Finding closed form solutions of differential equations

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- For example

$$\begin{aligned}(16x^4 + 48x^3 + 48x^2 + 18x + 2)f''(x)\\ - (16x^4 + 48x^3 + 52x^2 + 32x + 9)f'(x)\\ + (4x^2 + 14x + 7)f(x) = 0.\end{aligned}$$

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One independent variable x

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One independent variable x

One unknown function $f(x)$

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- In the example: $f(x) = \exp(x)$ and $f(x) = \frac{\sqrt{1+3x+2x^2}}{x+1}$.

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$$(2x^3 - 9x^2 - 5)f^{(3)}(x) - (2x^3 - 9x^2 - 5)f''(x) + (6x^2 - 24x + 18)f'(x) + (6 - 6x)f(x) = 0$$

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- In the example, a basis of the vector space of all polynomial solutions is given by $x - 3$ and $x^3 + 5$. (A third solution, linearly independent of those two, is not polynomial.)

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No matter what these constants are, we have

$$\begin{aligned} f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 \\ f''(x) &= 2c_2 + 6c_3 x \\ f'''(x) &= 6c_3 \end{aligned}$$

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The polynomial $f(x)$ solves the differential equation iff

$$\begin{aligned}(6c_0 + 18c_1 + 10c_2 - 30c_3)\\ (-6c_0 - 18c_1 + 36c_2 + 30c_3)x\\ -24c_2x^2\\ +2c_2x^3 = 0\end{aligned}$$

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The polynomial $f(x)$ solves the differential equation iff

$$\begin{pmatrix} 6 & 18 & 10 & -30 \\ -6 & -18 & 36 & 30 \\ 0 & 24 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

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The polynomial $f(x)$ solves the differential equation iff

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \alpha \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for some constants α, β.

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The polynomial $f(x)$ solves the differential equation iff

$$f(x) = \alpha(5 + 1x^3) + \beta(-3 + 1x) = (5\alpha - 3\beta) + \beta x + \alpha x^3$$

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There could still be polynomial solutions of higher degree.

In order to find **all** polynomial solutions, we need to know in advance how large their degree can get.

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No matter what the d and $c_d, c_{d-1}, \ldots$ are, we have

$$\begin{aligned} f(x) &= c_d\, x^d + \cdots \\ f'(x) &= c_d\, dx^{d-1} + \cdots \\ f''(x) &= c_d\, d(d-1)x^{d-2} + \cdots \\ f'''(x) &= c_d\, d(d-1)(d-2)x^{d-3} + \cdots \end{aligned}$$

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If $f(x) = c_d\, x^d + \cdots$ solves the differential equation, then

$$\begin{aligned} &(2x^3 - 9x^2 - 5) f^{(3)}(x) \\ &- (2x^3 - 9x^2 - 5) f''(x) \\ &+ (6x^2 - 24x + 18) f'(x) \\ &+ (6 - 6x) f(x) = 0 \end{aligned}$$

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If $f(x) = c_d\, x^d + \cdots$ solves the differential equation, then

$$d = 3 \quad \text{or} \quad d = 1.$$

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In general, plugging x^d with symbolic exponent d into an ODE gives $p(d)x^{d+i} + \cdots$ for some polynomial p (and some integer i).

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The possible degrees are integer roots of this polynomial.

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The possible degrees are integer roots of this polynomial.

The polynomial p is called the **indicial polynomial** of the differential equation.

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OUTPUT: A basis of the vector space of all its polynomial solutions.

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5. Solve the resulting linear system for $c_0, \ldots, c_d$.
6. The solutions of the system correspond to polynomial solutions of the equation.

Some possible meanings of “closed form”:

- polynomials e.g. $5x^2 + 3x - 2$
- rational functions e.g. $(5x-3)/(3x^2 - x + 5)$
- hyperexponential functions e.g. $\exp\left(\frac{2x+3}{x^2(x+1)}\right)\frac{(2x+5)^{1/3}}{(7x^2+x-3)^{1/2}}$
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- For example

$$(2x^4 - x^3 + 3x)f^{(3)}(x) - (2x^4 - 15x^3 + 15x^2 - 9x - 9)f''(x) \\ - (6x^3 - 30x^2 + 42x - 18)f'(x) + (6x - 18)f(x) = 0$$

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- In the example, a basis of the vector space of all rational solutions is given by $(3 - x)/x$ and $1/(1 + x)^2$. (A third solution, linearly independent of those two, is not a rational function.)

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No matter what $u(x)$ is, we have

$$\begin{aligned} f(x) &= \frac{u(x)}{x} \\ f'(x) &= \frac{u'(x)}{x} - \frac{u(x)}{x^2} \\ f''(x) &= \frac{u''(x)}{x} - 2\frac{u'(x)}{x^2} + 2\frac{u(x)}{x^3} \\ f'''(x) &= \frac{u'''(x)}{x} - 3\frac{u''(x)}{x^2} + 6\frac{u'(x)}{x^3} - 6\frac{u(x)}{x^4} \end{aligned}$$

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Plug $f(x) = u(x)/x$ into the differential equation. This gives

$$(2x^7 - x^6 + 3x^4)u^{(3)}(x) - (2x^7 - 9x^6 + 12x^5 - 9x^4)u''(x) \\ - (2x^6 - 12x^5 + 18x^4)u'(x) + (2x^5 - 6x^4)u(x) = 0.$$

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Determine the polynomial solutions of this equation. This gives $u(x) = 3 - x$ (up to constant multiples).

It follows that $f(x) = (3 - x)/x$ is (up to constant multiples) the only rational solution of the original equation with denominator x.

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$$f'(x) = \frac{\blacksquare}{\blacksquare\,p^{e+1}},\ \ f''(x) = \frac{\blacksquare}{\blacksquare\,p^{e+2}},\ \ f'''(x) = \frac{\blacksquare}{\blacksquare\,p^{e+3}},\text{ etc.}$$

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The factor p must divide the leading coefficient a_3 of the ODE.

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- Without loss of generality, $\alpha = 0$, so that $p = x$.

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- Looks familiar...

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- Also this polynomial is called **indicial polynomial.**

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6. Return the corresponding rational functions $f(x)$.

Some possible meanings of "closed form":

- polynomials ✓ — e.g. $5x^2 + 3x - 2$
- rational functions — e.g. $(5x - 3)/(3x^2 - x + 5)$
- hyperexponential functions — e.g. $\exp\left(\frac{2x+3}{x^2(x+1)}\right)\frac{(2x+5)^{1/3}}{(7x^2+x-3)^{1/2}}$
- algebraic functions — e.g. $x - \sqrt{x^2 + 1}$
- elementary functions — e.g. $\sin(x)/\sqrt{1 + \log(1 - e^x)}$
- special functions — e.g. $J_3(x^2+1) - {}_2F_1(2, 3; 1)(\frac{1}{x})$
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Examples.

$$x^{\sqrt{2}}(x+1) \sim x^{\sqrt{2}+4}(x+1)^{-3} \qquad x^{\sqrt{2}} \not\sim x^2 \qquad x^2 \not\sim \exp(x).$$

GIVEN: A linear ordinary differential equation with polynomial coefficients.

- For example

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\begin{aligned}
&(6x^5 - 60x^4 + 225x^3 - 386x^2 + 301x - 84)f(x) \\
&\quad + (x-1)^2(10x^5 - 86x^4 + 277x^3 - 411x^2 + 272x - 59)f'(x) \\
&\qquad + (x-2)^2(x-1)^4(2x^2 - 8x + 7)f''(x) = 0.
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FIND: its hyperexponential solutions.

- In the example, there are two hyperexponential solutions $\exp\left(\frac{x-3}{(x-1)(x-2)}\right)$ and $\exp\left(\frac{1}{x-1}\right)\frac{x^3-3x^2+2x-1}{(x-1)^3}$. (Here, all solutions can be written as linear combinations of hyperexponential terms. In general, this is not possible.)

