Finding hyperexponential solutions of differential equations

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$$\begin{aligned}(16x^4 + 48x^3 + 48x^2 + 18x + 2)f''(x)\\ - (16x^4 + 48x^3 + 52x^2 + 32x + 9)f'(x)\\ + (4x^2 + 14x + 7)f(x) = 0.\end{aligned}$$

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One unknown function $f(x)$

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- In the example: $f(x) = \exp(x)$ and $f(x) = \frac{\sqrt{1+3x+2x^2}}{x+1}$.

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Examples.

$$x^{\sqrt{2}}(x+1) \sim x^{\sqrt{2}+4}(x+1)^{-3} \qquad x^{\sqrt{2}} \not\sim x^2 \qquad x^2 \not\sim \exp(x).$$

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$$\begin{aligned}&(6x^5 - 60x^4 + 225x^3 - 386x^2 + 301x - 84)f(x)\\&\quad + (x-1)^2(10x^5 - 86x^4 + 277x^3 - 411x^2 + 272x - 59)f'(x)\\&\quad\quad + (x-2)^2(x-1)^4(2x^2 - 8x + 7)f''(x) = 0.\end{aligned}$$

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FIND: its hyperexponential solutions.

- In the example, there are two hyperexponential solutions $\exp\left(\frac{x-3}{(x-1)(x-2)}\right)$ and $\exp\left(\frac{1}{x-1}\right)\frac{x^3-3x^2+2x-1}{(x-1)^3}$. (Here, all solutions can be written as linear combinations of hyperexponential terms. In general, this is not possible.)

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No matter what $u(x)$ is, we have

$$f(x) = u(x)\exp\bigl(\tfrac{1}{x-1}\bigr)$$
$$f'(x) = \Bigl(u'(x) - \tfrac{1}{(x-1)^2}u(x)\Bigr)\exp\bigl(\tfrac{1}{x-1}\bigr)$$
$$f''(x) = \Bigl(u''(x) - \tfrac{2}{(x-1)^2}u'(x) + \tfrac{2x-1}{(x-1)^4}u(x)\Bigr)\exp\bigl(\tfrac{1}{x-1}\bigr), \text{etc.}$$

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Plug $f(x) = u(x)\exp\bigl(\frac{1}{x-1}\bigr)$ into the differential equation, divide by $\exp\bigl(\frac{1}{x-1}\bigr)$, and clear denominators.

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\begin{aligned}
&(x-2)^2(x-1)^4(2x^2-8x+7)u''(x) \\
&+(x-1)^2(10x^5-90x^4+309x^3-505x^2+392x-115)u'(x) \\
&-(8x^3-50x^2+92x-53)(x-1)u(x)=0.
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Find its rational solutions. This gives $u(x) = \frac{x^3-3x^2+2x-1}{(x-1)^3}$.

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These are series expansions of the form

$$\exp\Big(\frac{p(x)}{(x-\xi)^d}\Big)(x-\xi)^\alpha\Big(1+c_1(x-\xi)+c_2(x-\xi)^2+\cdots\Big),$$

where $d \in \mathbb{N}$, $p(x)$ is a polynomial of degree $< d$, and $\alpha, c_1, c_2, \ldots$ are constants.

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Example. For the ODE above and $\xi = 1$, we get

$$\exp\left(\tfrac{2}{x-1}\right)\Big(1 + (x-1) + \tfrac{3}{2}(x-1)^2 + \tfrac{13}{6}(x-1)^3 + \cdots\Big)$$
$$\exp\left(\tfrac{1}{x-1}\right)\Big((x-1)^{-3} + (x-1)^{-2} - 1 + \cdots\Big)$$

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Example. For the ODE above and $\xi = 2$, we get

$$\exp\left(\tfrac{-1}{x-2}\right)\Big(1 - 2(x-2) + 4(x-2)^2 - \tfrac{22}{3}(x-2)^2 + \cdots\Big)$$
$$\exp(0)\Big(1 - 6(x-2) + \tfrac{31}{2}(x-2)^2 - \tfrac{98}{3}(x-2)^3 + \cdots\Big)$$

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Fact. Nontrivial local exponential parts can only appear at points ξ where the leading coefficient of the differential equation is zero (aka *"singularities"*).

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- Solutions at $\xi = 1$:

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- Candidate exponential parts:

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- Solutions at $\xi = 2$:

$$\exp\left(\frac{-1}{x-2}\right)\left(1 - 2(x-2) + 4(x-2)^2 - \frac{22}{3}(x-2)^2 + \cdots\right)$$

$$\exp(0)\left(1 - 6(x-2) + \frac{31}{2}(x-2)^2 - \frac{98}{3}(x-2)^3 + \cdots\right)$$

- Candidate exponential parts:

$$\exp\left(\frac{2}{x-1} + \frac{-1}{x-2}\right),\ \exp\left(\frac{2}{x-1} + 0\right),\ \exp\left(\frac{1}{x-1} + \frac{-1}{x-2}\right),\ \exp\left(\frac{1}{x-1} + 0\right).$$

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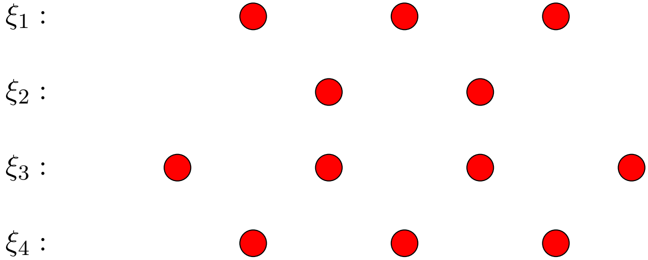
7. For each solution $u(x)$, output $f(x) = u(x)\, E$.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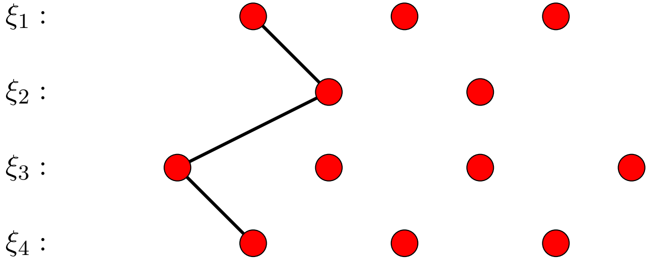
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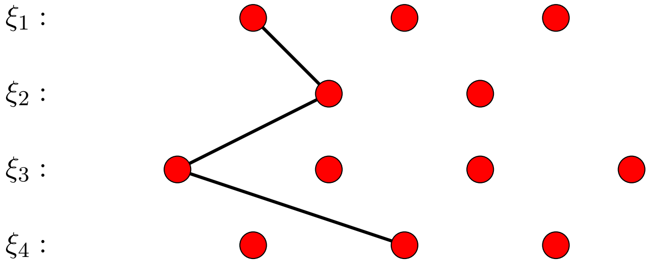
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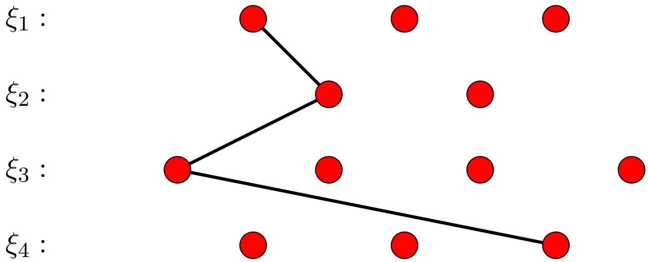
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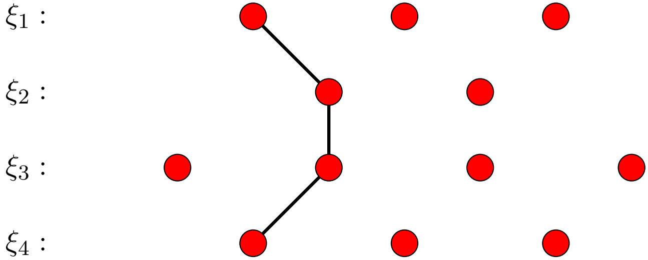
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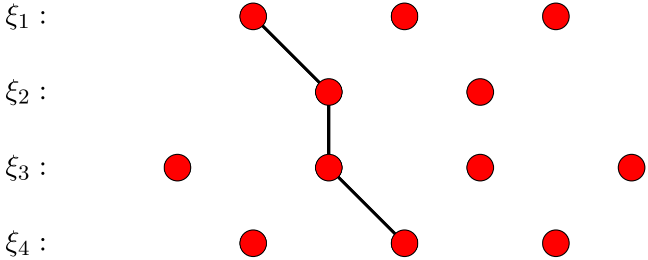

