Restricted Lattice walks in Three Dimensions

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joint work with
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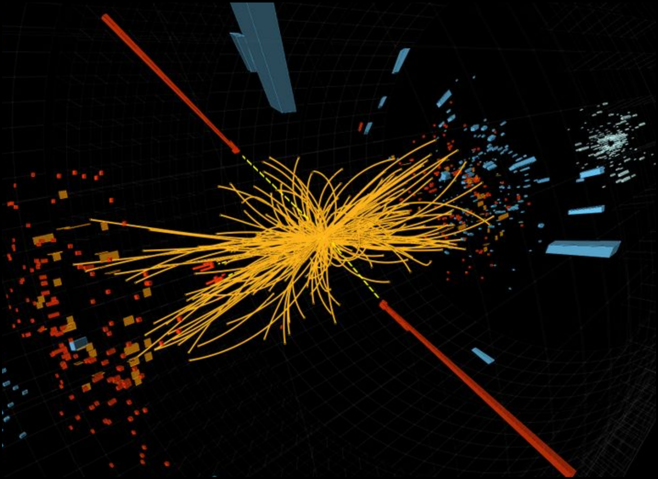

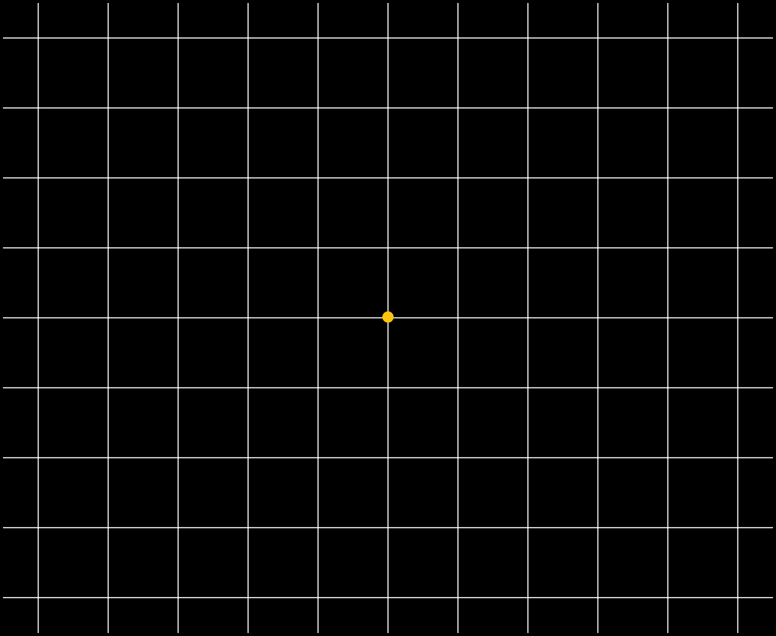

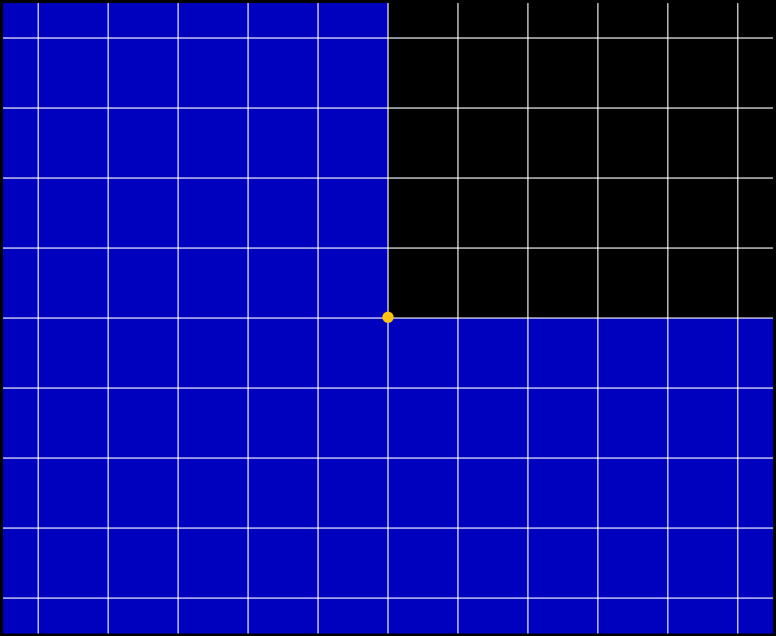

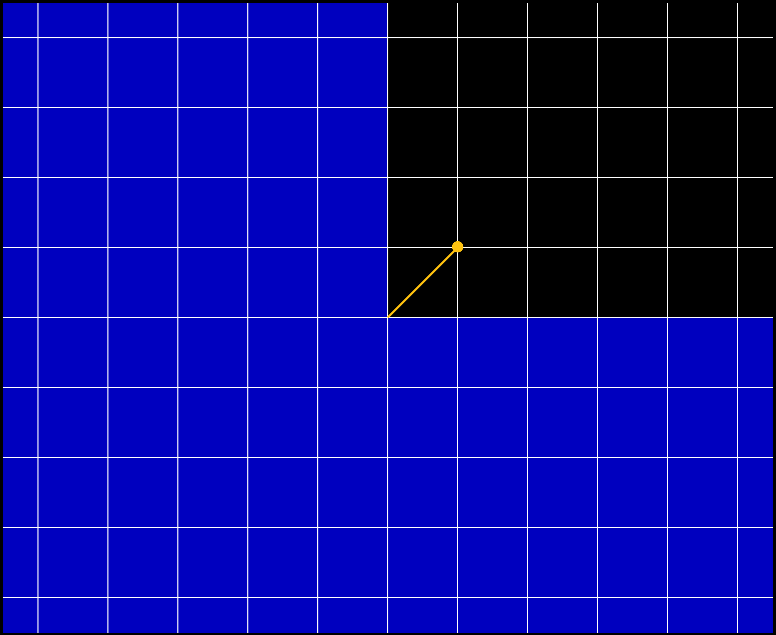

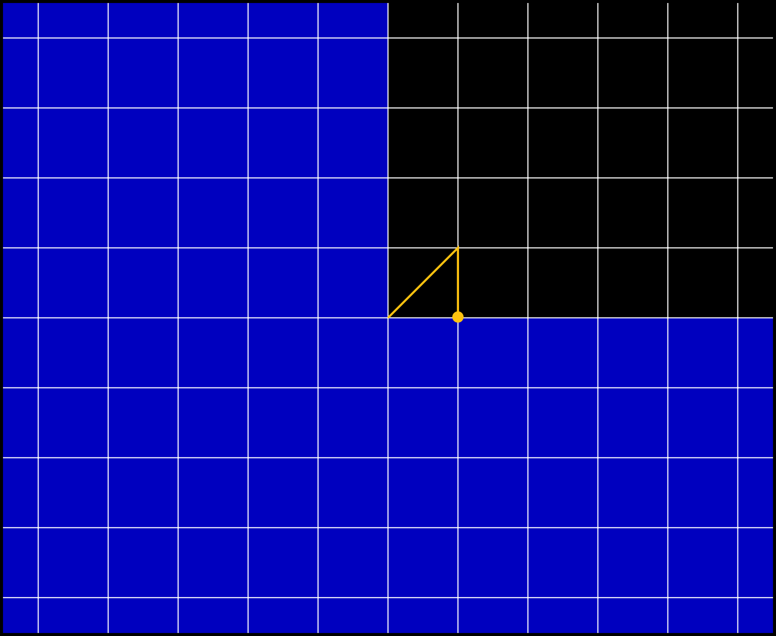

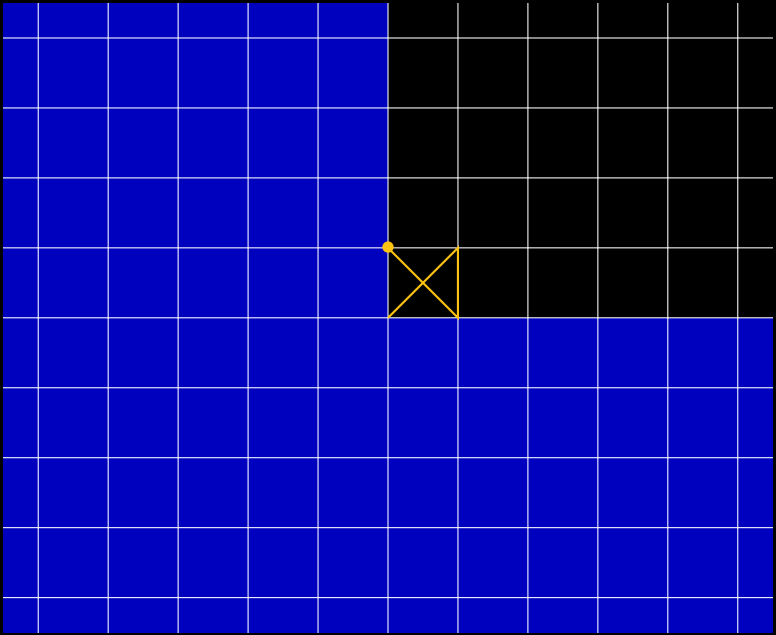

