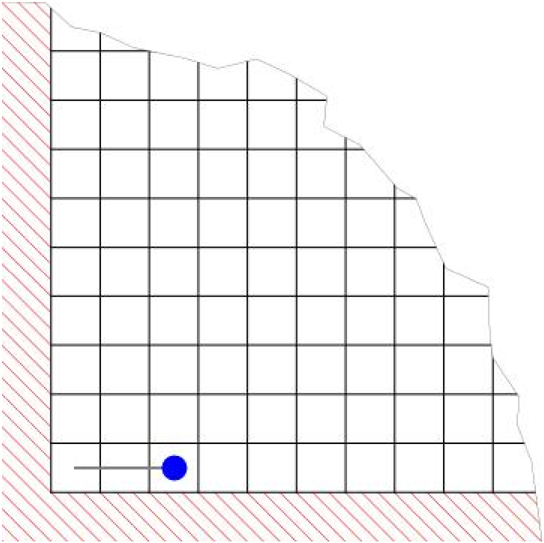


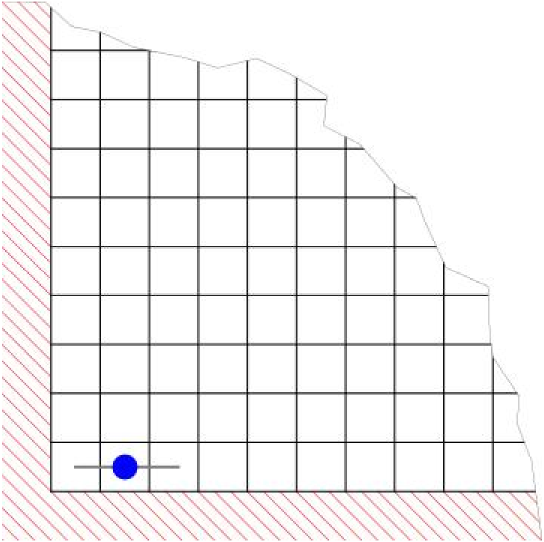


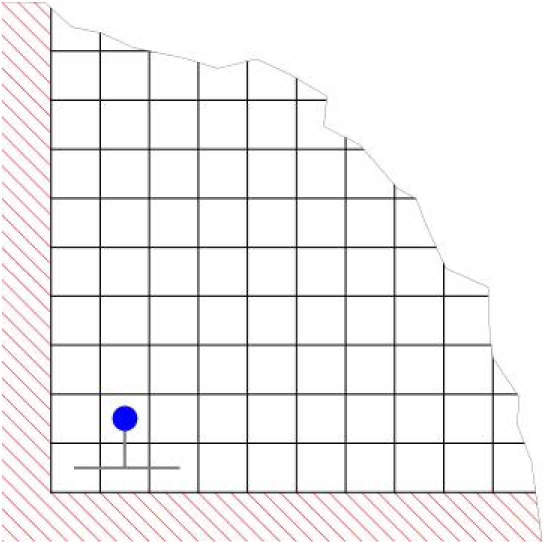


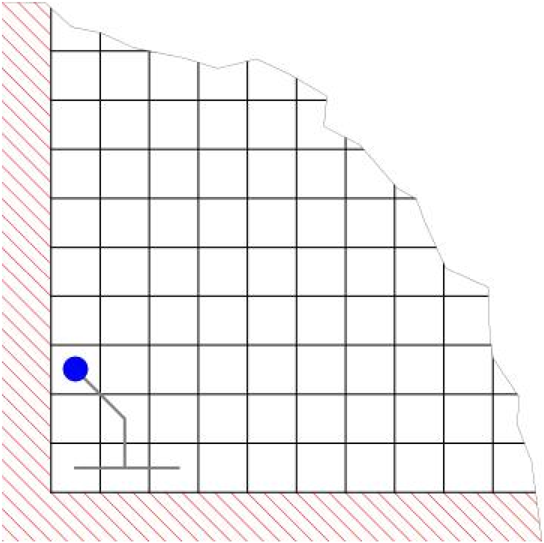


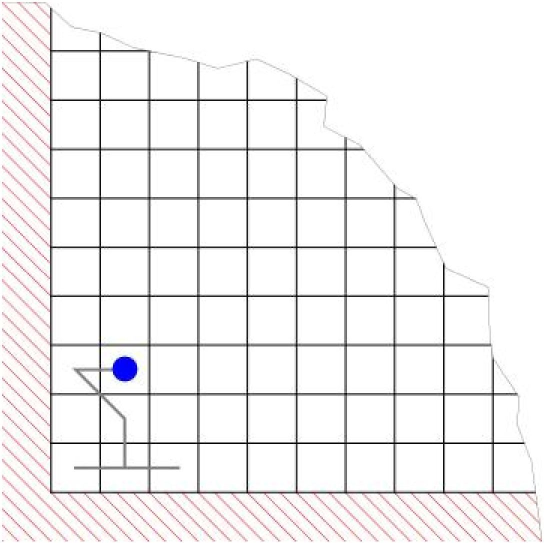


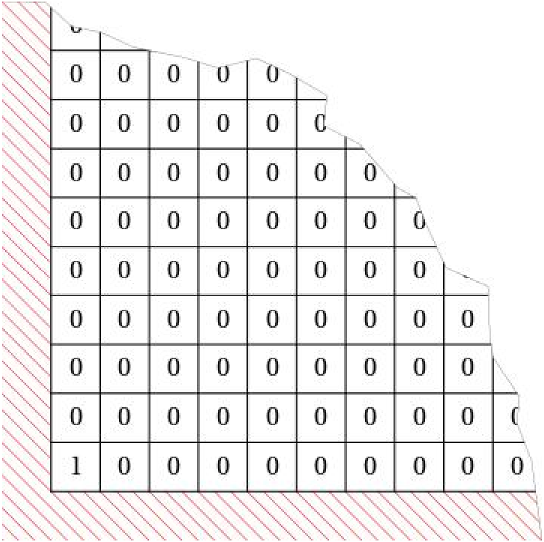








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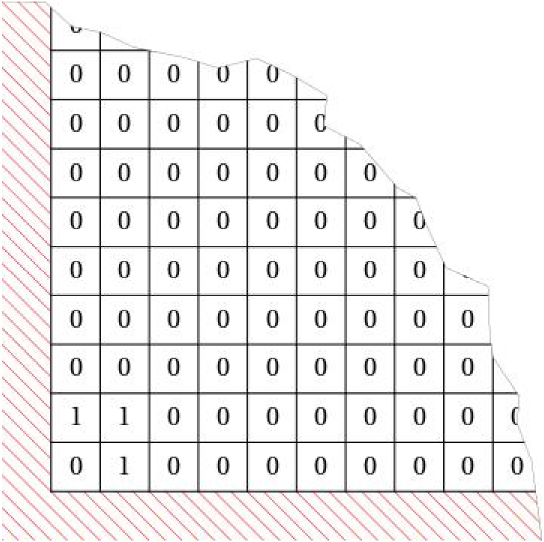
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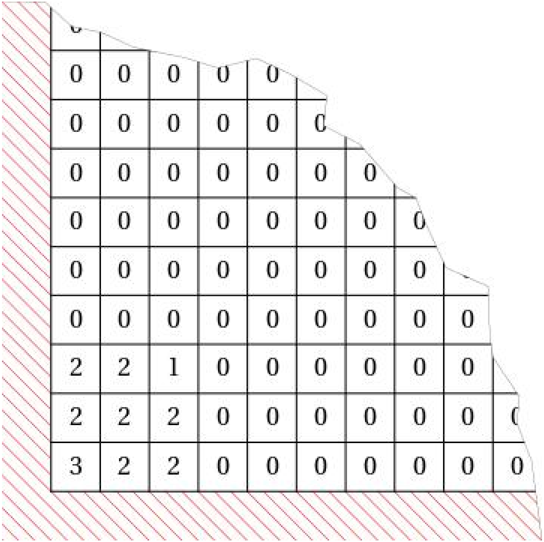
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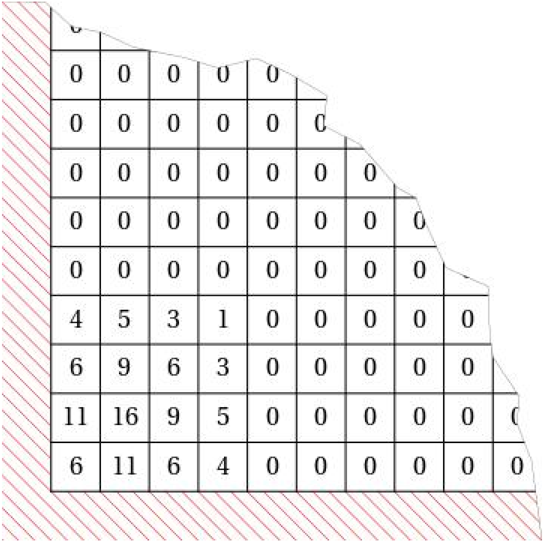
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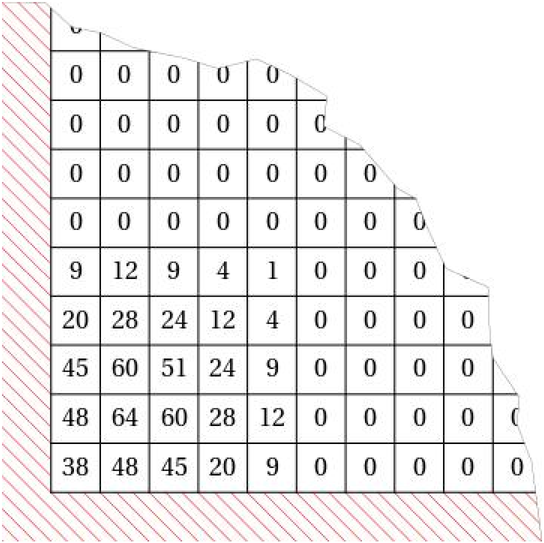
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21	30	25	14	5	1	0	0		
60	90	80	50	20	5	0	0		
154	230	200	126	50	14	0	0	0	
220	340	300	200	80	25	0	0	0	
255	395	340	230	90	30	0	0	0	0
160	255	220	154	60	21	0	0	0	0

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Let

$$a(t,x,y) := \sum_{n=0}^{\infty} \sum_{i,j=0}^{\infty} a_{n,i,j} x^i y^j t^n$$