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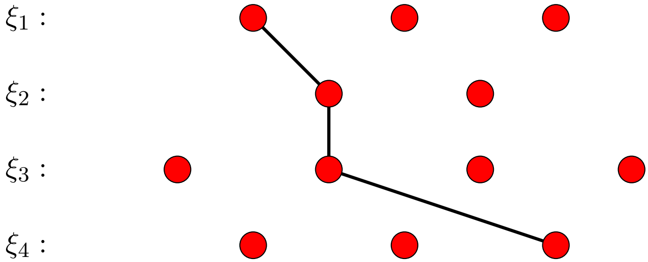
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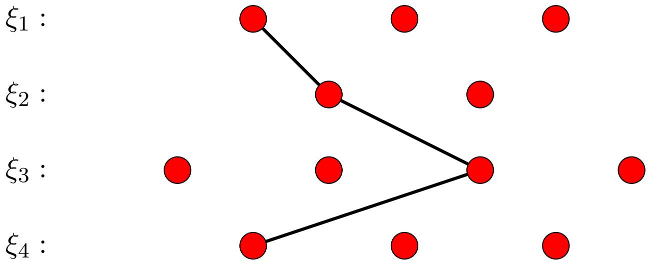

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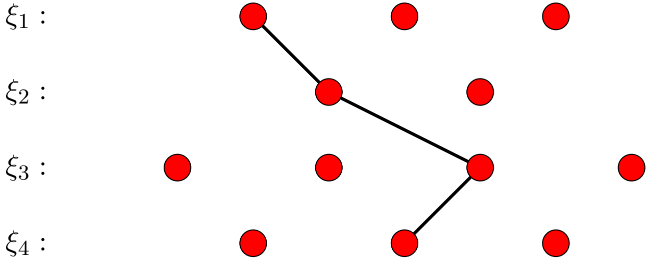
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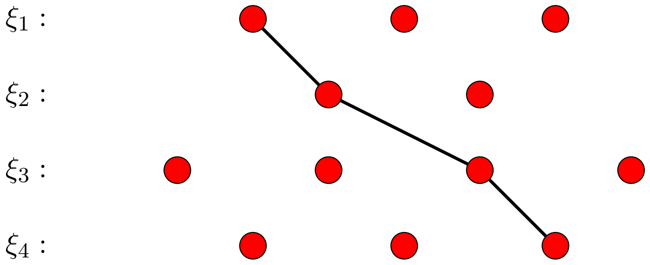

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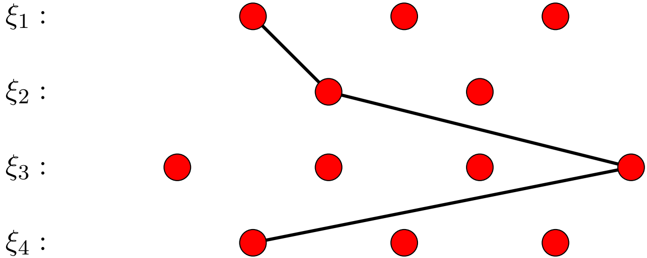
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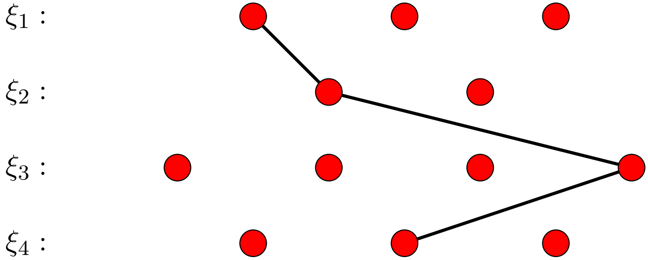
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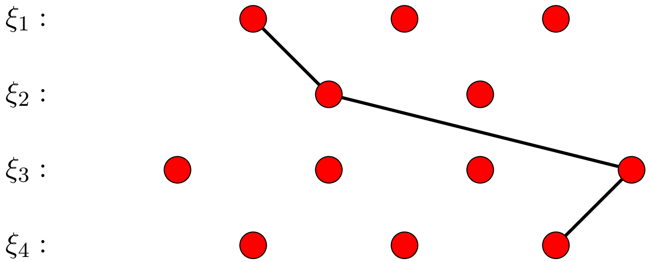
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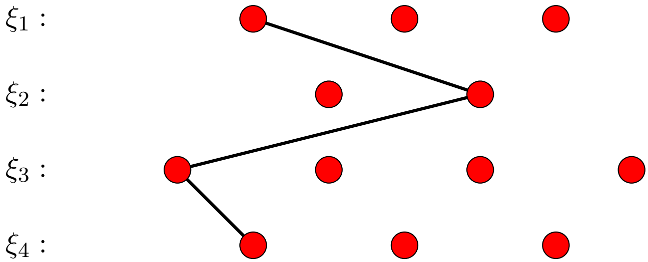
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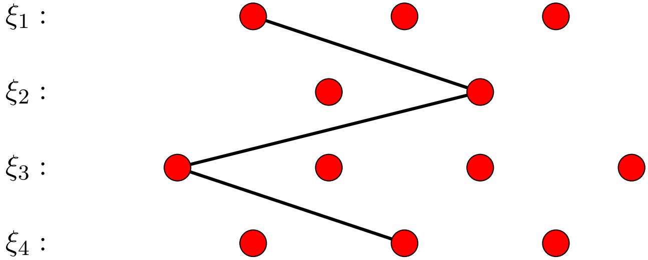
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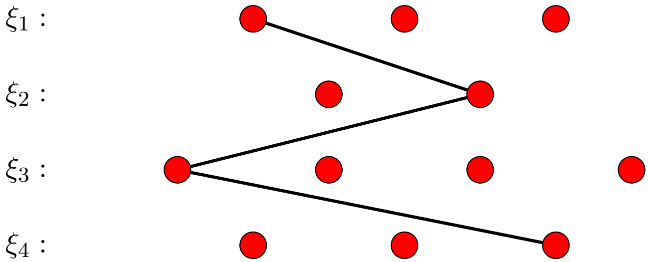
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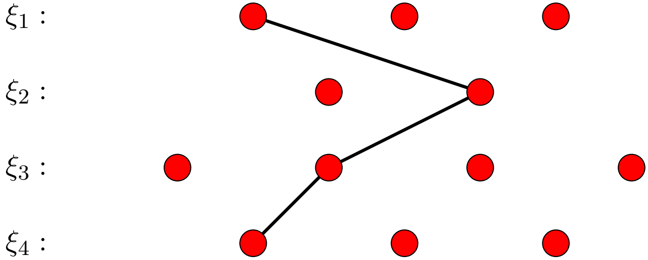
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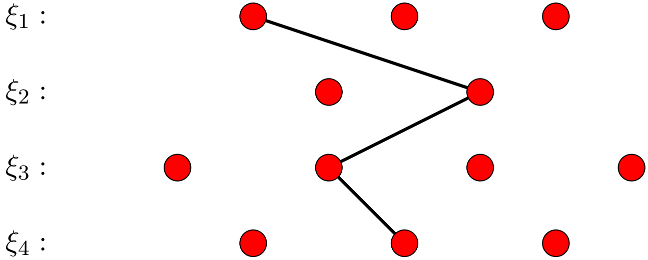
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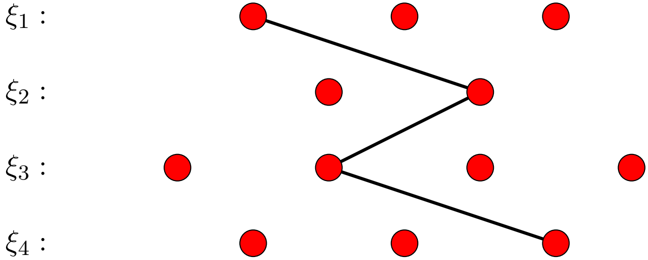
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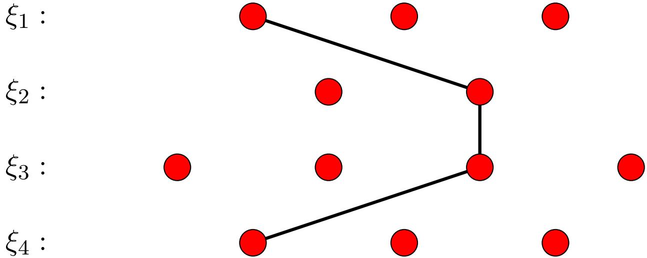
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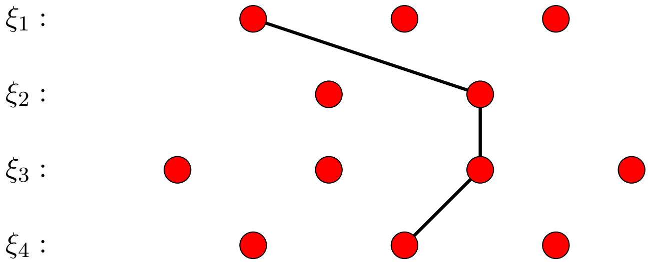
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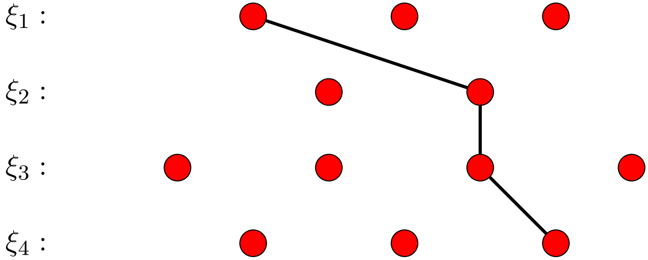
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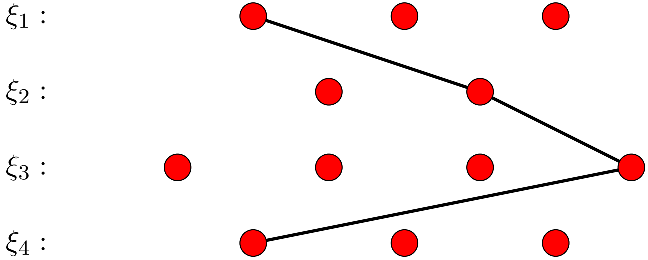
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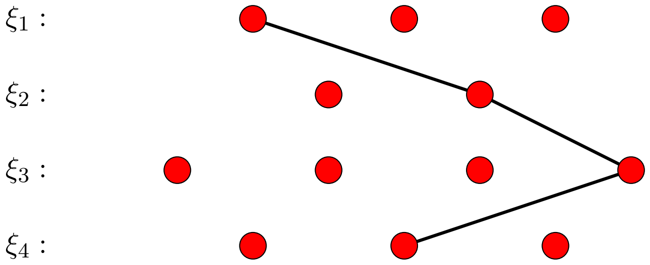
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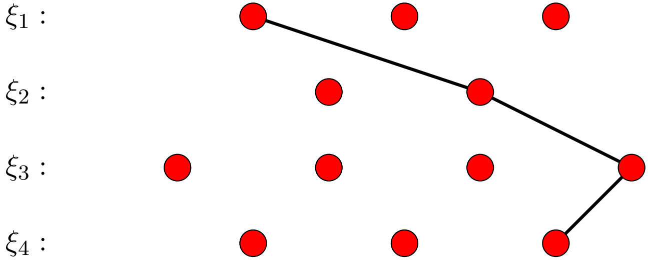