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No matter what $u(x)$ is, we have

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\begin{aligned}
f(x) &= u(x)\exp\bigl(\tfrac{1}{x-1}\bigr)\\
f'(x) &= \Bigl(u'(x) - \tfrac{1}{(x-1)^2}u(x)\Bigr)\exp\bigl(\tfrac{1}{x-1}\bigr)\\
f''(x) &= \Bigl(u''(x) - \tfrac{2}{(x-1)^2}u'(x) + \tfrac{2x-1}{(x-1)^4}u(x)\Bigr)\exp\bigl(\tfrac{1}{x-1}\bigr), \text{etc.}
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Plug $f(x) = u(x)\exp\bigl(\frac{1}{x-1}\bigr)$ into the differential equation, divide by $\exp\bigl(\frac{1}{x-1}\bigr)$, and clear denominators. This gives the equation

$$\begin{aligned}&(x-2)^2(x-1)^4(2x^2-8x+7)u''(x)\\&+(x-1)^2(10x^5-90x^4+309x^3-505x^2+392x-115)u'(x)\\&-(8x^3-50x^2+92x-53)(x-1)u(x)=0.\end{aligned}$$

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Find its rational solutions. This gives $u(x) = \frac{x^3-3x^2+2x-1}{(x-1)^3}$.

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These are series expansions of the form

$$\exp\Big(\frac{p(x)}{(x-\xi)^d}\Big)(x-\xi)^\alpha\Big(1+c_1(x-\xi)+c_2(x-\xi)^2+\cdots\Big),$$

where $d \in \mathbb{N}$, $p(x)$ is a polynomial of degree $< d$, and $\alpha, c_1, c_2, \dots$ are constants.

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Fact. There is a way to compute the *"local solutions"* of a given ODE at a given point ξ.

Example. For the ODE above and $\xi = 1$, we get

$$\exp\left(\tfrac{2}{x-1}\right)\Big(1 + (x-1) + \tfrac{3}{2}(x-1)^2 + \tfrac{13}{6}(x-1)^3 + \cdots\Big)$$
$$\exp\left(\tfrac{1}{x-1}\right)\Big((x-1)^{-3} + (x-1)^{-2} - 1 + \Big)$$

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Example. For the ODE above and $\xi = 2$, we get

$$\exp\left(\tfrac{-1}{x-2}\right)\Bigl(1 - 2(x-2) + 4(x-2)^2 - \tfrac{22}{3}(x-2)^2 + \cdots\Bigr)$$

$$\exp(0)\Bigl(1 - 6(x-2) + \tfrac{31}{2}(x-2)^2 - \tfrac{98}{3}(x-2)^3 + \cdots\Bigr)$$

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Example. For the ODE above, there are four candidates:

$$\begin{array}{ll} \exp\left(\frac{2}{x-1} - \frac{1}{x-2}\right) & \exp\left(\frac{2}{x-1} + 0\right) \\ \exp\left(\frac{1}{x-1} - \frac{1}{x-2}\right) & \exp\left(\frac{1}{x-1} + 0\right). \end{array}$$

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INPUT: A linear ordinary differential equation with polynomial coefficients.

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1. Let $\xi_1, \xi_2, \dots$ be the roots of the leading coefficient.
2. For each ξ_i, compute the exponential parts $\exp\left(\frac{p_j}{(x-\xi_i)^{d_j}}\right)$ $(j = 1, 2, \dots)$ of the local solutions at ξ_i.

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3. For each combination $E := \exp\left(\frac{p_{j_1}}{(x-\xi_1)^{d_{j_1}}} + \frac{p_{j_2}}{(x-\xi_2)^{d_{j_2}}} + \cdots\right)$ do:

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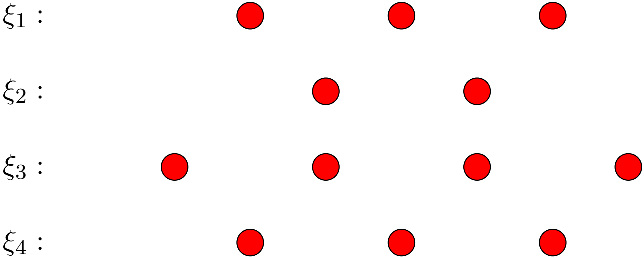
5. Construct an auxiliary equation for $u(x)$
6. Find its rational solutions
7. For each solution $u(x)$, output $f(x) = u(x)\,E$.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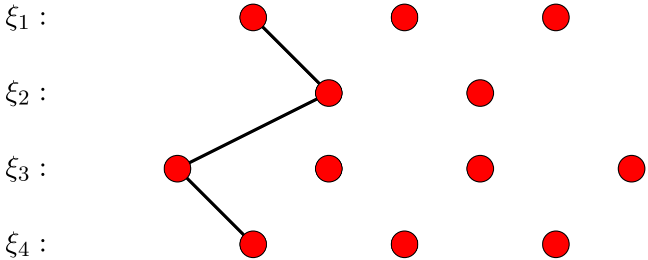
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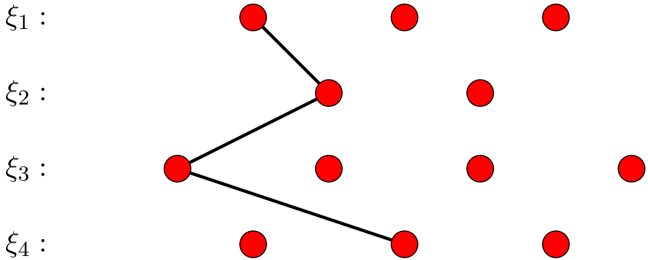
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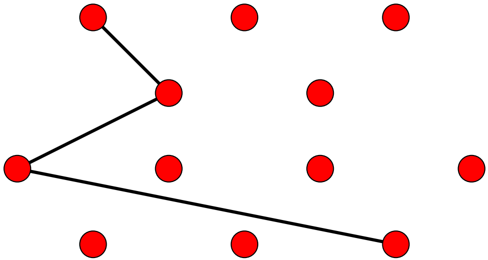

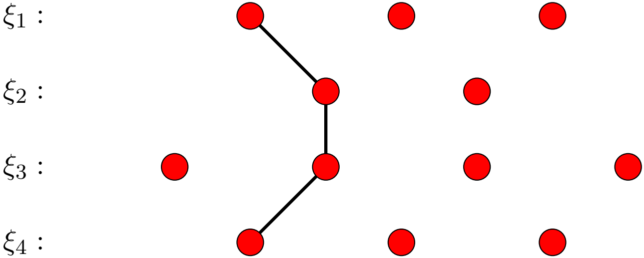
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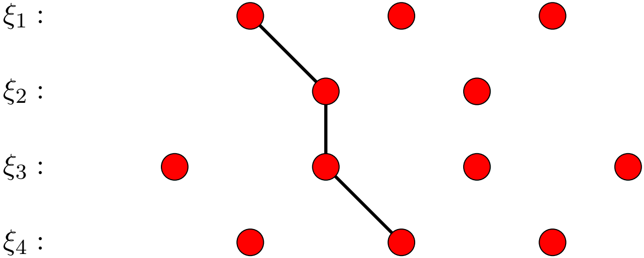
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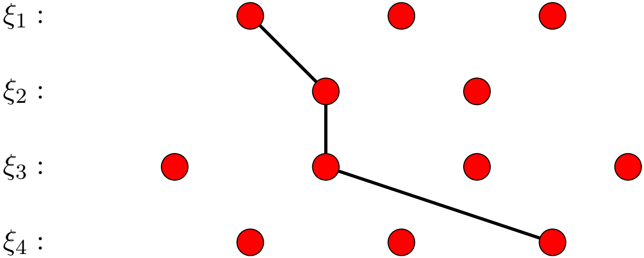
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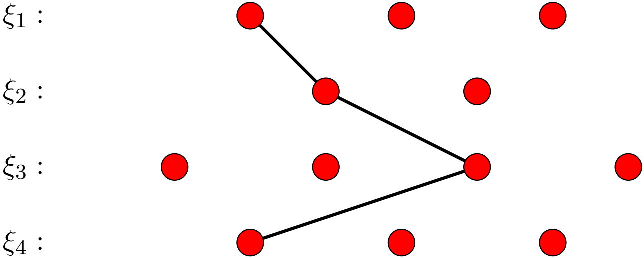

