Bounds for Creative Telescoping

Manuel Kauers

Based on joint work with Shaoshi Chen (Beijing), Christoph Koutschan (Linz), Lily Yen (Vancouver).

Task: Given a function $f(x, y)$,

Task: Given a function $f(x, y)$, find an operator of the form

$$L = T - \partial_y C$$

with $T \cdot f(x, y) = 0$, where

Task: Given a function $f(x, y)$, find an operator of the form

$$L = T - \partial_y C$$

with $T \cdot f(x, y) = 0$, where

- T is nonzero and free of y and ∂_y (the **telescoper**)

Task: Given a function $f(x, y)$, find an operator of the form

$$L = T - \partial_y C$$

with $T \cdot f(x, y) = 0$, where

- T is nonzero and free of y and ∂_y (the **telescoper**)
- C is an arbitrary operator (the **certificate**)

Task: Given a function $f(x,y)$, find an operator of the form

$$L = T - \partial_y C$$

with $T \cdot f(x,y) = 0$, where

- T is nonzero and free of y and ∂_y (the **telescoper**)
- C is an arbitrary operator (the **certificate**)
- ∂_y is a prescribed generator of the operator algebra (e.g., $\partial_y = \frac{d}{dy}$ "anti-integration" for integration problems, or $\partial_y = \sigma_y - 1$ "anti-summation" for summation problems)

Task: Given a function $f(x, y)$, find an operator of the form

$$L = T - \partial_y C$$

with $T \cdot f(x, y) = 0$, where

- T is nonzero and free of y and ∂_y (the **telescoper**)
- C is an arbitrary operator (the **certificate**)
- ∂_y is a prescribed generator of the operator algebra (e.g., $\partial_y = \frac{d}{dy}$ "anti-integration" for integration problems, or $\partial_y = \sigma_y - 1$ "anti-summation" for summation problems)

Why? Because such operators are useful for summation and integration ($\rightarrow$ talks of C. Koutschan or N. Takayama earlier today)

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$$

Example: For $f(x, y) = \frac{1}{3x+y^2} \exp(-xy)$ we can take

$$T = 4x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$$

Then $(T - \frac{d}{dy} C) \cdot f(x, y) = 0$ implies

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$$

Then $(T - \frac{d}{dy}C) \cdot f(x,y) = 0$ implies

$$T \cdot f(x,y) \quad = \quad \frac{d}{dy}(C \cdot f(x,y))$$

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$

Then $(T - \frac{d}{dy}C) \cdot f(x,y) = 0$ implies

$$\int_0^1 T \cdot f(x,y)\,dy = \int_0^1 \frac{d}{dy}(C \cdot f(x,y))\,dy$$

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$

Then $(T - \frac{d}{dy}C) \cdot f(x,y) = 0$ implies

$$T \cdot \int_0^1 f(x,y)\,dy = \int_0^1 \frac{d}{dy}(C \cdot f(x,y))\,dy$$

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$

Then $(T - \frac{d}{dy}C)\cdot f(x,y) = 0$ implies

$$T \cdot \int_0^1 f(x,y)\,dy = \Big[C \cdot f(x,y)\Big]_{y=0}^{1}$$

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$$

Then $(T - \frac{d}{dy}C) \cdot f(x,y) = 0$ implies

$$T \cdot \int_0^1 f(x,y)\,dy = \frac{(81x^2+39x-8)x-2}{(3x+1)^2}\exp(-x) - 9x$$

Example: For $f(x,y) = \frac{1}{3x+y^2}\exp(-xy)$ we can take

$$T = 4x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + (27x^3 - 2), \quad C = \frac{81x^3 - 12xy + 39x^2y^2 - 2y^3 + 4xy^4}{3x+y^2}$$

Then $(T - \frac{d}{dy}C) \cdot f(x,y) = 0$ implies

$$T \cdot \int_0^1 f(x,y)\,dy = \frac{(81x^2+39x-8)x-2}{(3x+1)^2}\exp(-x) - 9x$$

So we obtain an explicit (inhomogeneous) linear differential equation with respect to x for the integral $\int_0^1 f(x,y)dy$.

Typical classes of functions $f(x, y)$:

- $f(x, y)$ is called *hyperexponential* if it can be written in the form

$$f(x, y) = c_0(x, y) \exp\Big(\frac{a(x, y)}{b(x, y)}\Big) \prod_{i=1}^{m} c_i(x, y)^{e_i}$$

  for certain polynomials $a, b, c_0, c_1, \dots, c_m$ and constants $e_1, \dots, e_m$ (not necessarily integers).

Typical classes of functions $f(x, y)$:

- $f(x, y)$ is called *hyperexponential* if it can be written in the form

$$f(x, y) = c_0(x, y) \exp\Big(\frac{a(x, y)}{b(x, y)}\Big) \prod_{i=1}^{m} c_i(x, y)^{e_i}$$

  for certain polynomials $a, b, c_0, c_1, \dots, c_m$ and constants $e_1, \dots, e_m$ (not necessarily integers).
- *Example:* $f(x, y) = \frac{1}{3x+y^2} \exp(-xy)$

Typical classes of functions $f(x, y)$:

Typical classes of functions $f(x,y)$:

- $f(x,y)$ is called *(proper) hypergeometric* if it can be written in the form

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

  for a certain polynomial c, certain constants $p, q, a_i'', b_i'', u_i'', v_i''$ and certain fixed nonnegative integers $a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'$.