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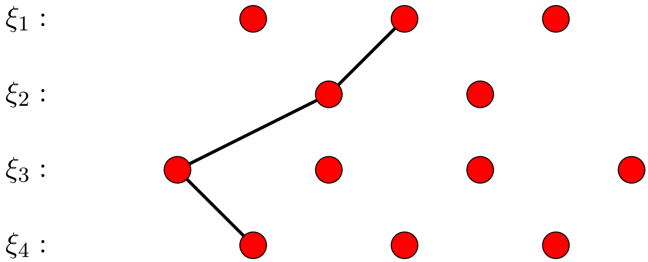
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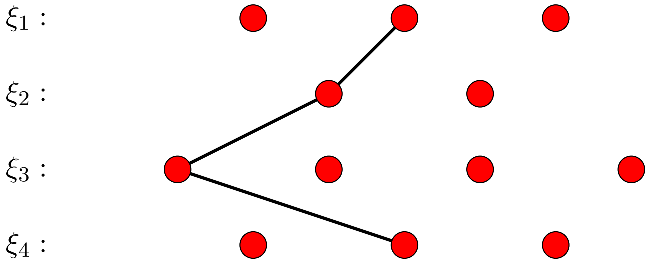
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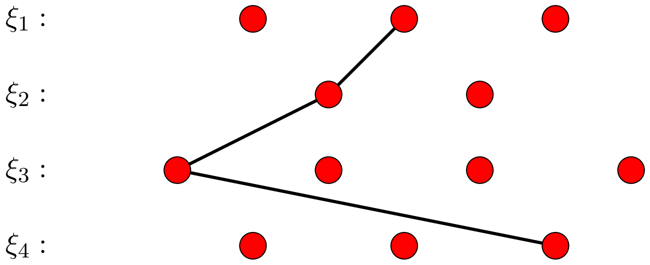
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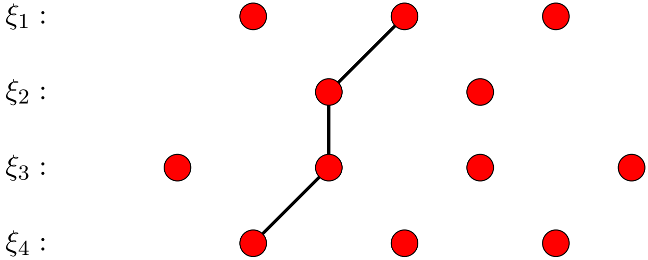
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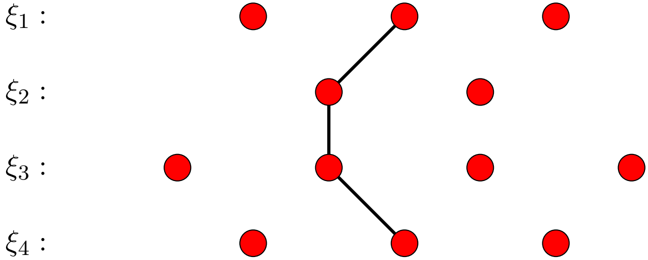
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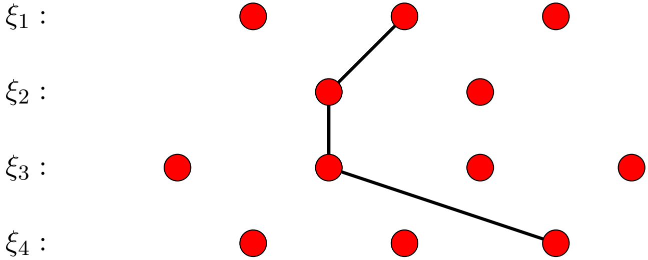
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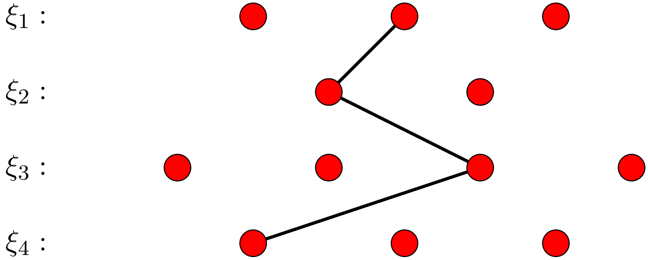
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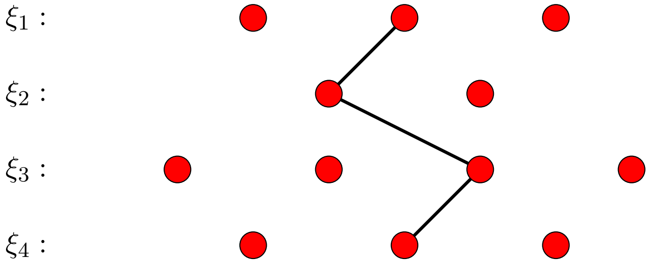
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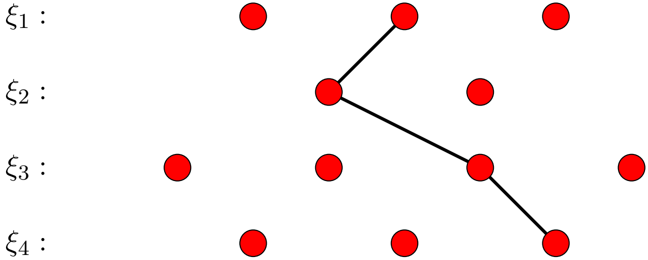