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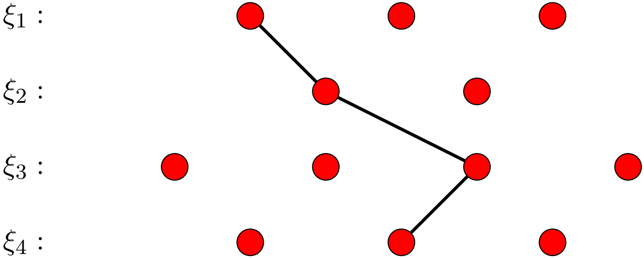
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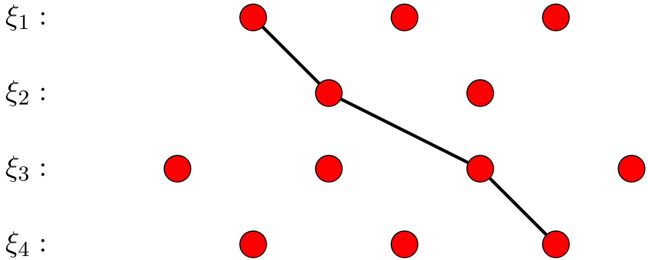
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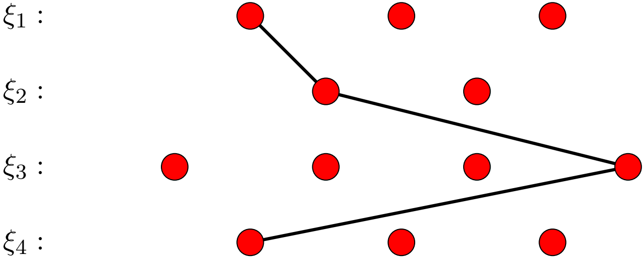
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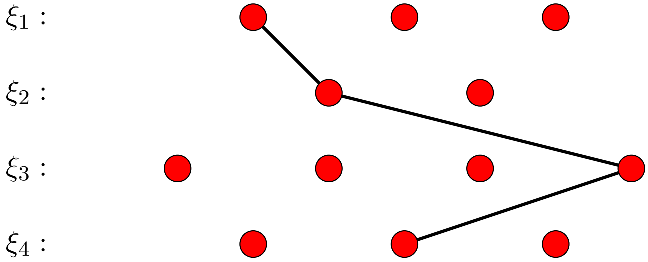
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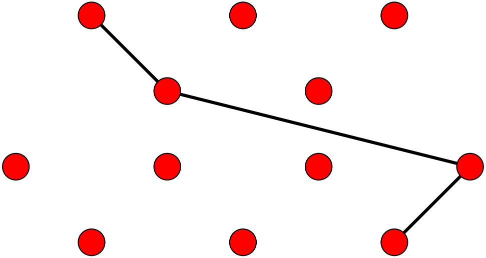

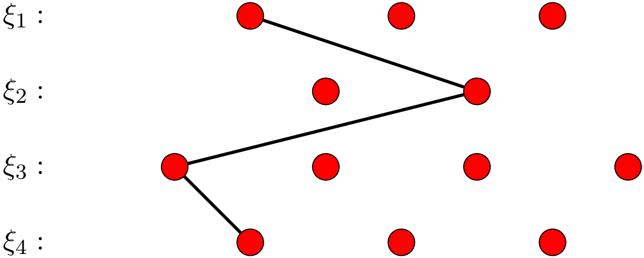
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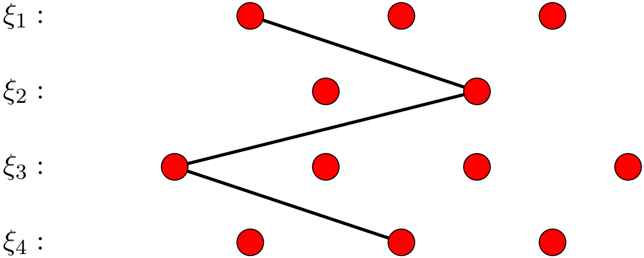

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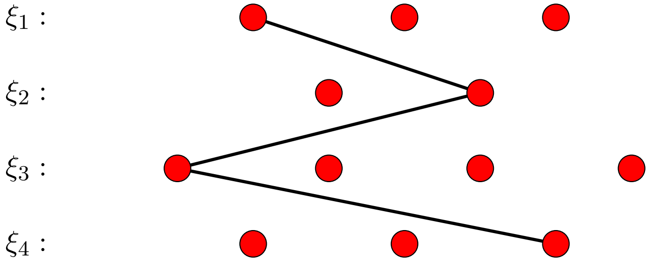
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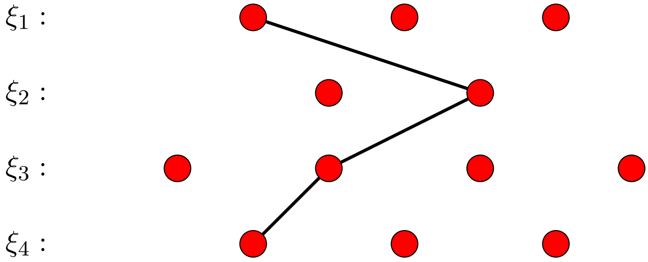
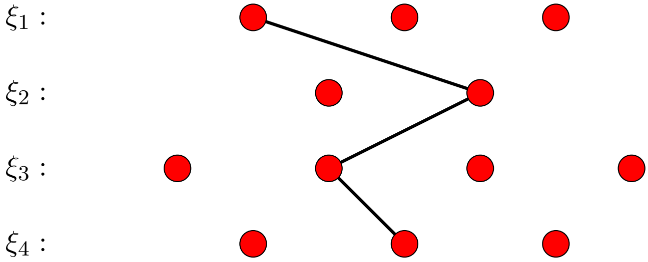
▶ Hyperexponential Solutions





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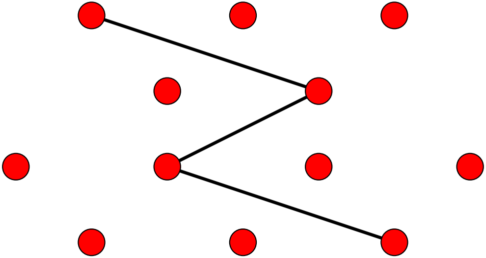

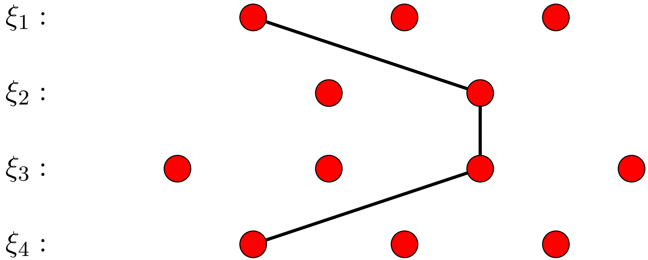
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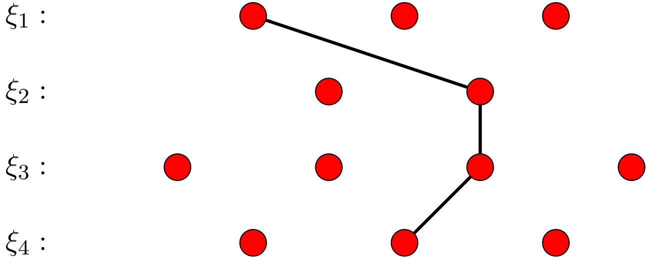
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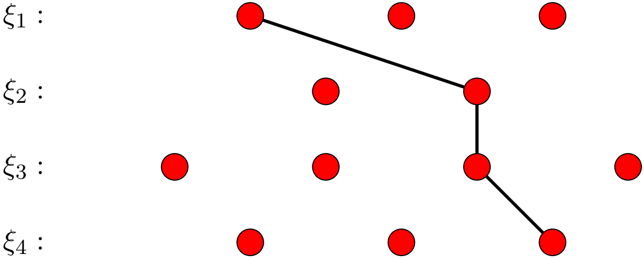
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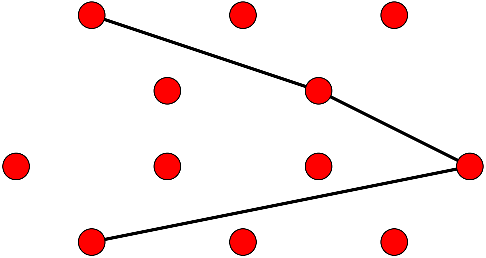