Typical classes of functions $f(x, y)$:

- $f(x, y)$ is called *(proper) hypergeometric* if it can be written in the form

$$f(x, y) = c(x, y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a'_i y + a''_i)\Gamma(b_i x - b'_i y + b''_i)}{\Gamma(u_i x + u'_i y + u''_i)\Gamma(v_i x - v'_i y + v''_i)}$$

  for a certain polynomial c, certain constants $p, q, a''_i, b''_i, u''_i, v''_i$ and certain fixed nonnegative integers $a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i$.
- *Example:* $f(x, y) = (x + y)2^x(-1)^y \frac{(x+y)!(2x-y)!(2x-2y)!}{(x+2y)!^2}$

Typical classes of functions $f(x,y)$:

- $f(x,y)$ is called *D-finite* if there exists an operator algebra $\mathbb{A} = K(x,y)[\partial_x, \partial_y]$ acting on $f(x,y)$ and the left ideal

$$\operatorname{ann}(f) := \{L \in \mathbb{A} : L \cdot f = 0\}$$

is such that the quotient algebra $\mathbb{A}/\operatorname{ann}(f)$ is a finite dimensional $K(x,y)$-vector space.

Typical classes of functions $f(x,y)$:

- $f(x,y)$ is called *D-finite* if there exists an operator algebra $\mathbb{A} = K(x,y)[\partial_x, \partial_y]$ acting on $f(x,y)$ and the left ideal

  $$\operatorname{ann}(f) := \{L \in \mathbb{A} : L \cdot f = 0\}$$

  is such that the quotient algebra $\mathbb{A}/\operatorname{ann}(f)$ is a finite dimensional $K(x,y)$-vector space.

  This class contains the previous two as special cases, and it covers **many** additional functions.

Typical classes of functions $f(x, y)$:

- $f(x, y)$ is called *D-finite* if there exists an operator algebra $\mathbb{A} = K(x, y)[\partial_x, \partial_y]$ acting on $f(x, y)$ and the left ideal

  $$\operatorname{ann}(f) := \{L \in \mathbb{A} : L \cdot f = 0\}$$

  is such that the quotient algebra $\mathbb{A}/\operatorname{ann}(f)$ is a finite dimensional $K(x, y)$-vector space.

  This class contains the previous two as special cases, and it covers **many** additional functions.

  D-finite is closely related to *holonomic* ($\rightarrow$ talks of C. Koutschan or N. Takayama earlier today)

Main Question in Today's Talk:

What can we say about the **size** of T for a specific function $f(x, y)$ without computing it?

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$
\begin{aligned}
T = {} & (34045 + 60101x - 15377x^2) \\
& + (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\
& + (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^4)\partial_x^2 \\
& + (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^4)\partial_x^3 \\
& + (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^4)\partial_x^4 \\
& + (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\
& + (41662 - 44140x + 13204x^2)\partial_x^6
\end{aligned}
$$

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$\begin{aligned} T = &(34045 + 60101x - 15377x^2) \\ &+ (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\ &+ (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^4)\partial_x^2 \\ &+ (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^4)\partial_x^3 \\ &+ (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^4)\partial_x^4 \\ &+ (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\ &+ (41662 - 44140x + 13204x^2)\partial_x^6 \end{aligned}$$

- $\operatorname{ord}(T)$:= maximal ∂_x-exponent in T ("order")

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$\begin{aligned} T = {} & (34045 + 60101x - 15377x^2) \\ & + (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\ & + (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^4)\partial_x^2 \\ & + (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^4)\partial_x^3 \\ & + (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^4)\partial_x^4 \\ & + (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\ & + (41662 - 44140x + 13204x^2)\partial_x^{6} \end{aligned}$$

- $\mathrm{ord}(T)$:= maximal ∂_x-exponent in T ("order")

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$\begin{aligned} T = &(34045 + 60101x - 15377x^2) \\ &+ (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\ &+ (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^4)\partial_x^2 \\ &+ (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^4)\partial_x^3 \\ &+ (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^4)\partial_x^4 \\ &+ (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\ &+ (41662 - 44140x + 13204x^2)\partial_x^6 \end{aligned}$$

- $\mathrm{ord}(T)$:= maximal ∂_x-exponent in T ("order")
- $\deg(T)$:= maximal x-exponent in T ("degree")

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$
\begin{aligned}
T = {} & (34045 + 60101x - 15377x^2) \\
& + (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\
& + (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^{4})\partial_x^2 \\
& + (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^{4})\partial_x^3 \\
& + (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^{4})\partial_x^4 \\
& + (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\
& + (41662 - 44140x + 13204x^2)\partial_x^6
\end{aligned}
$$

- $\mathrm{ord}(T)$:= maximal ∂_x-exponent in T ("order")
- $\deg(T)$:= maximal x-exponent in T ("degree")