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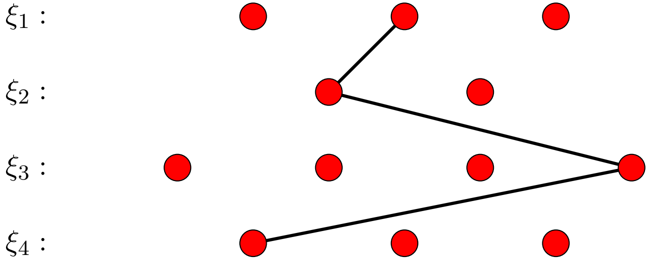
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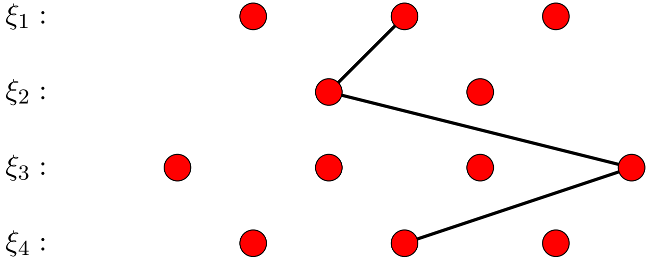
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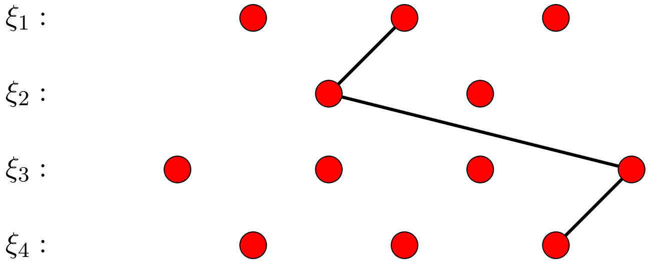
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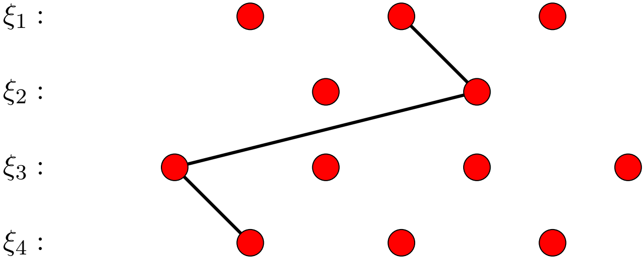
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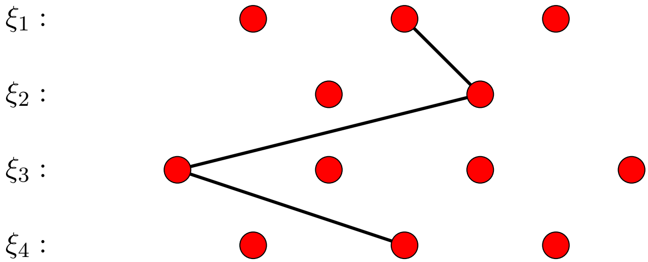
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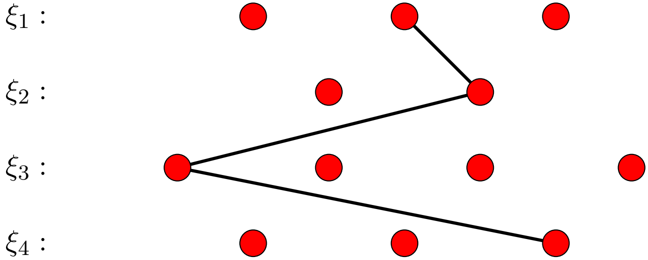
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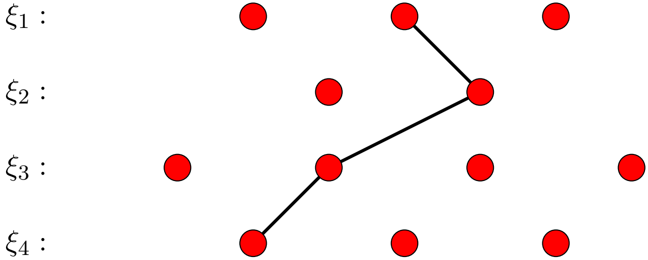
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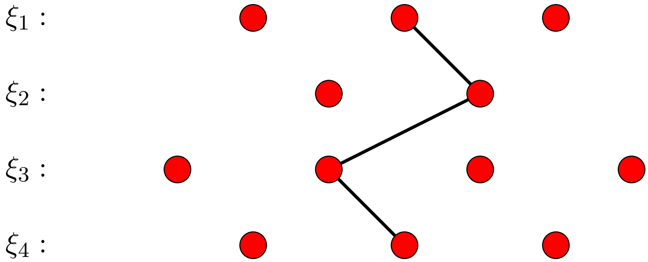
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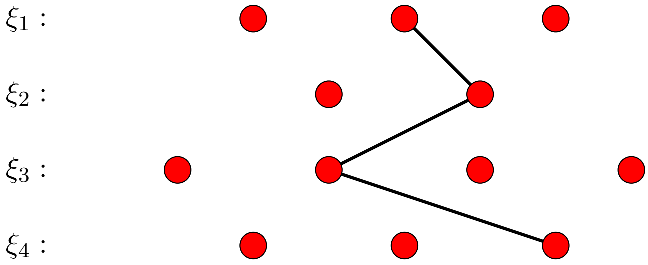
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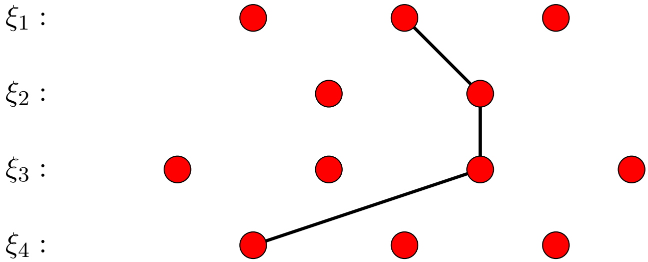