be the *generating function* of $a_{n,i,j}$.

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Question: What is $a(t,x,y)$?

Starting point: The combinatorial definition.

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It immediately implies the *recurrence equation*

$$a_{n+1,i,j} = a_{n,i-1,j+1} + a_{n,i,j+1} + a_{n,i+1,j+1} + a_{n,i-1,j} \\ + a_{n,i+1,j} + a_{n,i-1,j-1} + a_{n,i,j-1} + a_{n,i+1,j-1}$$

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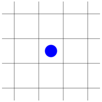

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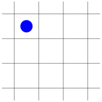

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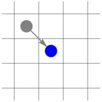

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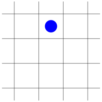

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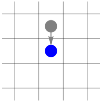

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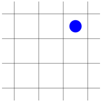

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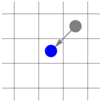

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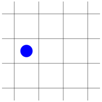

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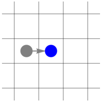

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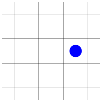

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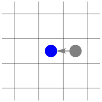

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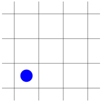

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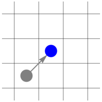

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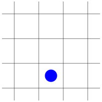

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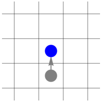

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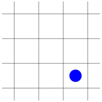