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$\begin{aligned} T = {} & (34045 + 60101x - 15377x^2) \\ & + (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\ & + (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^4)\partial_x^2 \\ & + (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^4)\partial_x^3 \\ & + (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^4)\partial_x^4 \\ & + (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\ & + (41662 - 44140x + 13204x^2)\partial_x^6 \end{aligned}$$

- $\mathrm{ord}(T) :=$ maximal ∂_x-exponent in T ("order")
- $\deg(T) :=$ maximal x-exponent in T ("degree")
- $\mathrm{ht}(T) :=$ maximal integer appearing in T ("height")

How to measure the "size" of an operator $T \in \mathbb{Z}[x][\partial_x]$?

$$\begin{aligned} T = {} & (34045 + 60101x - 15377x^2) \\ & + (-68071 - 62604x - 93961x^2 + 54058x^3)\partial_x \\ & + (-35079 - 54446x + 5324x^2 + 94790x^3 + 55527x^4)\partial_x^2 \\ & + (92795 + 13448x - 97390x^2 - 81011x^3 + 55462x^4)\partial_x^3 \\ & + (-86626 + 83267x + 82406x^2 - 76639x^3 + 29278x^4)\partial_x^4 \\ & + (-96781 + 45676x + 40203x^2 + 59197x^3)\partial_x^5 \\ & + (41662 - 44140x + 13204x^2)\partial_x^6 \end{aligned}$$

- $\mathrm{ord}(T)$:= maximal ∂_x-exponent in T ("order")
- $\deg(T)$:= maximal x-exponent in T ("degree")
- $\mathrm{ht}(T)$:= maximal integer appearing in T ("height")

problem	order	degree	height
D-finite closure properties			
hypergeometric summation			
hyperexponential integration			
holonomic summation/integration			

problem	order	degree	height
D-finite closure properties			
hypergeometric summation			
hyperexponential integration			
holonomic summation/integration			

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper T with

$$\operatorname{ord}(T) \leq \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper T with

$$\operatorname{ord}(T) \leq \max\Big\{\sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i')\Big\}$$

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper T with

$$\operatorname{ord}(T) \leq \max\Big\{\sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i')\Big\}$$

Usually there is no telescoper of lower order.

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓		
hyperexponential integration			
holonomic summation/integration			

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	√		
hyperexponential integration			
holonomic summation/integration			

Theorem (Apagodu-Zeilberger) For every (non-rational) hyperexponential term

$$f(x,y) = c_0(x,y)\exp\Big(\frac{a(x,y)}{b(x,y)}\Big)\prod_{i=1}^{m} c_i(x,y)^{e_i}$$

there exists a telescoper T with

$$\operatorname{ord}(T) \leq \deg_y(b) + \max\{\deg_y(a), \deg_y(b)\} + \sum_{i=1}^{m} \deg_y(c_i),$$

Theorem (Apagodu-Zeilberger) For every (non-rational) hyperexponential term

$$f(x,y) = c_0(x,y)\exp\Bigl(\frac{a(x,y)}{b(x,y)}\Bigr)\prod_{i=1}^{m} c_i(x,y)^{e_i}$$

there exists a telescoper T with

$$\operatorname{ord}(T) \leq \deg_y(b) + \max\{\deg_y(a), \deg_y(b)\} + \sum_{i=1}^{m} \deg_y(c_i),$$

The first $\deg_y(b)$ can be replaced by $\deg_y(\operatorname{sqfp}_y(b))$. That changed, there is usually no telescoper of lower order.

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓		
hyperexponential integration	✓		
holonomic summation/integration			

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓		
hyperexponential integration	✓		
holonomic summation/integration			

Theorem (Apagodu-Zeilberger)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper T with

$$\operatorname{ord}(T) \leq \max\left\{\sum_{i=1}^{m}(a_i' + v_i'),\ \sum_{i=1}^{m}(u_i' + b_i')\right\}$$

Theorem (Apagodu-Zeilberger; Chen-Kauers)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper T with

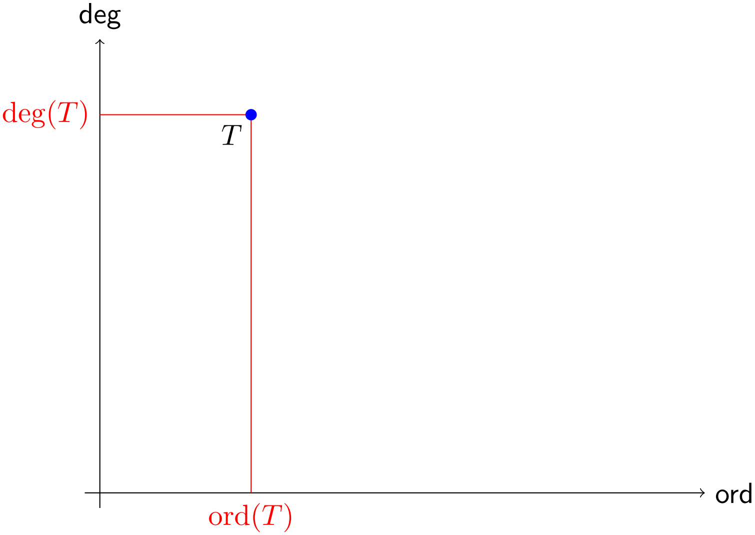
$$\operatorname{ord}(T) \le \max\Big\{\sum_{i=1}^{m}(a_i' + v_i'),\ \sum_{i=1}^{m}(u_i' + b_i')\Big\}$$