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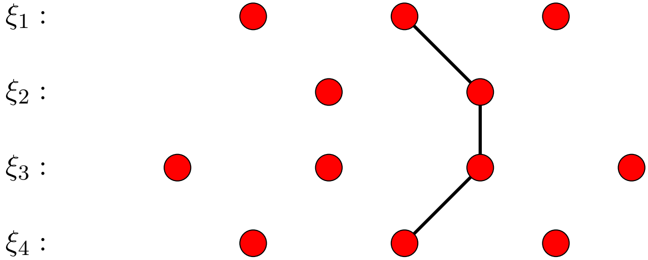
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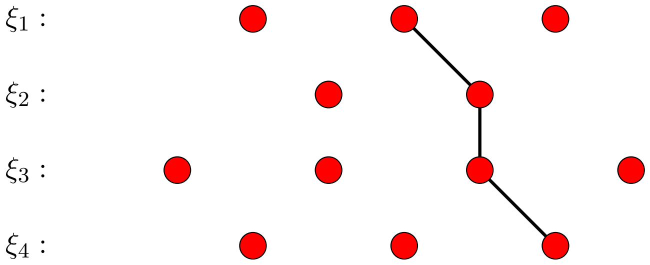
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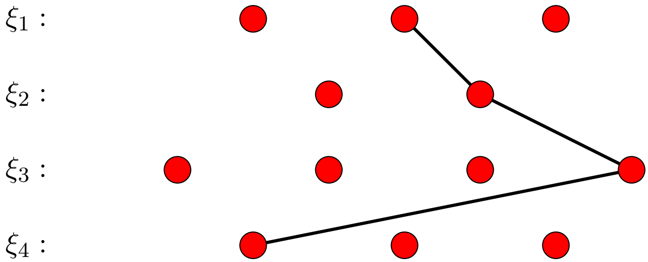
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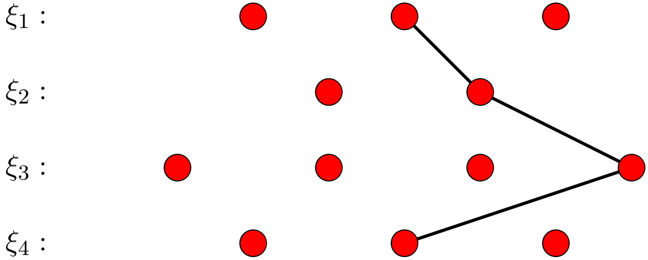
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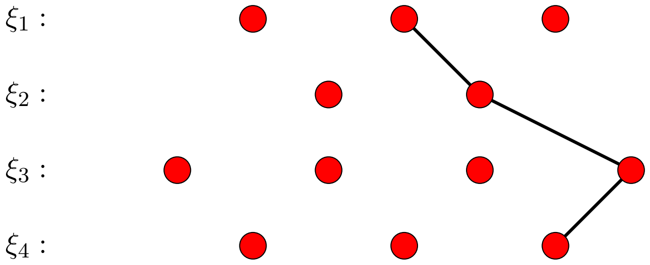
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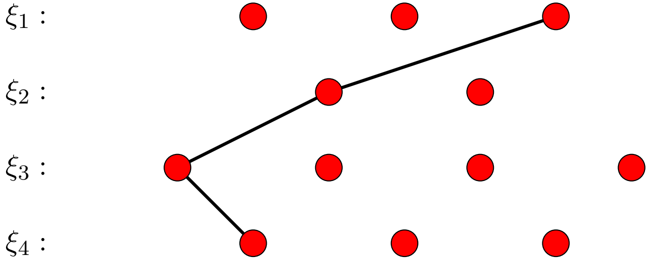

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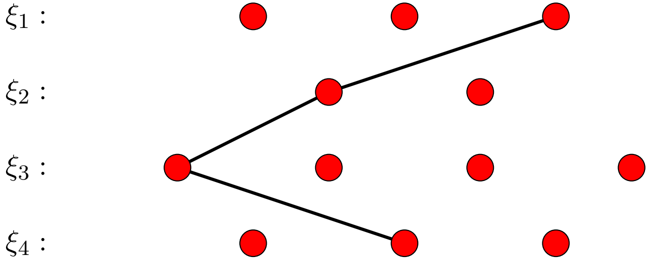

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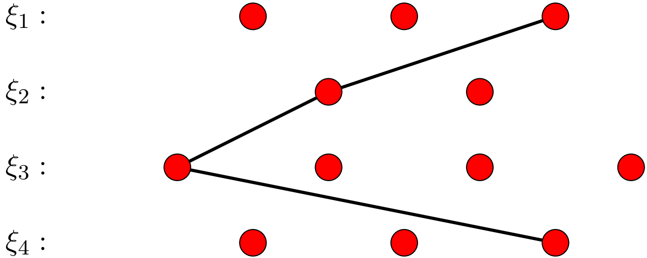
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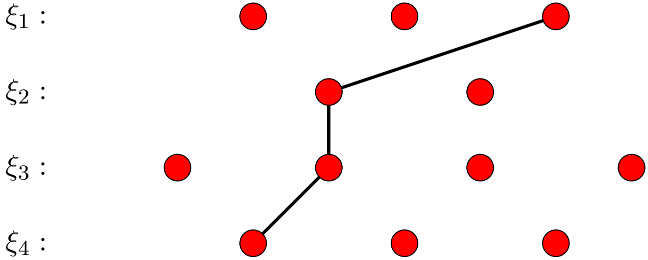

ξ_1 :
ξ_2 :
ξ_3 :
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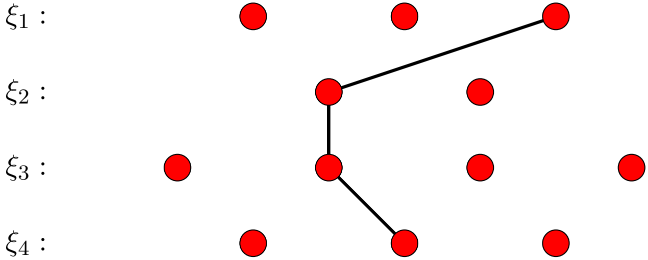