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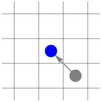
$$a_{n+1,i,j} = a_{n,i-1,j+1} + a_{n,i,j+1} + a_{n,i+1,j+1} + a_{n,i-1,j} + a_{n,i+1,j} + a_{n,i-1,j-1} + a_{n,i,j-1} + a_{n,i+1,j-1}$$



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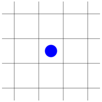

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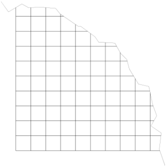
which, together with the *boundary conditions*

$$a_{n,i,-1} = 0 \qquad a_{n,-1,j} = 0$$



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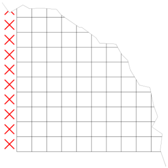

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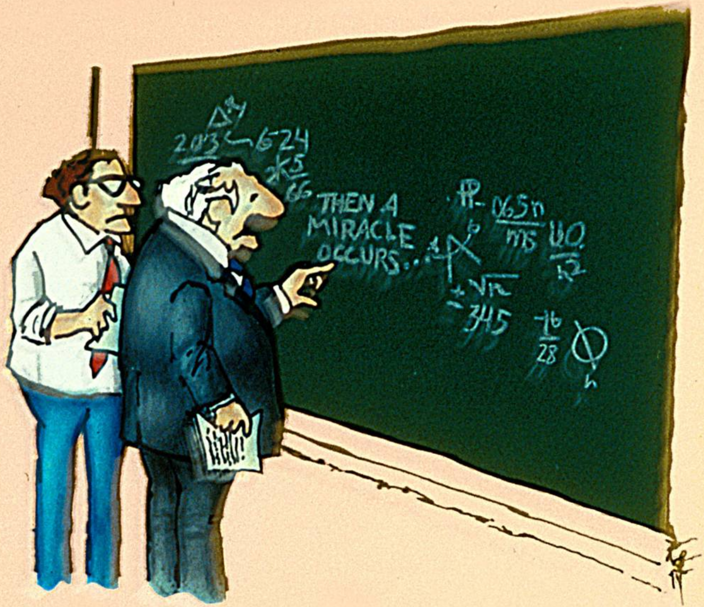
which, together with the *boundary conditions*

$$a_{n,i,-1} = 0 \qquad\qquad a_{n,-1,j} = 0$$

and the *initial value*

$$a_{0,0,0} = 1$$

determines all the numbers $a_{n,i,j}$.


THEN A
MIRACLE
OCCURS...

It follows for the *generating function* that

$$a(t,x,y) = \frac{1}{xy}[x^{>}][y^{>}]\frac{(x-x^{-1})(y-y^{-1})}{1-t((x+1+x^{-1})y^{-1}+(x+x^{-1})+(x+1+x^{-1})y)},$$

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(This miracle was performed by the combinatorial wizards M. Bousquet-Melou and M. Mishna.)

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It follows from here that $a(t,x,y)$ is also equal to the *formal residue*

$$\operatorname{res}_{u,v} \frac{1}{(1-xu)(1-yv)} \frac{(u-u^{-1})(v-v^{-1})}{1-t((u+1+u^{-1})v^{-1}+(u+u^{-1})+(u+1+u^{-1})v)},$$

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A *differential equation* can be computed with *creative telescoping.*

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A *differential equation* can be computed with *creative telescoping.*

Write

$$R = \frac{1}{(1-xu)(1-yv)}\frac{(u-u^{-1})(v-v^{-1})}{1-t((u+1+u^{-1})v^{-1}+(u+u^{-1})+(u+1+u^{-1})v)}$$

so that $a(t,x,y) = \operatorname{res}_{u,v} R$.

Observe: $\operatorname{res}_u D_u c(u) = 0$ for every series $c(u)$.