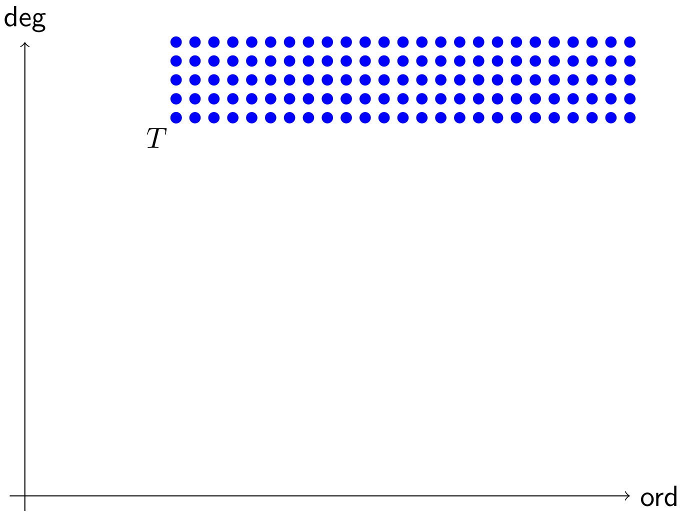
and

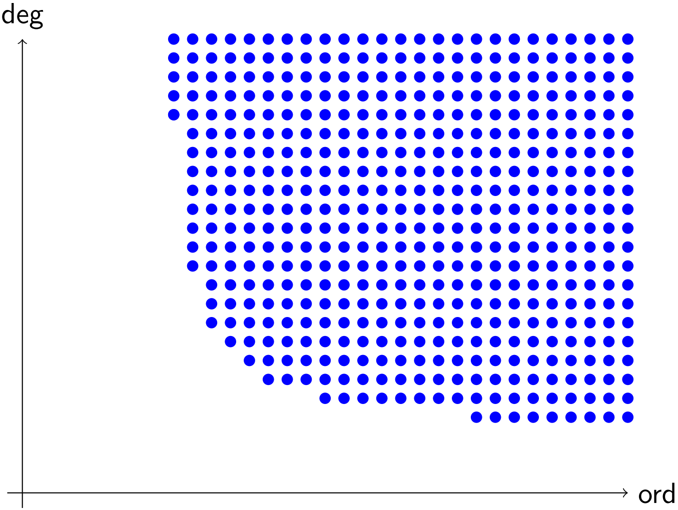
$$\deg(T) \le \left\lceil \tfrac{1}{2}\nu(2\delta + 2\nu\vartheta + |\mu| - \nu|\mu|)\right\rceil$$

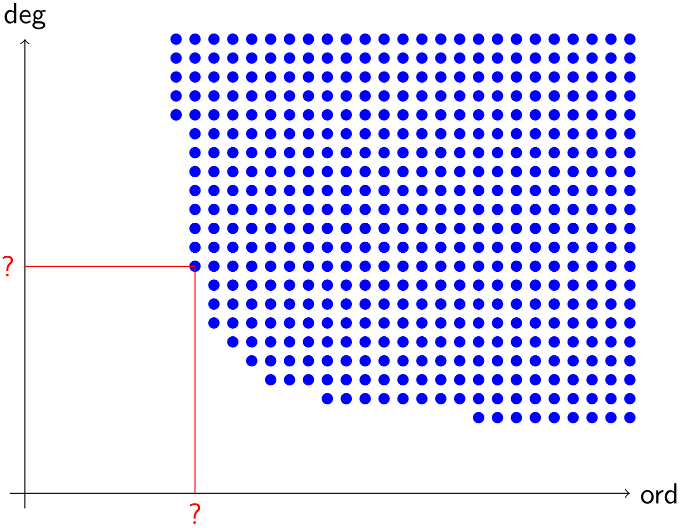
where

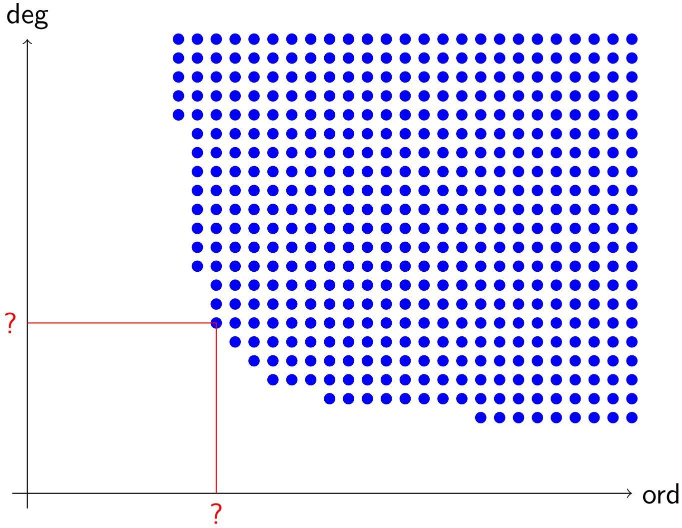
- $\delta = \deg(c)$
- $\nu = \max\left\{\sum_{i=1}^{m}(a'_i + v'_i),\ \sum_{i=1}^{m}(u'_i + b'_i)\right\}$
- $\vartheta = \max\left\{\sum_{i=1}^{m}(a_i + b_i),\ \sum_{i=1}^{m}(u_i + v_i)\right\}$
- $\mu = \sum_{i=1}^{m}\big((a_i + b_i) - (u_i + v_i)\big)$