ξ_1 :
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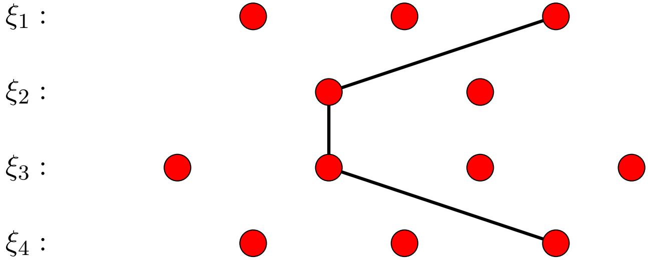
ξ_1 :
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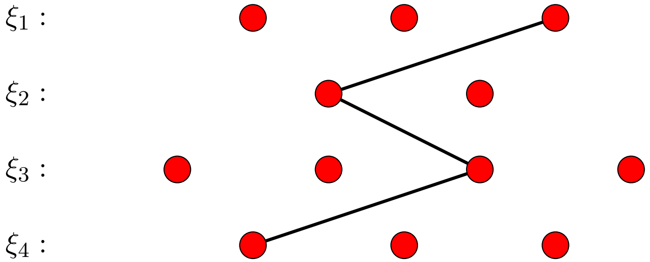
ξ_1 :
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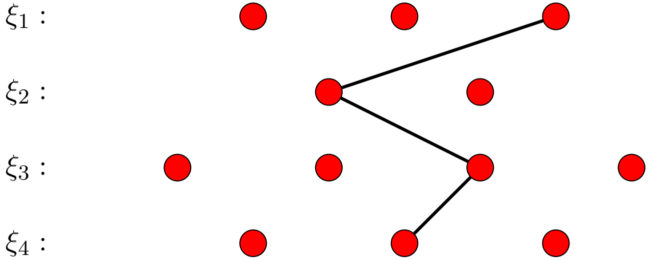
ξ_1 :
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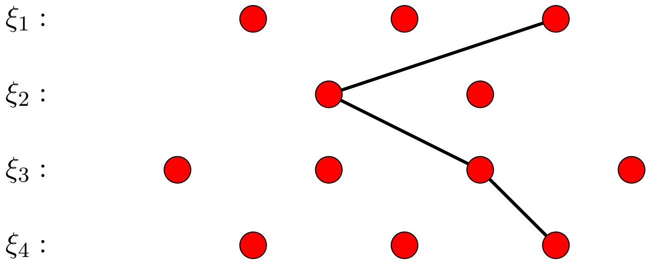
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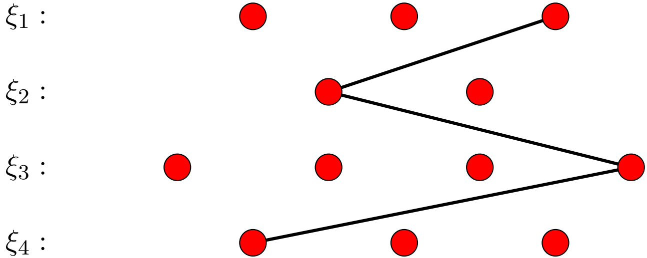
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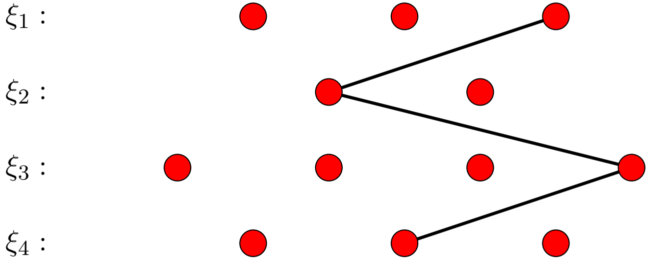
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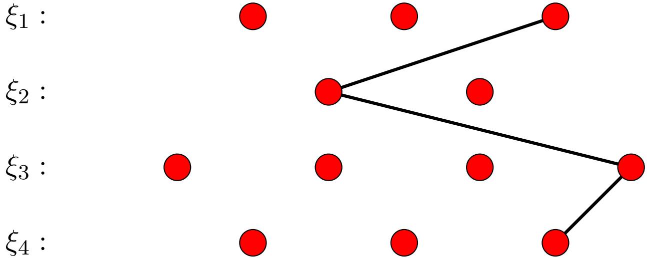
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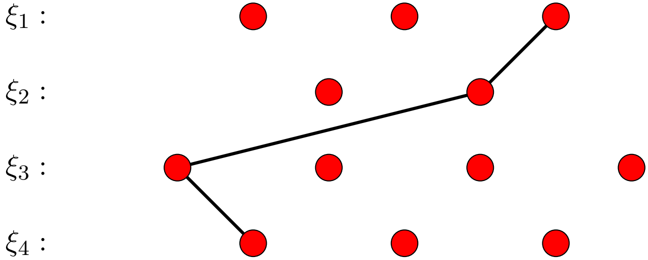
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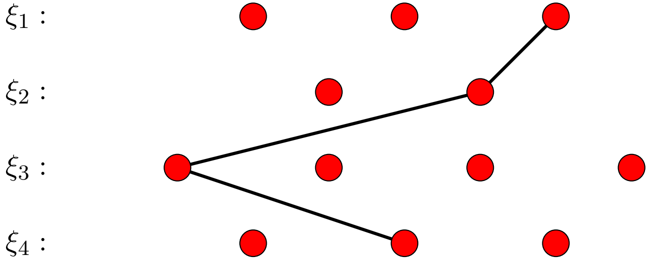
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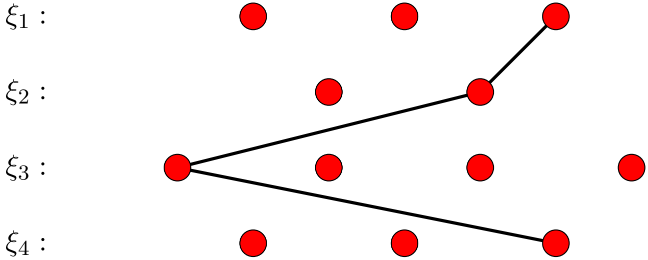
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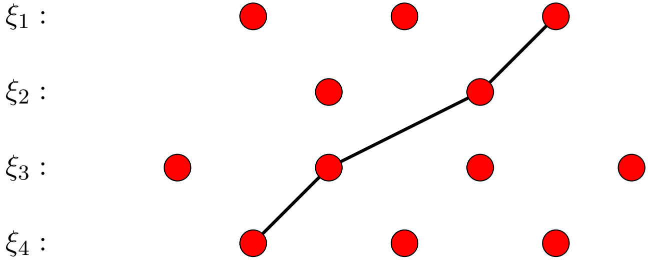
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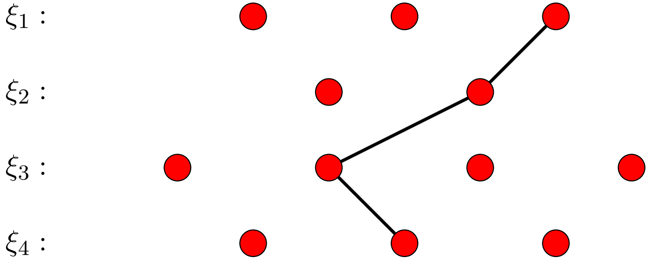
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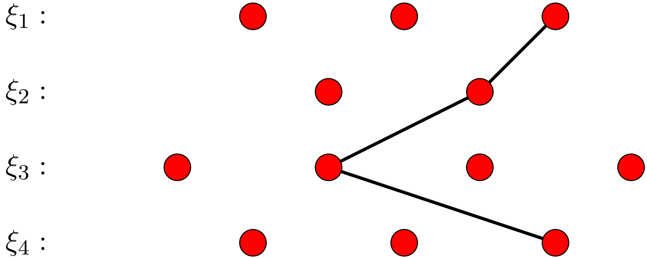
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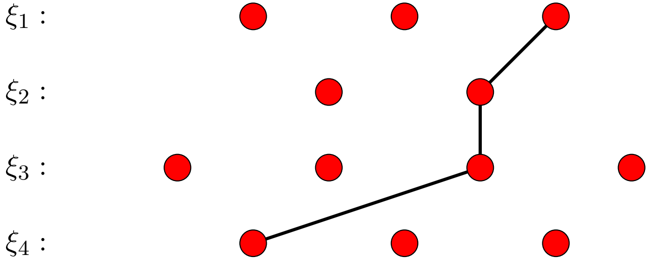
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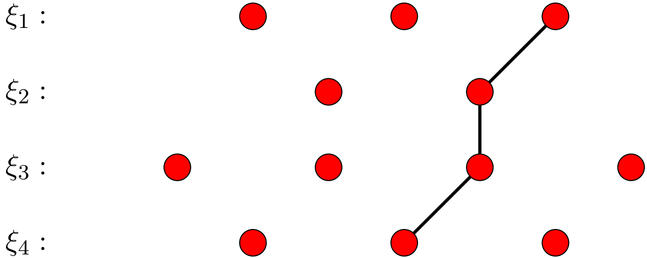
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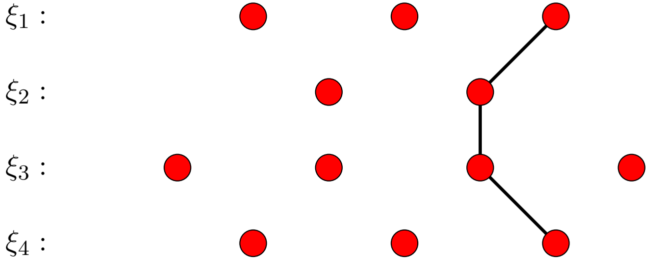
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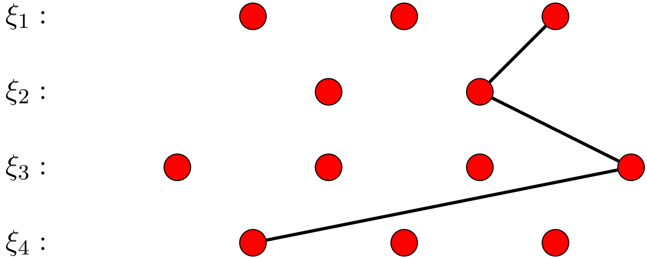
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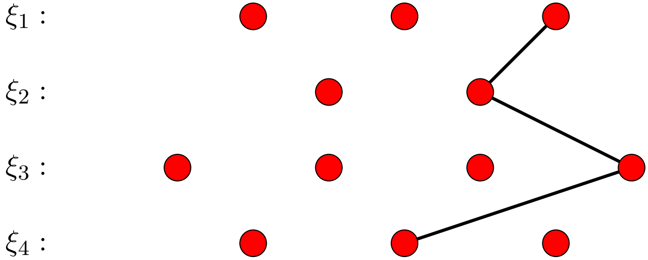
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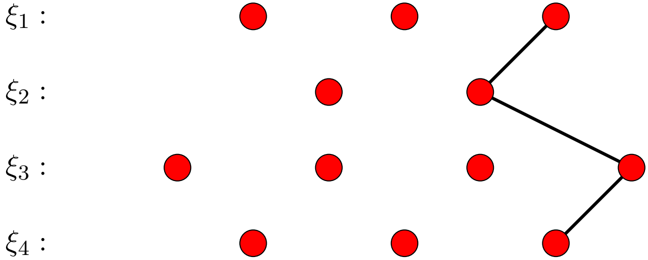
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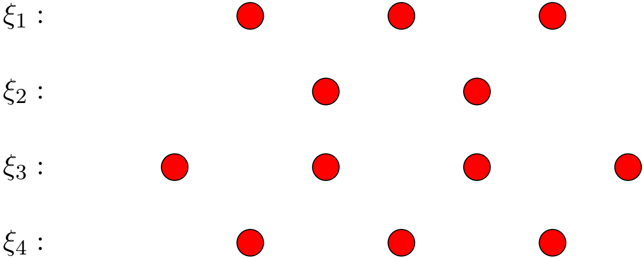
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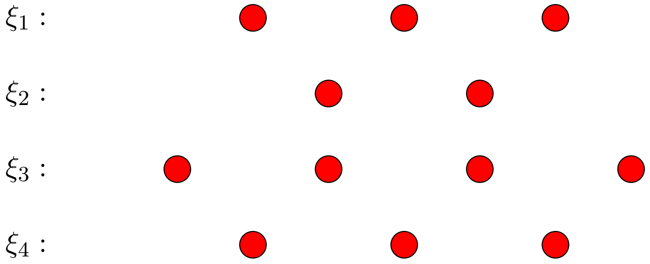
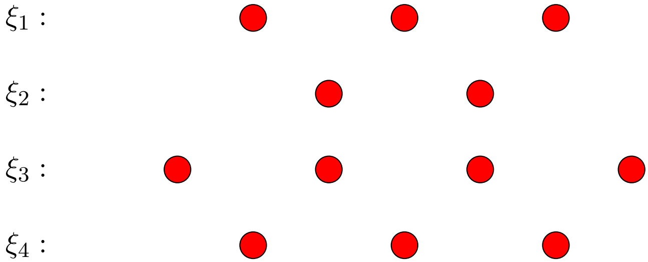