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Therefore: if we can find a differential operator

$$P = p_0(t,x,y) + p_1(x,y,t)D_t + p_2(t,x,y)D_t^2 + \cdots + p_r(t,x,y)D_t^r$$

and two rational functions $Q_1, Q_2 \in \mathbb{Q}(t,u,v,x,y)$

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$$\operatorname{res}_{u,v}(P\,R + D_u Q_1 + D_v Q_2) = 0$$

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and two rational functions $Q_1, Q_2 \in \mathbb{Q}(t,u,v,x,y)$ with

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then

$$\operatorname{res}_{u,v}(P\,R) + \operatorname{res}_{u,v}(D_u Q_1) + \operatorname{res}_{u,v}(D_v Q_2) = 0$$

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Note: Knowing P, we can compute a *closed form* for $a(t,x,y)$.

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But: Computing P, Q_1, Q_2 is quite costly.

Simplify the problem by setting $x = y = 1$.

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Note: The coefficients of t^n in $a(t, 1, 1)$ count the number of walks with n steps and arbitrary endpoint.

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Integration software (e.g., by C. Koutschan) finds the equation

$$\begin{aligned}
&(t+1)(2t-1)(4t+1)(8t-1)t^2 D_t^3 a(t,1,1)\\
&\quad + (576t^4 + 200t^3 - 252t^2 - 33t + 5) D_t^2 a(t,1,1)\\
&\quad\quad + (288t^4 + 22t^3 - 117t^2 - 12t + 1) D_t a(t,1,1)\\
&\quad\quad\quad + 12(32t^3 - 6t^2 - 12t - 1) a(t,1,1) = 0.
\end{aligned}$$

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From here follows the *final result*

$$a(t,1,1) = -\frac{1}{t} \int_t \frac{16t^2+24t-1}{(1+4x)^5} \, {}_2F_1\left(\begin{matrix} 5/4 \quad 5/4 \\ 2 \end{matrix} \;\middle|\; \frac{-2t(t+1)(t-1/8)}{(t+1/4)^4} \right).$$

Variation: What happens if we forbid steps into certain directions?

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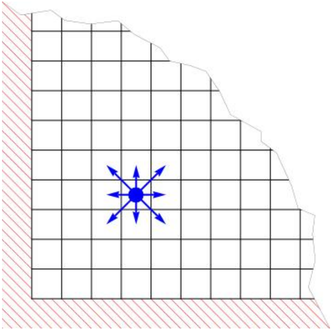

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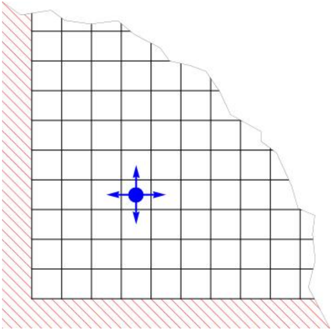

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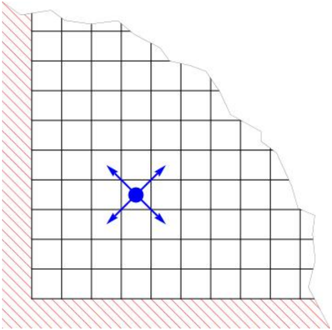

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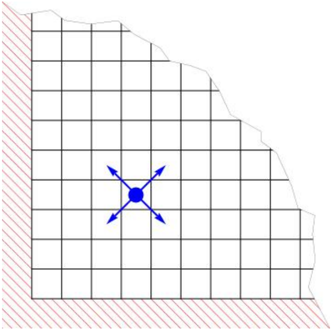

- Different step sets lead to different generating functions.

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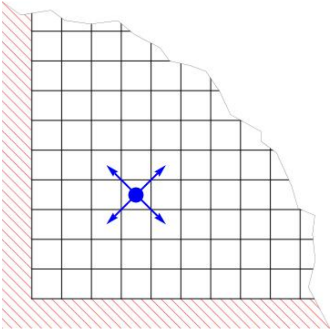

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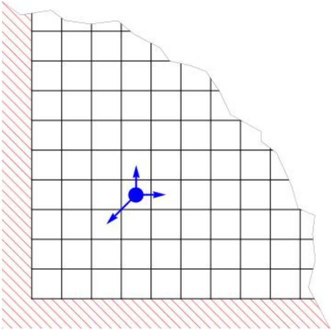

- Different step sets lead to different generating functions.
- Different generating functions have different algebraic properties.
- *Kreweras walks:* The generating function is algebraic.