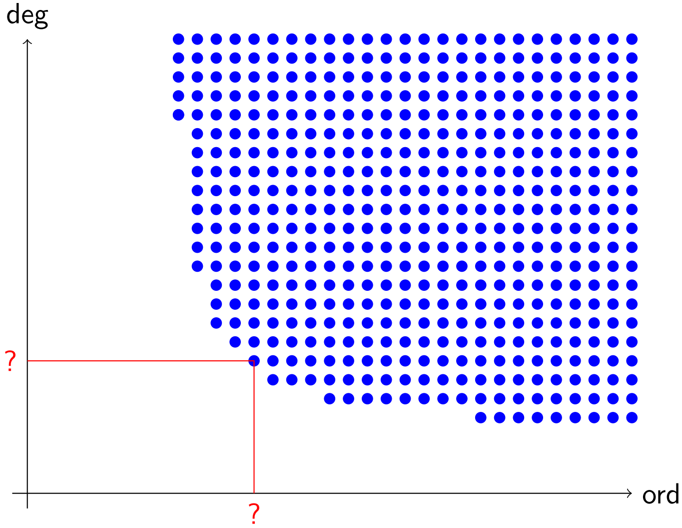

deg
$\deg(T)$
T
ord
$\mathrm{ord}(T)$

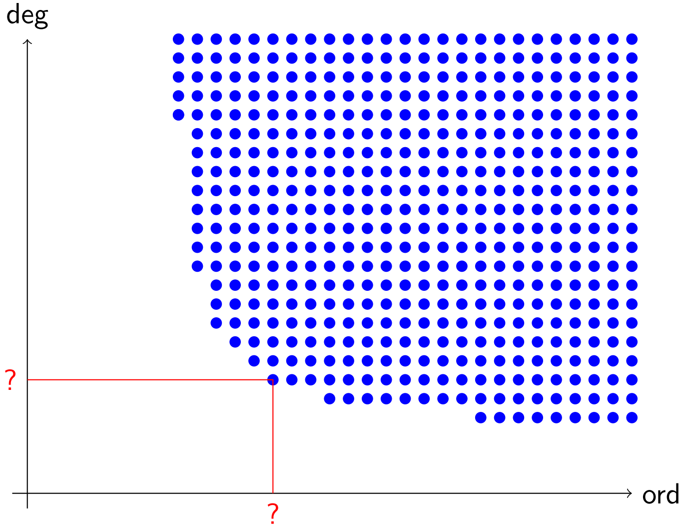

deg
T
ord

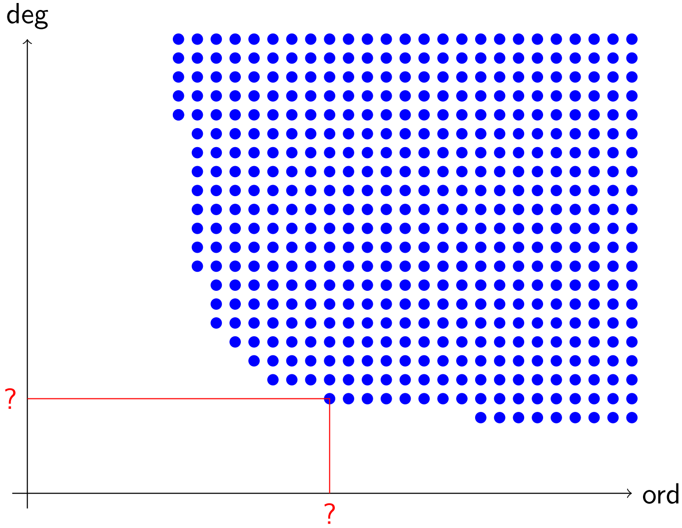

deg
ord

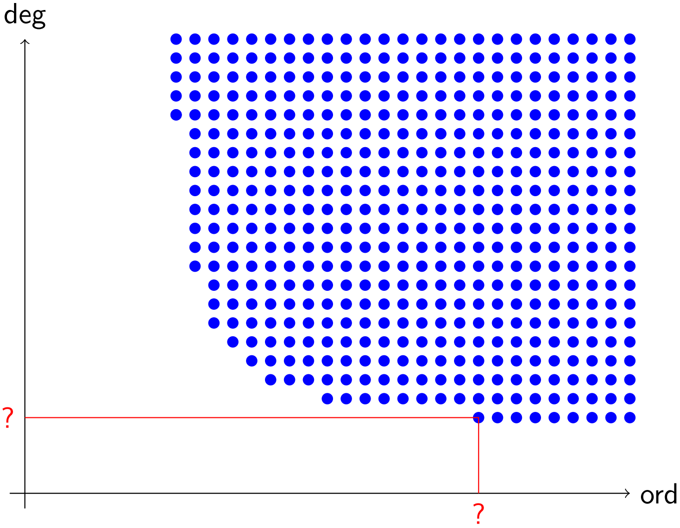

deg
?
ord
?