For an order r equation with n singular points, there are r^n combinations.



For an order r equation with n singular points, there are r^n combinations. **That's a lot.**

NEW

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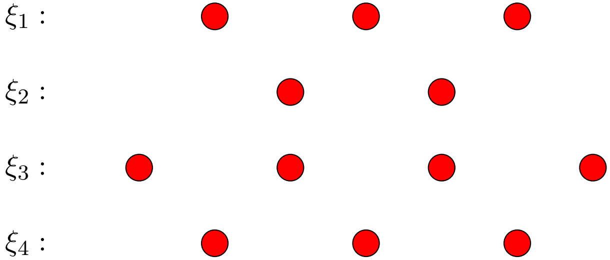
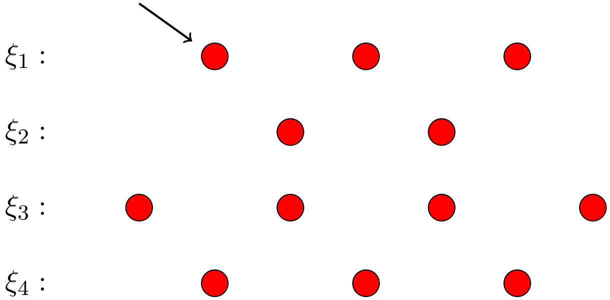
- Is based on the principle of *dynamic programming.*
- Also makes use of *effective analytic continuation.*

NEW



NEW

vector space of all
series solution at ξ_1 with
a certain exponential part



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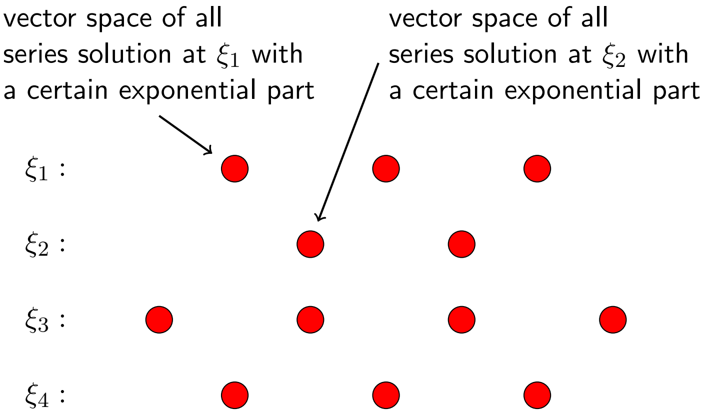
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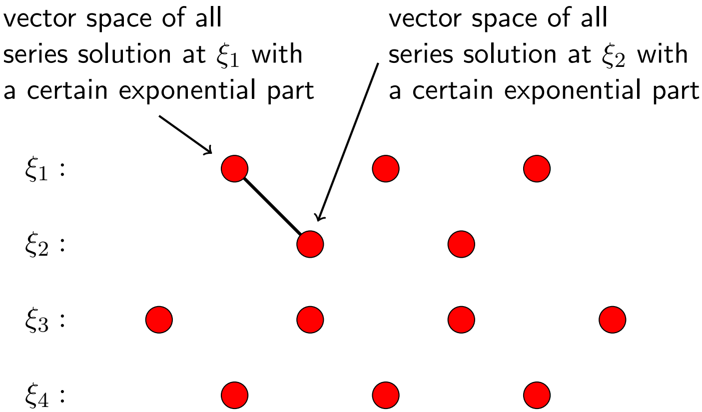
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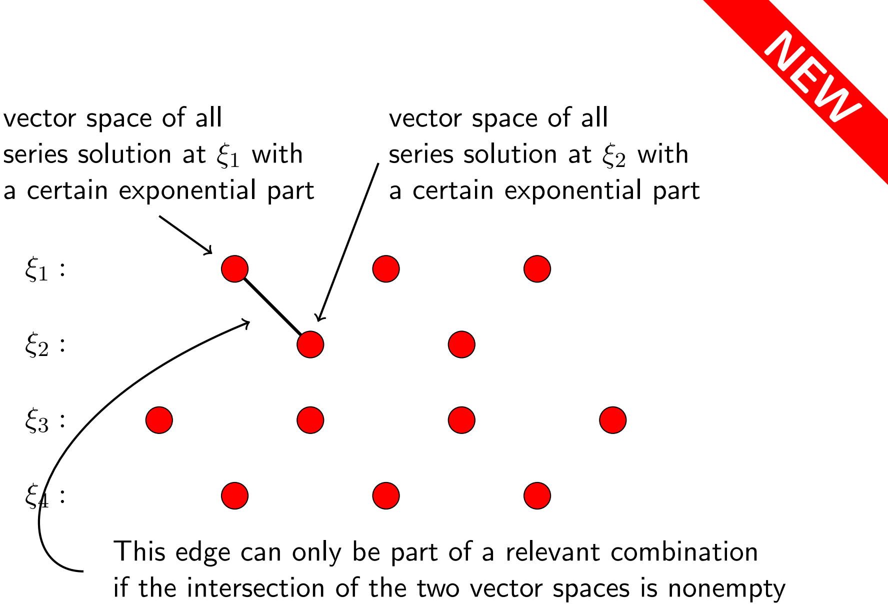
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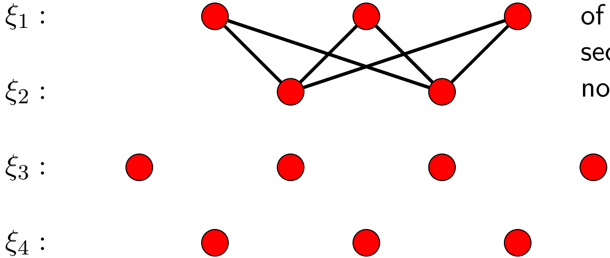
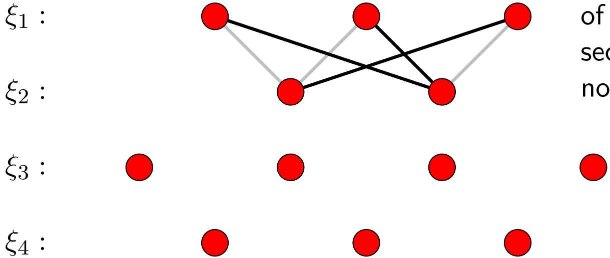
ξ_2 :
ξ_3 :
ξ_4 :
This edge can only be part of a relevant combination
if the intersection of the two vector spaces is nonempty