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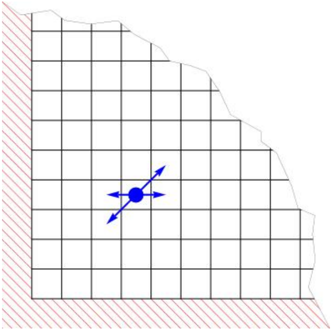

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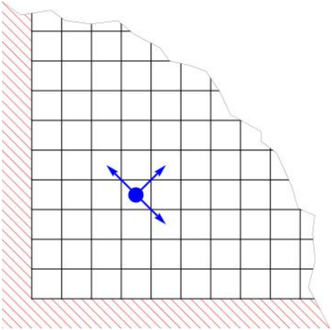

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- *Our contribution:* For the cases where the generating function is D-finite transcendental, we find an explicit ${}_2F_1$ representation.

2. The Computer Algebra Part.

Fine Tuning Creative Telescoping

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- There are *algorithms* for computing (P,Q) for given R.

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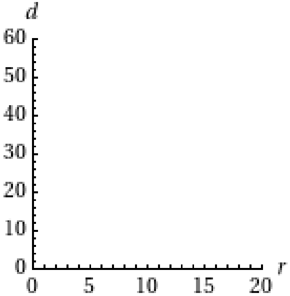

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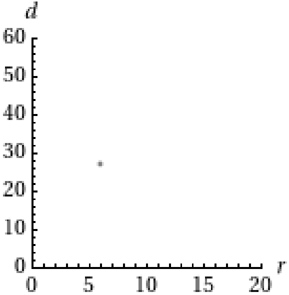

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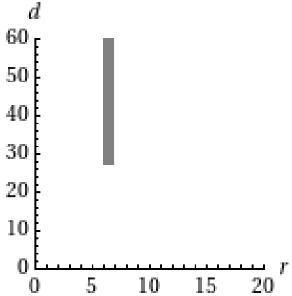

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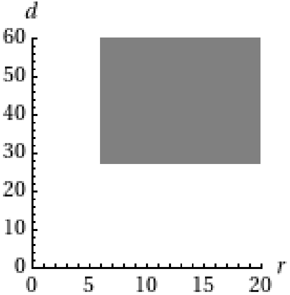

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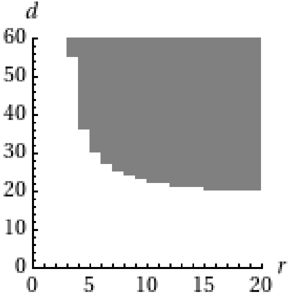

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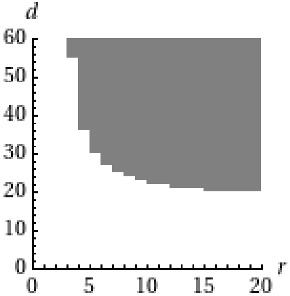

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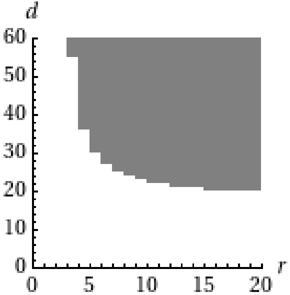

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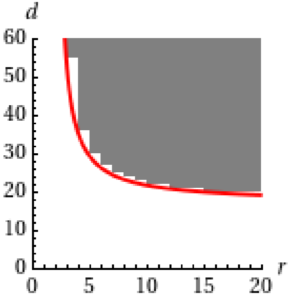

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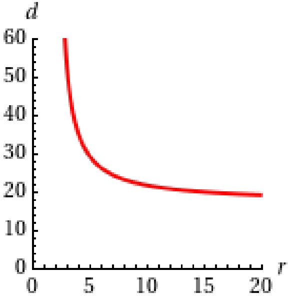