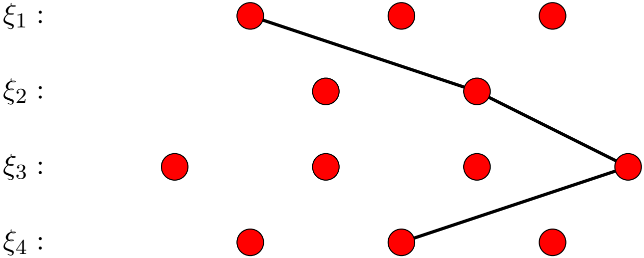
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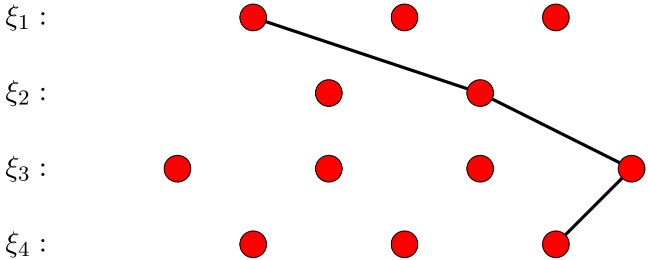
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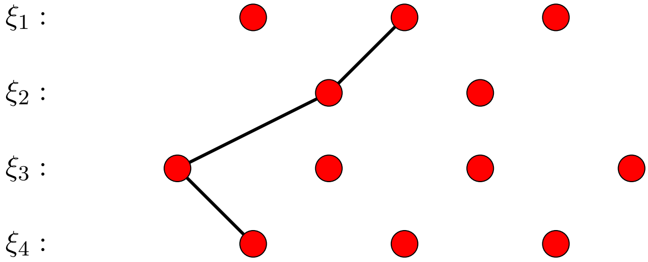

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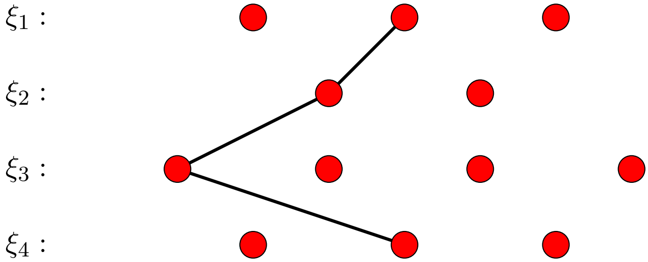
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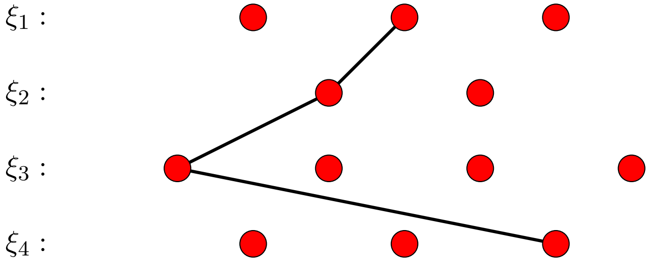

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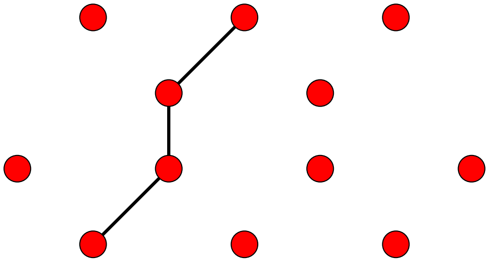

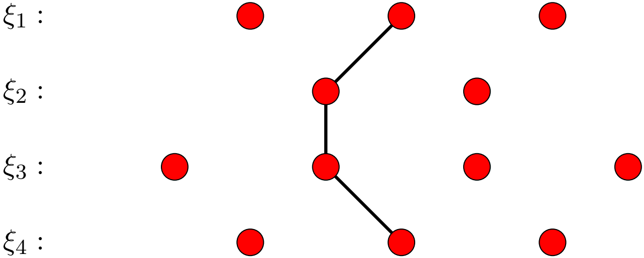
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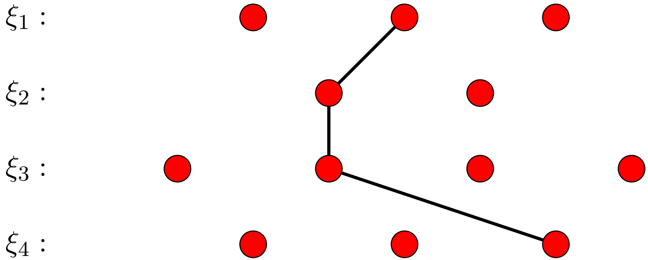
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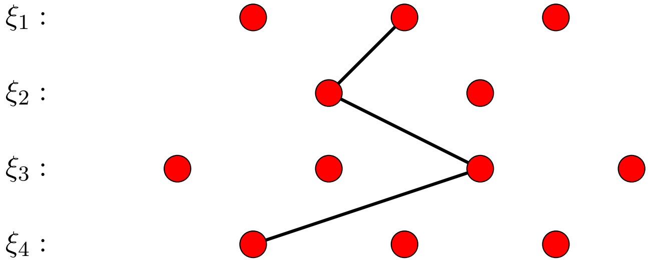
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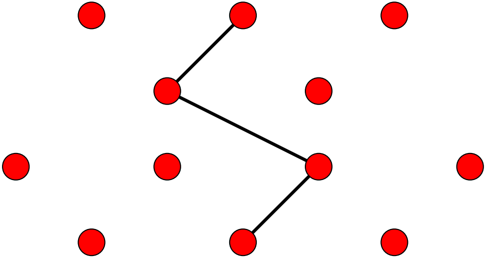

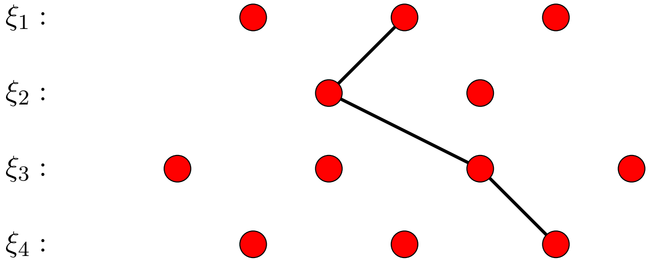
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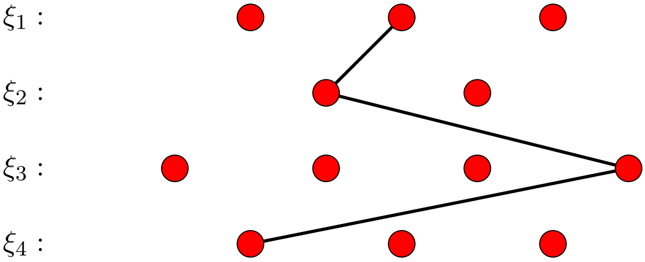
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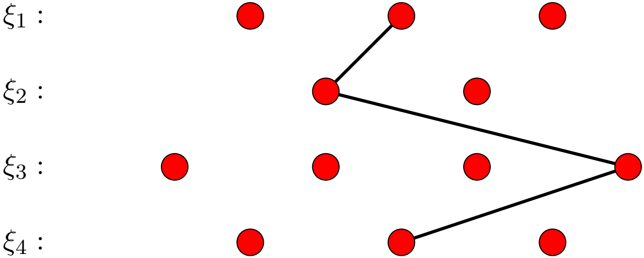
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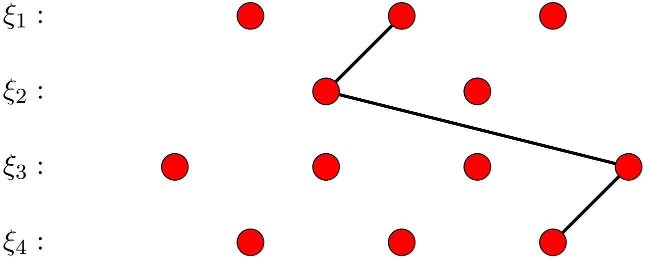
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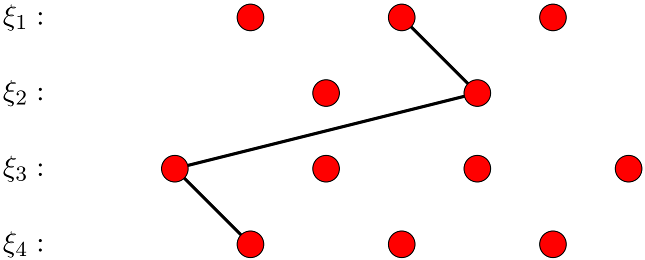
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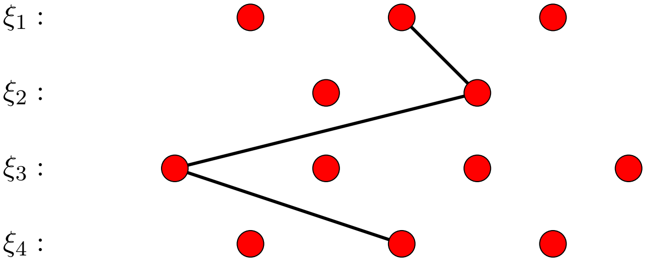
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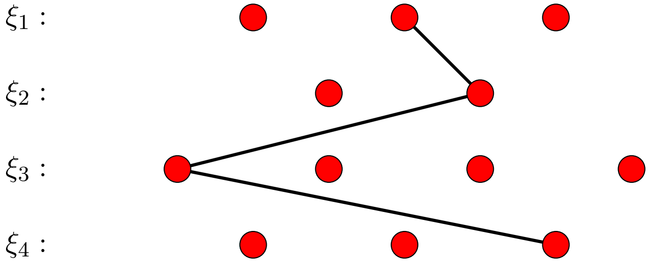
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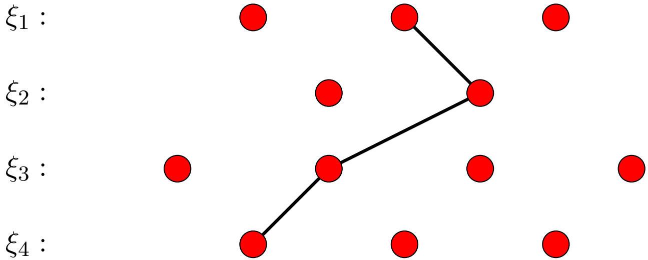