Fact. At most r of these $O(r^2)$ intersections can be nonempty.

NEW



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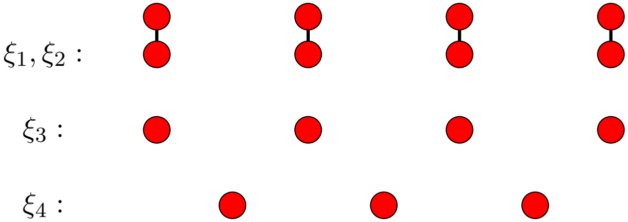

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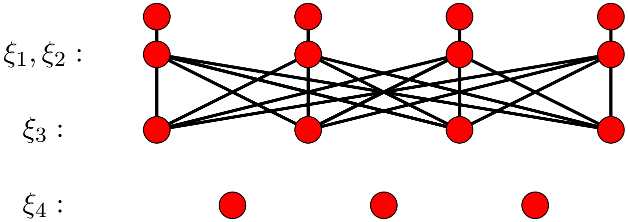
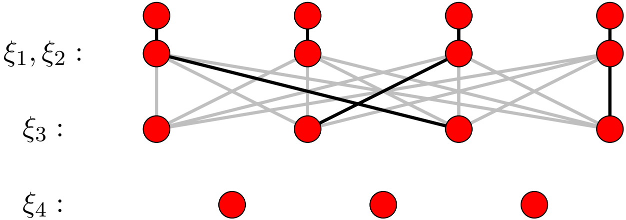


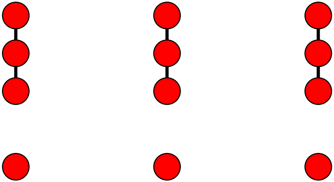

ξ_1, ξ_2 :
ξ_3 :
ξ_4 :

NEW

NEW

ξ_1, ξ_2, ξ_3 :

ξ_4 :



NEW

$\xi_1, \xi_2, \xi_3 :$

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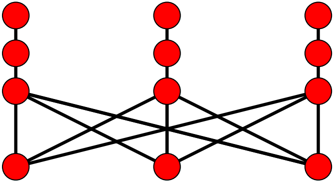

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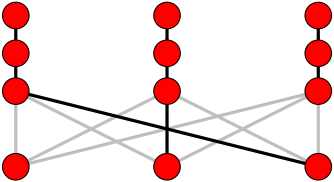

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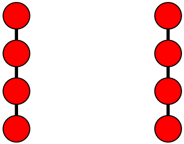

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Example: What is

$$\Big[\exp\big(\tfrac{1}{x-1}\big)P_1(x-1),\quad \exp\big(\tfrac{1}{x-1}\big)P_2(x-1)\Big]$$
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The (formal) series expansions at $\xi = 1$ and those at $\xi = 2$ don't live in the same ring.

NEW

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Fact. There is an algorithm for doing this.

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- For every generalized series solution $F(x-\xi)$ at ξ and (almost) every open sector $S \subseteq \mathbb{C}$ with vertex ξ there exist $r > 0$ and a unique analytic function $f\colon S \cap U_r(\xi) \to \mathbb{C}$ so that F is the asymptotic expansion of f for $x \to \xi$ within S.

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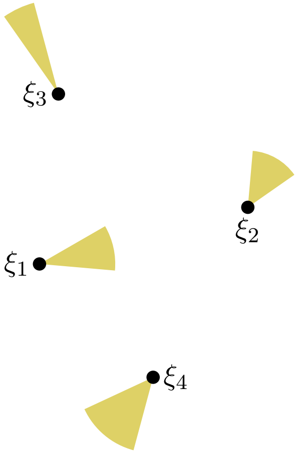
- Given F, S, a path P from ξ to some $z \in \mathbb{C}$ leaving ξ through S, and $N \in \mathbb{N}$, there is an algorithm due to J. van der Hoeven which computes the first N digits of the analytic continuation along P of f evaluated at z.



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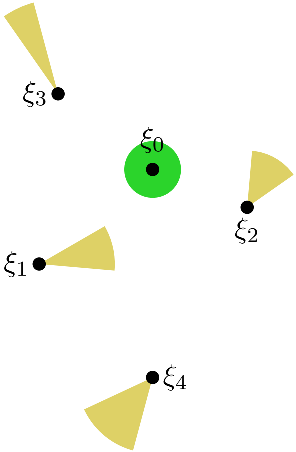

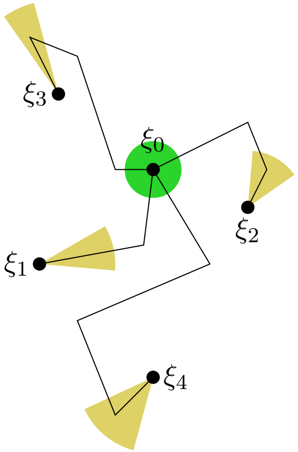

ξ_3
ξ_0
ξ_2
ξ_1
ξ_4

NEW

NEW

$$
\begin{aligned}
& \left[\exp\left(\tfrac{1}{x-1}\right)P_1(x-1), \quad \exp\left(\tfrac{1}{x-1}\right)P_2(x-1)\right] \\
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\end{aligned}
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NEW

$$\left[\tilde{P}_1(x-0), \quad \tilde{P}_2(x-0)\right]$$
$$\cap \;\left[\tilde{Q}_1(x-0), \quad \tilde{Q}_2(x-0)\right]$$

NEW

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Use interval arithmetic to do the linear algebra. Then:

- If $V \cap W$ appears to be empty, then it is **really** empty. The corresponding edge can be safely discarded.
- If $V \cap W$ appears to be non-empty, then it may be really empty or the precision was too low. Keep the corresponding edge, to be on the safe side.

Algorithm summary

NEW

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INPUT: A linear ordinary differential equation with polynomial coefficients.

OUTPUT: A short list of candidates for the exponential parts of its hyperexponential solutions.

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5. Return the resulting list of candidates.

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2. For each ξ_i, compute the exponential parts $E_{i,1}, E_{i,2}, \ldots$ of the local solutions at ξ_i.
3. For each combination $E := E_{1,j_1} E_{2,j_2} \cdots$ do:
4. Make an ansatz $f(x) = u(x)\,E$
5. Construct an auxiliary equation for $u(x)$
6. Find its rational solutions
7. For each solution $u(x)$, output $f(x) = u(x)\,E$.

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