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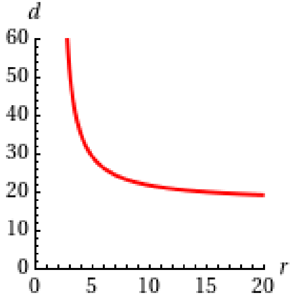

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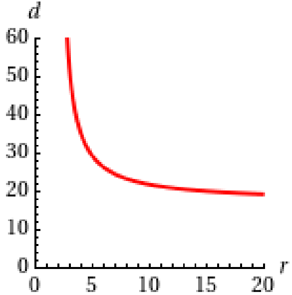

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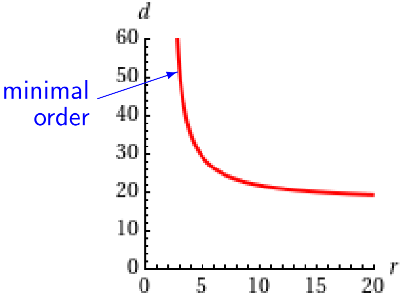

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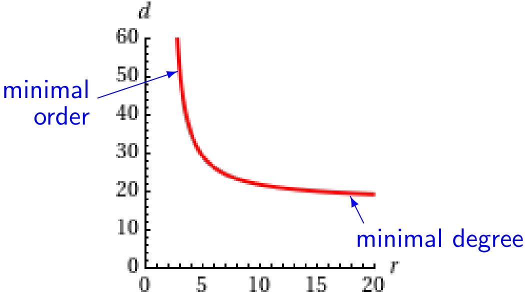

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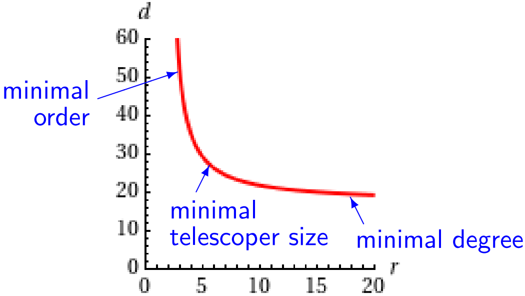

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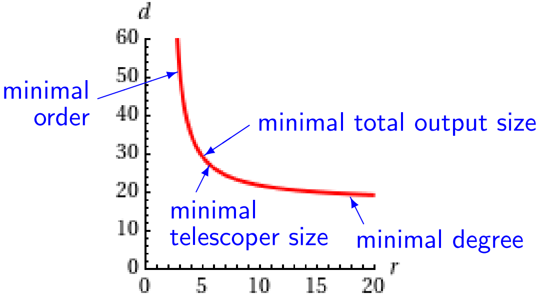

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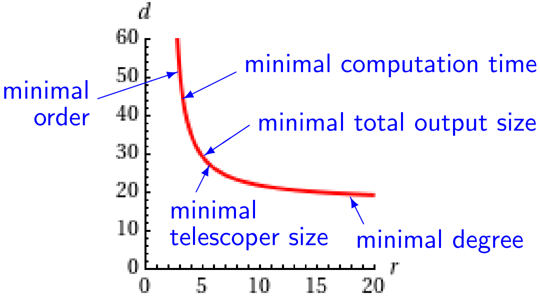

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- For *small input,* the minimal order operator is the cheapest.
- For *"industrial size input",* operators of [slightly] nonminimal order are cheaper.
- For *astronomic input,* it is most efficient to compute the operator of order $\frac{1}{4}(1+\sqrt{17})r_{\min}$, where $r_{\min}$ is the size of the minimal operator.

3. Conclusion.

Symbolic Computation

$+$ *Enumerative Combinatorics*

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- The existence of powerful computational machinery suggests to rephrase a combinatorial problem as input for them.
- Unexpected output may lead to combinatorial insight or raise new questions.