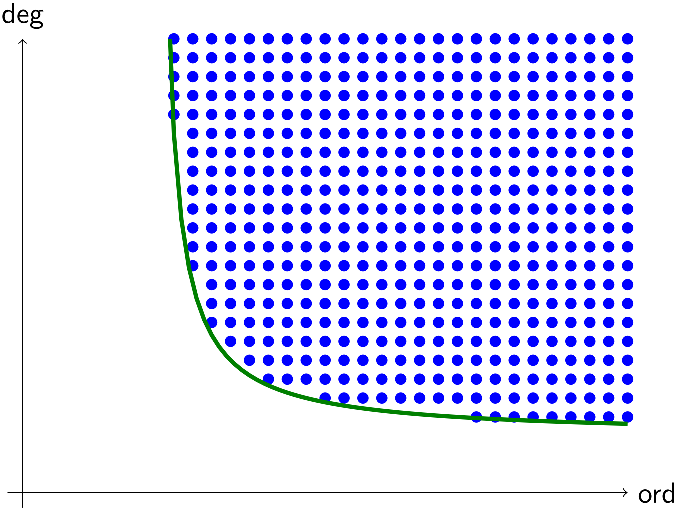

deg
ord
?
?


deg
ord
?
?


deg
?
ord
?


deg
?
ord
?


deg
?
ord
?


deg
ord

Theorem (Chen-Kauers)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exist telescopers T with $\operatorname{ord}(T) \leq r$ and $\deg(T) \leq d$ for all $(r,d) \in \mathbb{N}^2$ with

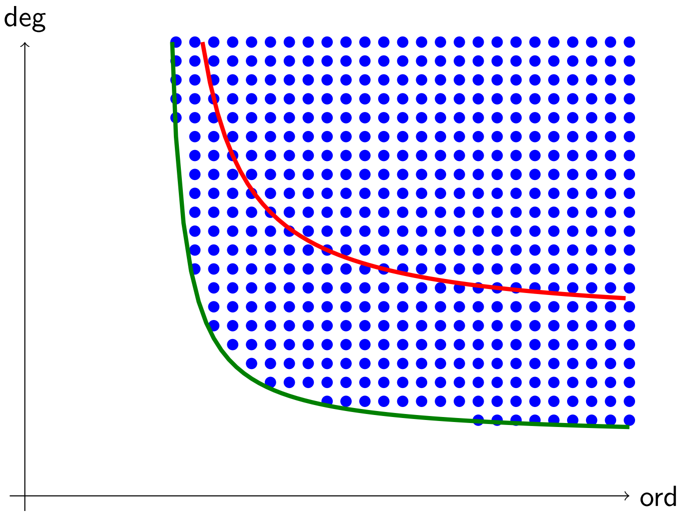
$$r \geq \nu \text{ and } d > \frac{\bigl(\vartheta\nu - 1\bigr)r + \frac{1}{2}\nu\bigl(2\delta + |\mu| + 3 - (1 + |\mu|)\nu\bigr) - 1}{r - \nu + 1}.$$

Theorem (Chen-Kauers)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exist telescopers T with $\operatorname{ord}(T) \leq r$ and $\deg(T) \leq d$ for all $(r,d) \in \mathbb{N}^2$ with

$$r \geq \nu \text{ and } d > \frac{(\vartheta\nu - 1)r + \frac{1}{2}\nu\big(2\delta + |\mu| + 3 - (1 + |\mu|)\nu\big) - 1}{r - \nu + 1}.$$


deg
ord

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓	✓	
hyperexponential integration	✓		
holonomic summation/integration			

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓	✓	
hyperexponential integration	✓	✓	
holonomic summation/integration			

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓	✓	
hyperexponential integration	✓	✓	
holonomic summation/integration			

Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term $f(x,y)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper T with $\operatorname{ord}(T) \leq \nu$ and

$$\begin{aligned}\operatorname{ht}(T) \leq\ & \max\bigl\{|p|^{\nu}, |q|+1\bigr\}\operatorname{ht}(c)^{\nu+1}(\delta+\vartheta\nu+1)!^{\nu+1}(\nu+1)^{\delta(\nu+1)}\\ &\times (|y|+1)^{\delta+(\vartheta-1)\nu+1}\delta!^{2(\nu+1)}|x|^{\nu^2}\\ &\times (\delta+\vartheta\nu+1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2}\\ &\times (2(\nu+2)\Omega-2)^{(\delta+\vartheta+1)\nu+(2\vartheta-1)\nu^2}\end{aligned}$$

where ν, ϑ, δ are as before, and

$$\Omega = \max_{i=1}^{m}\bigl\{|a_i|, |a_i'|, |a_i''|, |b_i|, |b_i'|, |b_i''|, |u_i|, |u_i'|, |u_i''|, |v_i|, |v_i'|, |v_i''|\bigr\}.$$

Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term $f(x,y)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper T with $\operatorname{ord}(T) \leq \nu$ and

$$\begin{aligned}\operatorname{ht}(T) \leq\ & \max\{|p|^{\nu}, |q|+1\}\operatorname{ht}(c)^{\nu+1}(\delta+\vartheta\nu+1)!^{\nu+1}(\nu+1)^{\delta(\nu+1)} \\ & \times (|y|+1)^{\delta+(\vartheta-1)\nu+1}\delta!^{2(\nu+1)}|x|^{\nu^2} \\ & \times (\delta+\vartheta\nu+1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2} \\ & \times (2(\nu+2)\Omega-2)^{(\delta+\vartheta+1)\nu+(2\vartheta-1)\nu^2}\end{aligned}$$

$$\mathbf{exp}(\mathbf{O}(\boldsymbol{\Omega}^3 \log(\boldsymbol{\Omega})))$$

where ν, ϑ, δ are as before, and

$$\Omega = \max_{i=1}^{m}\{|a_i|, |a_i'|, |a_i''|, |b_i|, |b_i'|, |b_i''|, |u_i|, |u_i'|, |u_i''|, |v_i|, |v_i'|, |v_i''|\}.$$

How good is this bound?

How good is this bound?

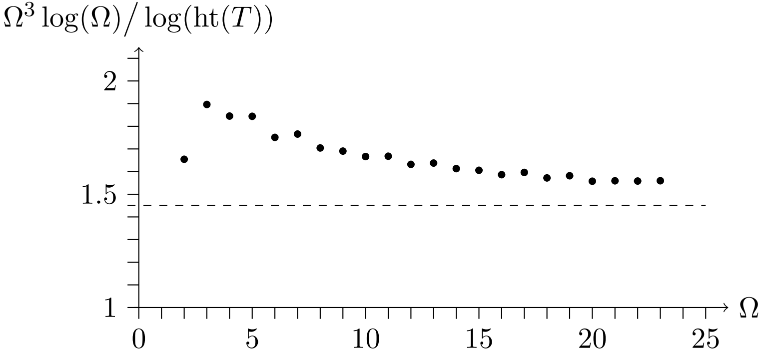
Consider $f_\Omega(x, y) = \Gamma(\Omega y)/\Gamma(\Omega x - y)$ for $\Omega = 1, 2, 3, \ldots$.

How good is this bound?

Consider $f_\Omega(x, y) = \Gamma(\Omega y)/\Gamma(\Omega x - y)$ for $\Omega = 1, 2, 3, \ldots$.
We compare the actual height to the bound.

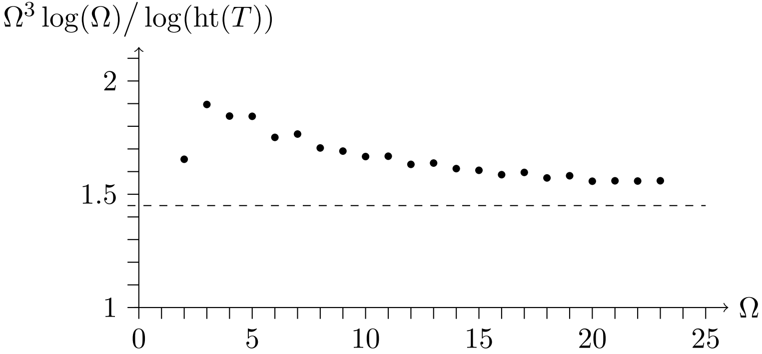
How good is this bound?

Consider $f_\Omega(x,y) = \Gamma(\Omega y)/\Gamma(\Omega x - y)$ for $\Omega = 1, 2, 3, \ldots$.
We compare the actual height to the bound.



How good is this bound?

Consider $f_\Omega(x,y) = \Gamma(\Omega y)/\Gamma(\Omega x - y)$ for $\Omega = 1, 2, 3, \ldots$.
We compare the actual height to the bound.



The asymptotics seems to be right.

The previous theorem only bounds the height of the telescoper of order ν.

The previous theorem only bounds the height of the telescoper of order ν.

Recall that higher order operators may have lower degree.

The previous theorem only bounds the height of the telescoper of order ν.

Recall that higher order operators may have lower degree.

What is their height?

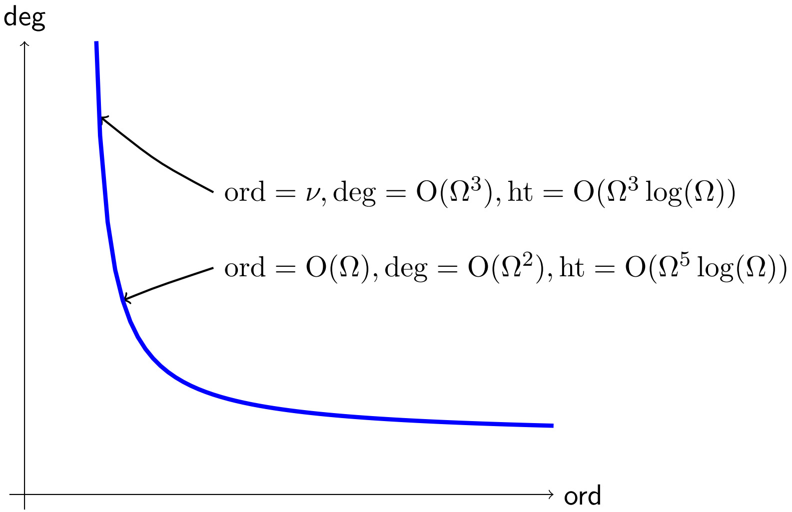
The previous theorem only bounds the height of the telescoper of order ν.