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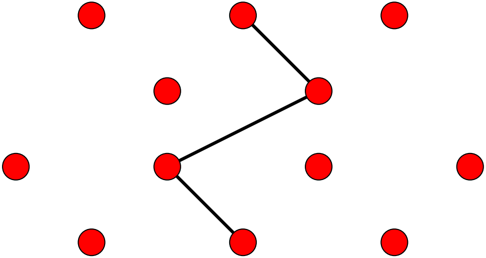

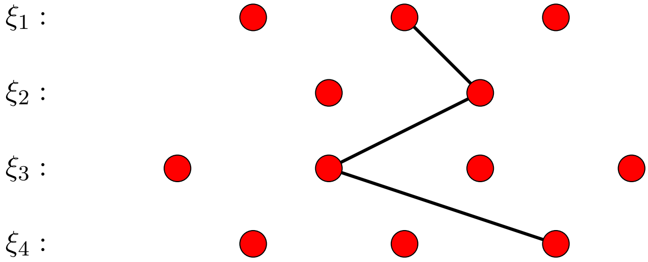
ξ_1 :
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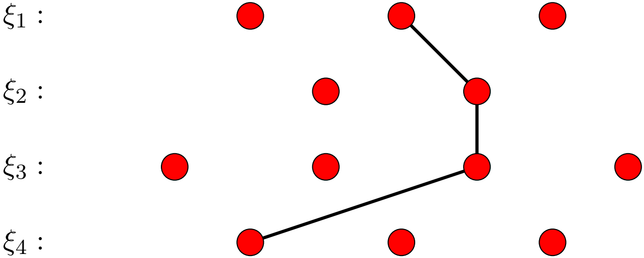
ξ_3 :
ξ_4 :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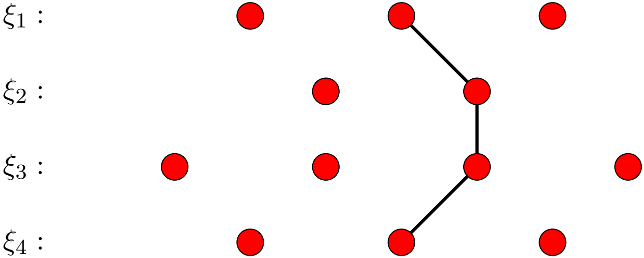
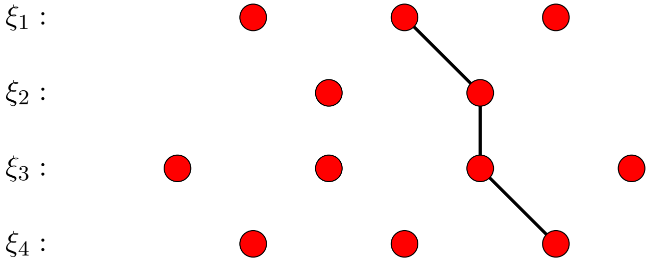

ξ_1 :
ξ_2 :
ξ_3 :
ξ_4 :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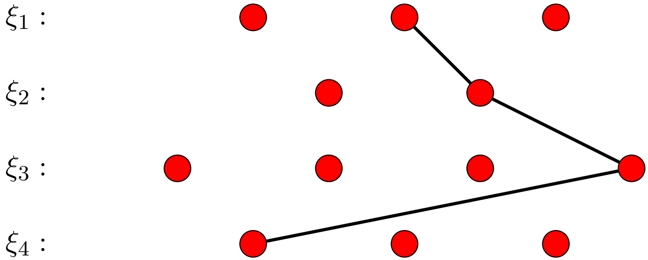
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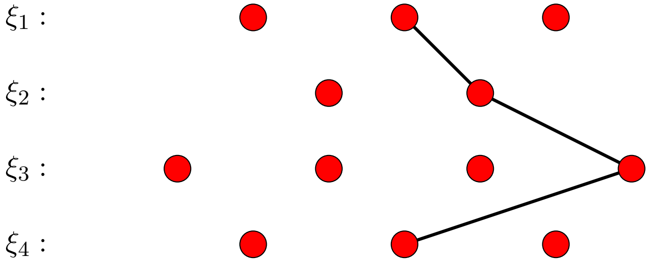
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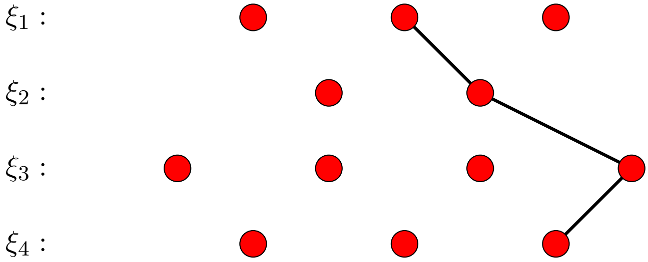
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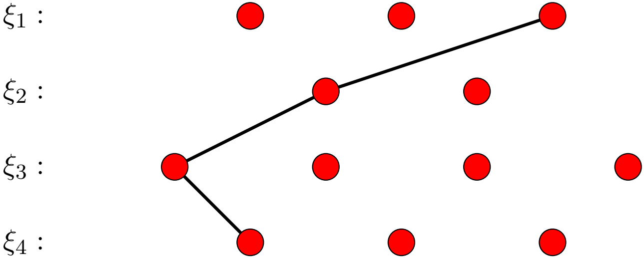
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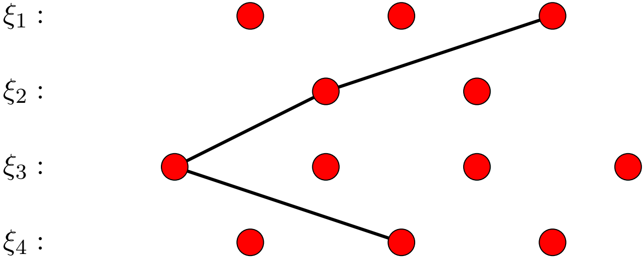
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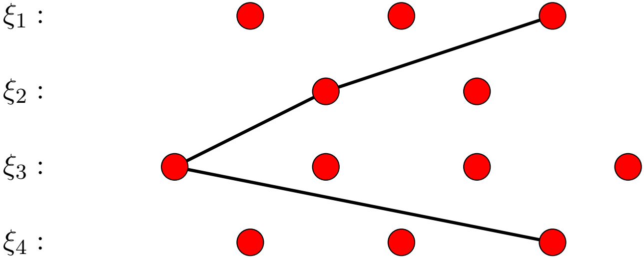
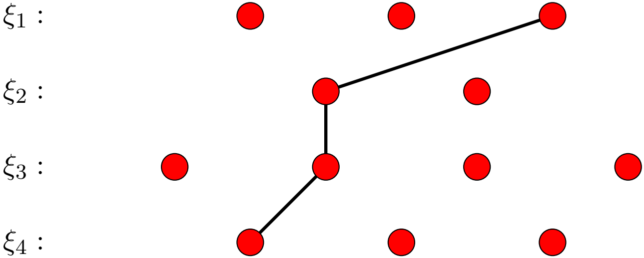
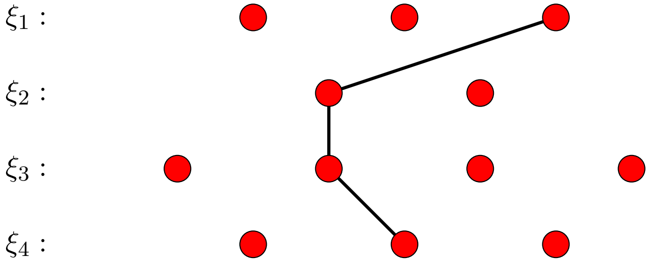
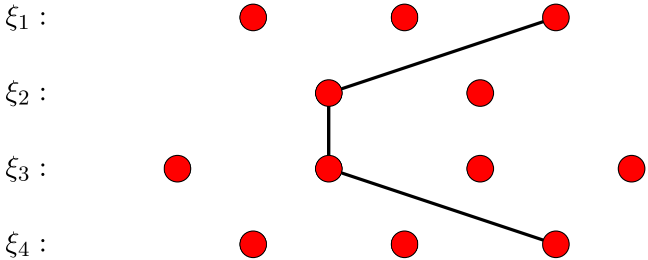