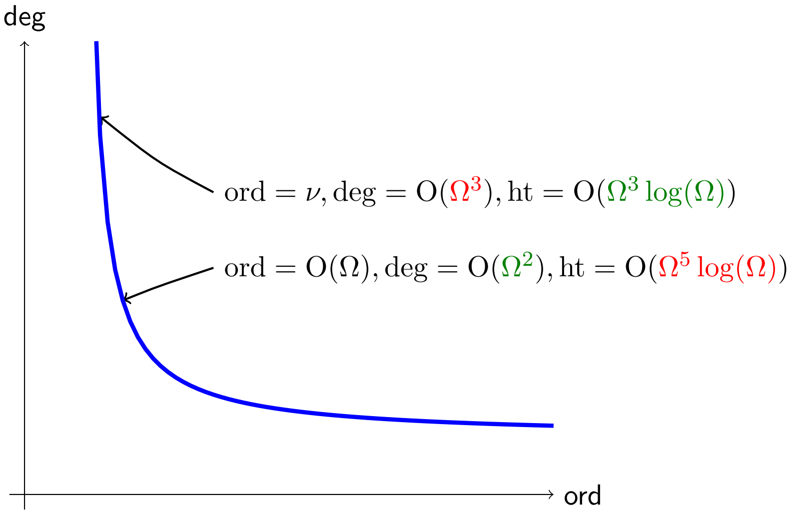
Recall that higher order operators may have lower degree.

What is their height?

Theorem (Kauers-Yen)
Every (non-rational) proper hypergeometric term $f(x, y)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper T with

$$\begin{aligned} \operatorname{ord}(T) &= \mathrm{O}(\Omega) \\ \deg(T) &= \mathrm{O}(\Omega^2) \\ \operatorname{ht}(T) &= \mathrm{O}(\Omega^5 \log(\Omega)) \end{aligned}$$


deg
$\mathrm{ord} = \nu, \deg = \mathrm{O}(\Omega^3), \mathrm{ht} = \mathrm{O}(\Omega^3 \log(\Omega))$
$\mathrm{ord} = \mathrm{O}(\Omega), \deg = \mathrm{O}(\Omega^2), \mathrm{ht} = \mathrm{O}(\Omega^5 \log(\Omega))$
ord


deg
$\mathrm{ord} = \nu, \deg = O(\Omega^3), \mathrm{ht} = O(\Omega^3 \log(\Omega))$
$\mathrm{ord} = O(\Omega), \deg = O(\Omega^2), \mathrm{ht} = O(\Omega^5 \log(\Omega))$
ord

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓	✓	✓
hyperexponential integration	✓	✓	
holonomic summation/integration			

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	√	√	√
hyperexponential integration	√	√	
holonomic summation/integration			

Theorem (Chen-Kauers-Koutschan)
Let $f(x,y)$ be a D-finite function, so that $K(x,y)[\frac{d}{dx},\frac{d}{dy}]/\mathrm{ann}(f) \cong K(x,y)^d$, and let $M \in K[x,y]^{d\times d}$ and $m \in K[x,y]$ be such that for all $v \in K(x,y)^d$ we have

$$\frac{d}{dy}v = \frac{1}{m}Mv + v'.$$

Then there exists a telescoper T for $f(x,y)$ with

$$\mathrm{ord}(T) \leq d \max\{\deg_y(m), \deg_y(M)\}.$$

Theorem (Chen-Kauers-Koutschan)
Let $f(x, y)$ be a D-finite function, so that $K(x, y)[\frac{d}{dx}, \frac{d}{dy}]/\mathrm{ann}(f) \cong K(x, y)^d$, and let $M \in K[x, y]^{d\times d}$ and $m \in K[x, y]$ be such that for all $v \in K(x, y)^d$ we have

$$\frac{d}{dy}v = \frac{1}{m}Mv + v'.$$

Then there exists a telescoper T for $f(x, y)$ with

$$\mathrm{ord}(T) \leq d \max\{\deg_y(m), \deg_y(M)\}.$$

There is also a more general version for when ∂_x or ∂_y are not the partial derivatives.

problem	order	degree	height
D-finite closure properties			
hypergeometric summation	✓	✓	✓
hyperexponential integration	✓	✓	
holonomic summation/integration	✓		

problem	order	degree	height
D-finite closure properties	✓	✓	✓
hypergeometric summation	✓	✓	✓
hyperexponential integration	✓	✓	
holonomic summation/integration	✓		

problem	order	degree	height
D-finite closure properties	✓	✓	✓
hypergeometric summation	✓	✓	✓
hyperexponential integration	✓	✓	?
holonomic summation/integration	✓	?	?