ξ_1 :
ξ_2 :
ξ_3 :
ξ_4 :

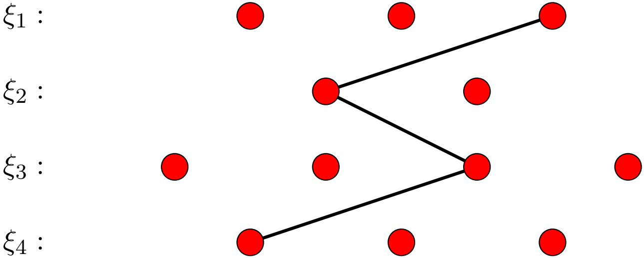
▶ Hyperexponential Solutions




ξ_1 :
ξ_2 :
ξ_3 :
ξ_4 :



ξ_1 :
ξ_2 :
ξ_3 :
ξ_4 :


ξ_1 :
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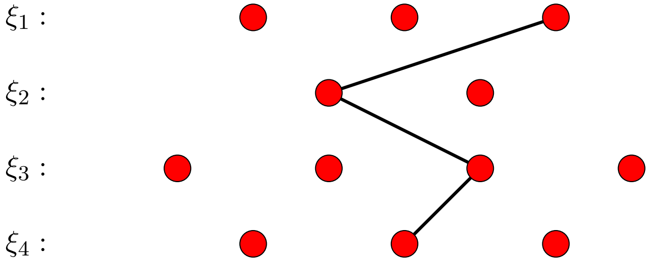
ξ_1 :
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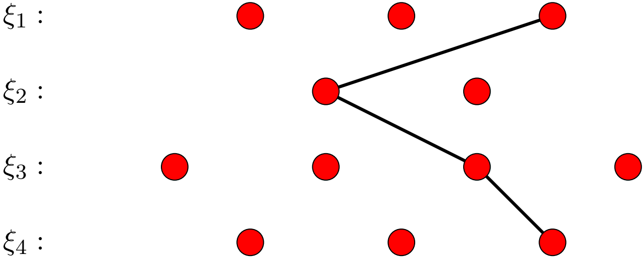
ξ_3 :
ξ_4 :


ξ_1 :
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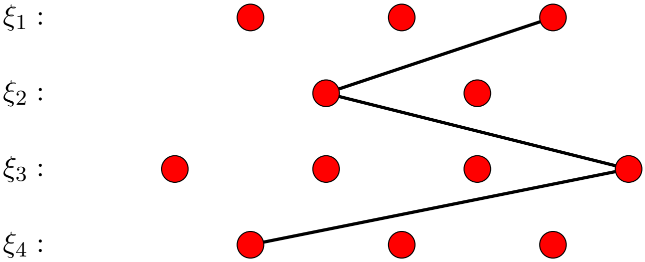
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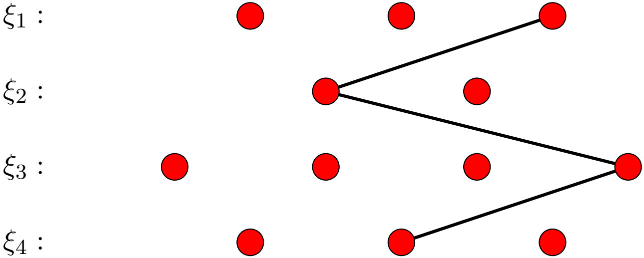
ξ_1 :
ξ_2 :
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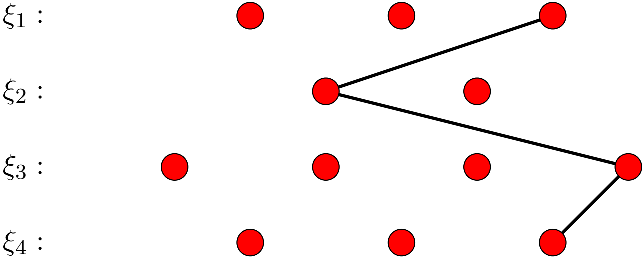
▸ Hyperexponential Solutions



ξ_1 :

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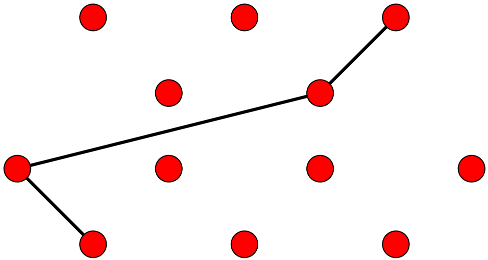
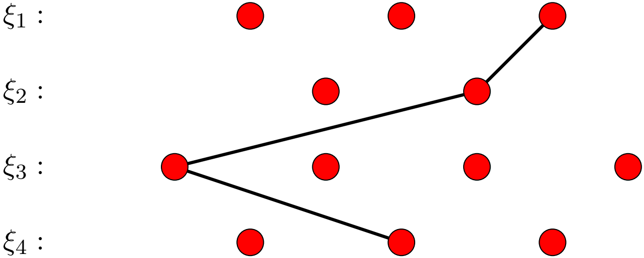
ξ_4 :




ξ_1 :
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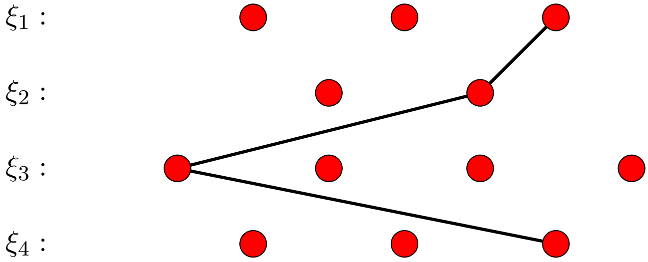
ξ_1 :
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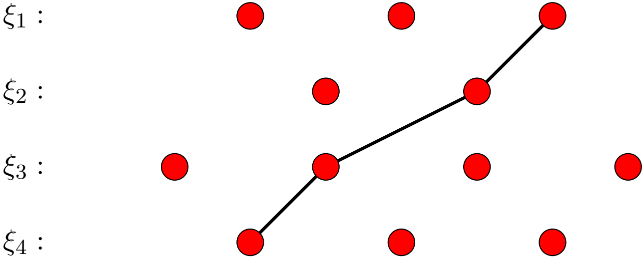
ξ_1 :
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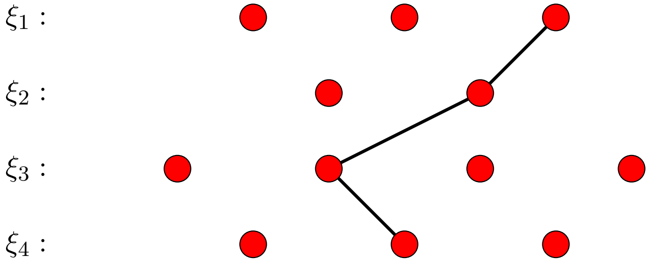

ξ_1 :
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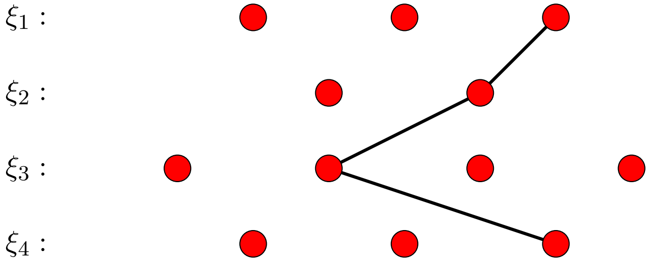
ξ_3 :
ξ_4 :


ξ_1 :
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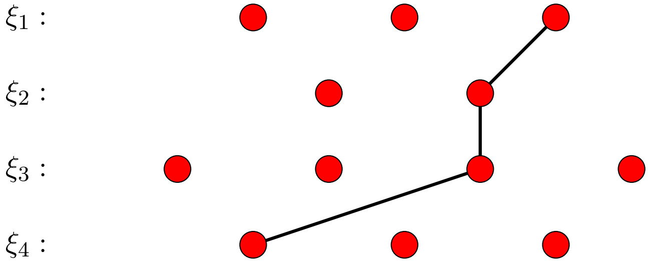
ξ_3 :
ξ_4 :


ξ_1 :
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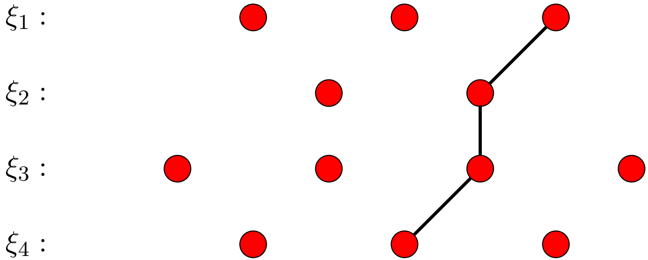
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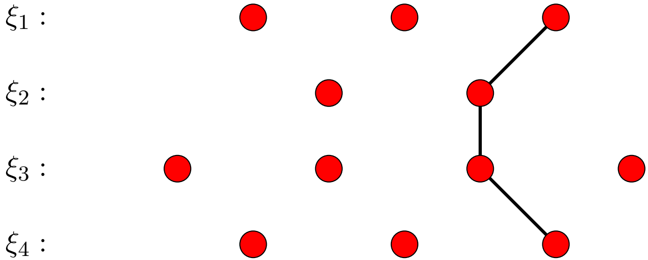
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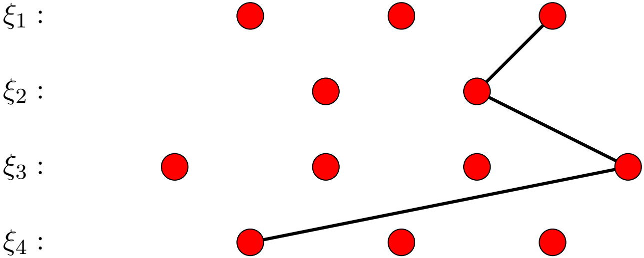
ξ_1 :
ξ_2 :
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ξ_4 :

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ξ_3 :

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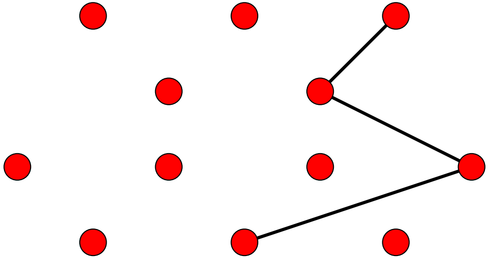

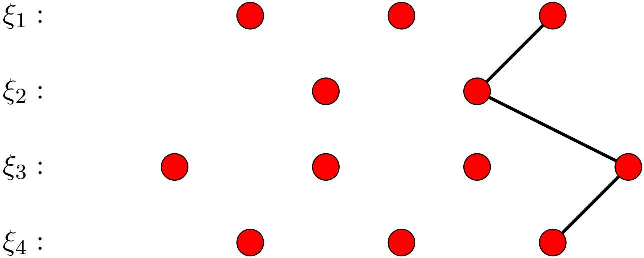
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ξ_3 :
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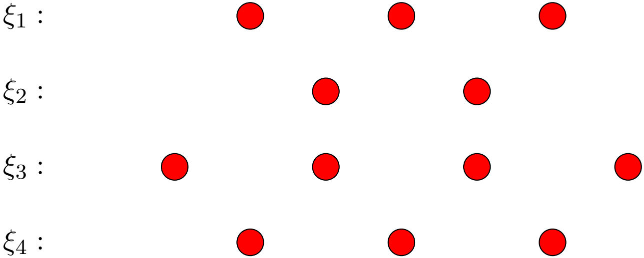
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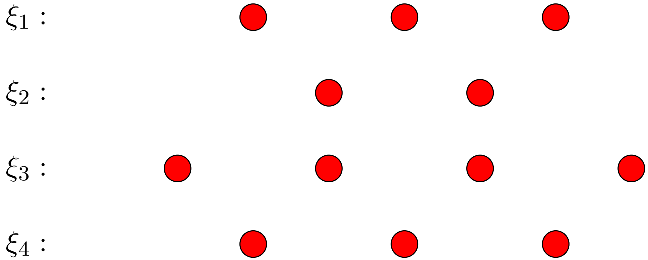
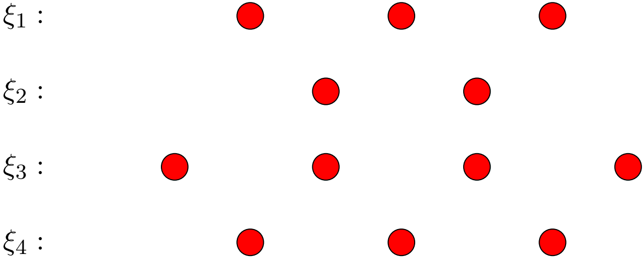


For an order r equation with n singular points, there are r^n combinations.



For an order r equation with n singular points, there are r^n combinations. **That's a lot.**

NEW

Our contribution (Johansson, MK, Mezzarobba; ISSAC'13):

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- Is based on the principle of *dynamic programming.*

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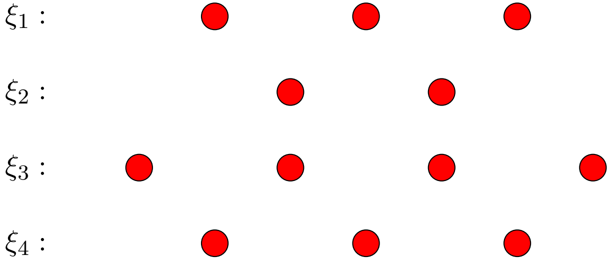
- Returns at most r candidates (instead of r^n).
- Needs at most $n^4 r$ arithmetic operations to find them.
- Is based on the principle of *dynamic programming.*
- Also requires *effective analytic continuation.*

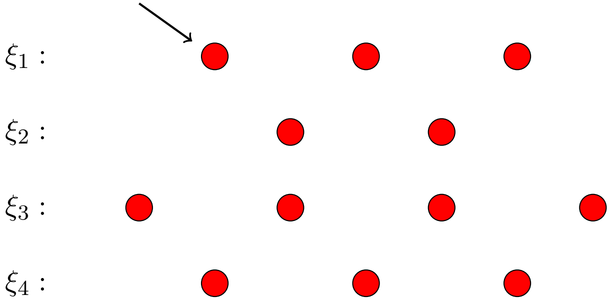
NEW



▸ Hyperexponential Solutions

NEW

vector space of all
series solution at ξ_1 with
a certain exponential part



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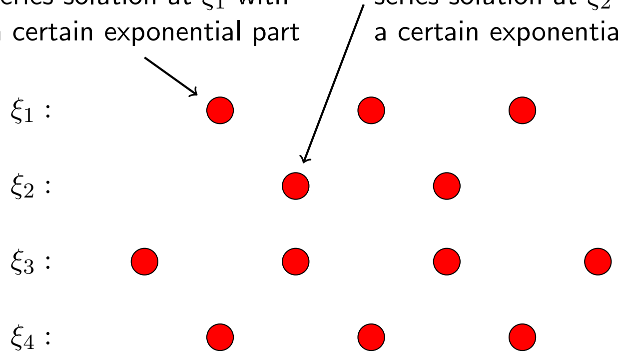

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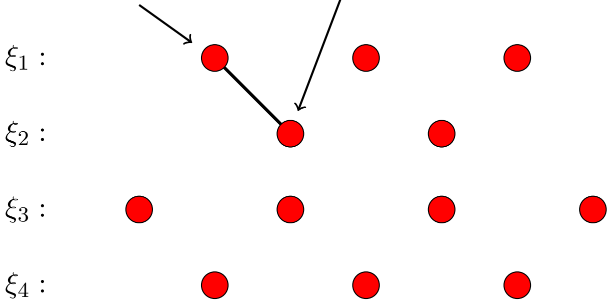

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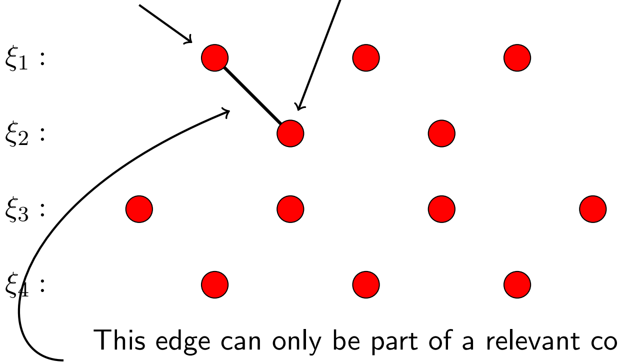
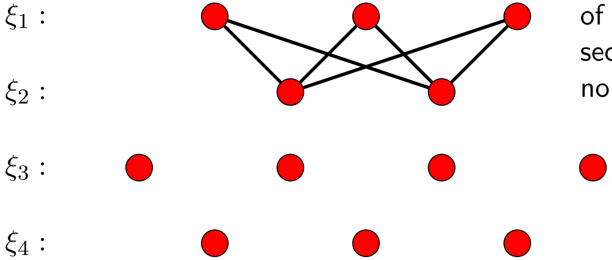


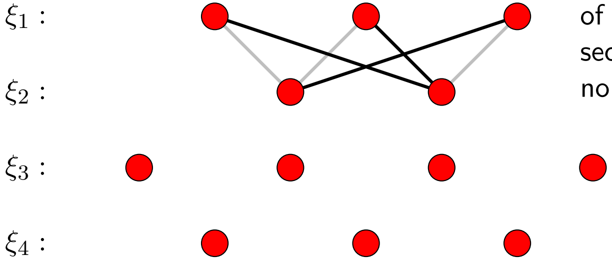
This edge can only be part of a relevant combination
if the intersection of the two vector spaces is nonempty

NEW



Fact. At most r of these $O(r^2)$ intersections can be nonempty.

NEW



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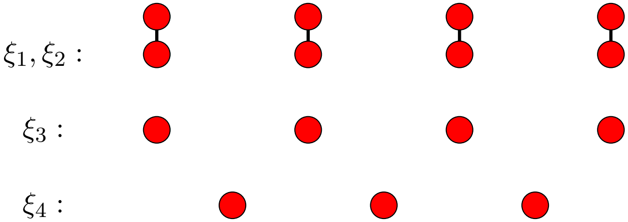

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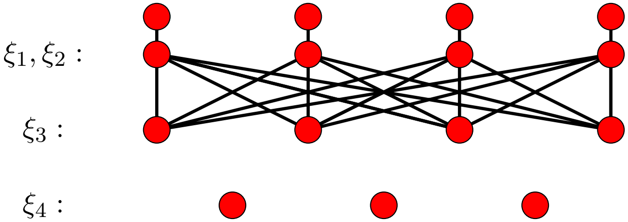

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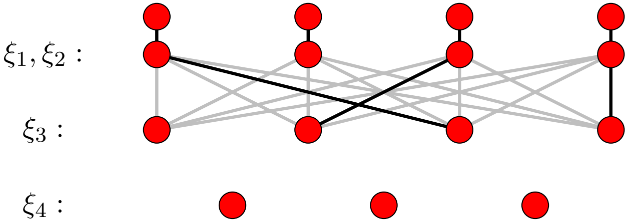

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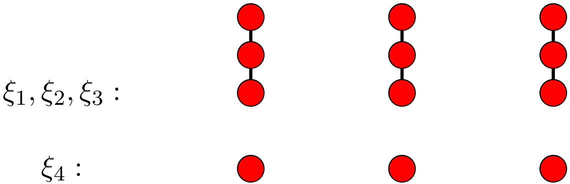
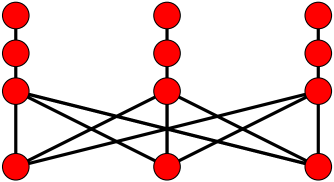


NEW

ξ_1, ξ_2, ξ_3 :

ξ_4 :



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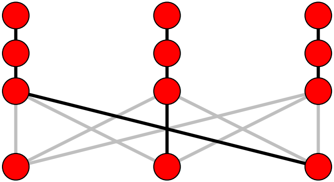
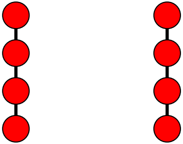


NEW

$\xi_1, \xi_2, \xi_3, \xi_4 :$



NEW

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A priori, spaces for different ξ_i are not comparable.

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Example: What is

$$\begin{aligned} &\Bigl[\exp\bigl(\tfrac{1}{x-1}\bigr)P_1(x-1), \exp\bigl(\tfrac{1}{x-1}\bigr)P_2(x-1)\Bigr] \\ \cap\ &\Bigl[\exp\bigl(\tfrac{1}{x-2}\bigr)Q_1(x-2), \exp\bigl(\tfrac{1}{x-2}\bigr)Q_2(x-2)\Bigr] \end{aligned}$$

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Idea: Interpret the series as asymptotic expansions of actual complex functions, and determine their expansions at some fixed common reference point using certified numerical approximation.

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Idea: Interpret the series as asymptotic expansions of actual complex functions, and determine their expansions at some fixed common reference point using certified numerical approximation.

This is not an easy thing to do, but efficient algorithms for this task are known.

NEW

Some possible meanings of “closed form”:

- polynomials ✓ e.g. $5x^2 + 3x - 2$
- rational functions ✓ e.g. $\bigl(5x-3\bigr)/\bigl(3x^2 - x + 5\bigr)$
- hyperexponential functions e.g. $\exp\bigl(\frac{2x+3}{x^2(x+1)}\bigr)\frac{(2x+5)^{1/3}}{(7x^2+x-3)^{1/2}}$
- ~~algebraic functions e.g. $x - \sqrt{x^2+1}$~~
- ~~elementary functions e.g. $\sin(x)/\sqrt{1+\log(1-e^x)}$~~
- ~~special functions e.g. $J_3(x^2+1) - {}_2F_1(2,3;1)(\frac{1}{x})$~~
- holonomic functions ✓

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- holonomic functions ✓