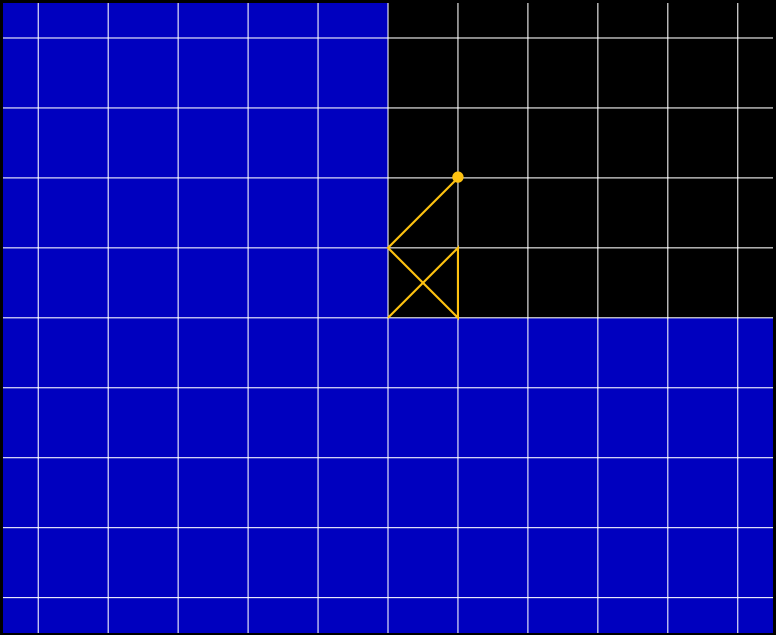

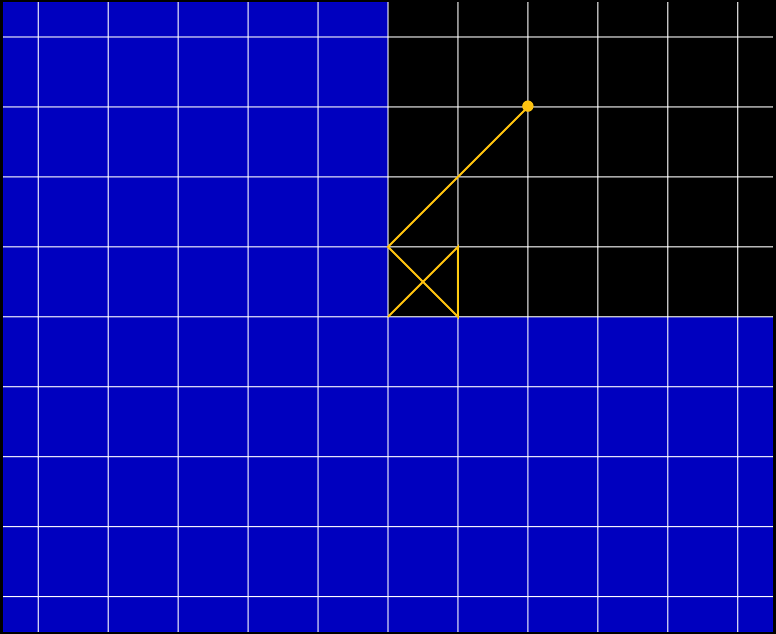

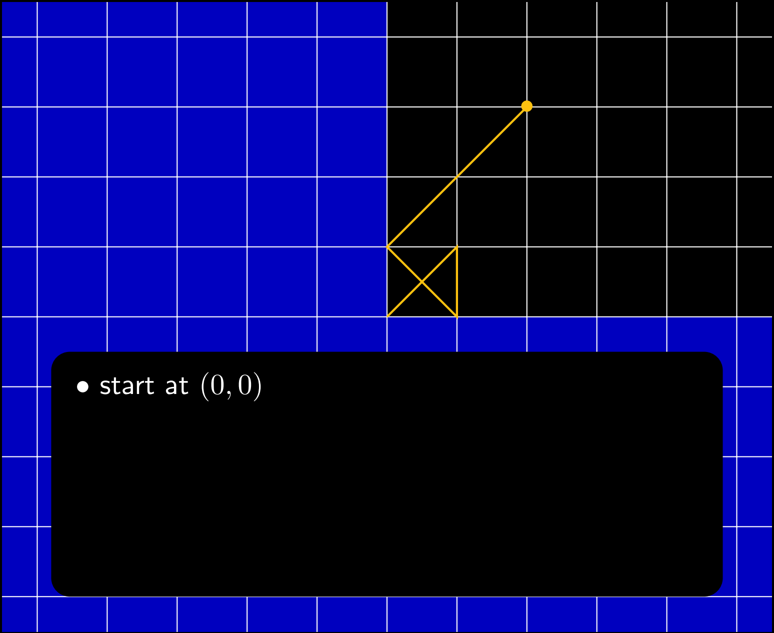

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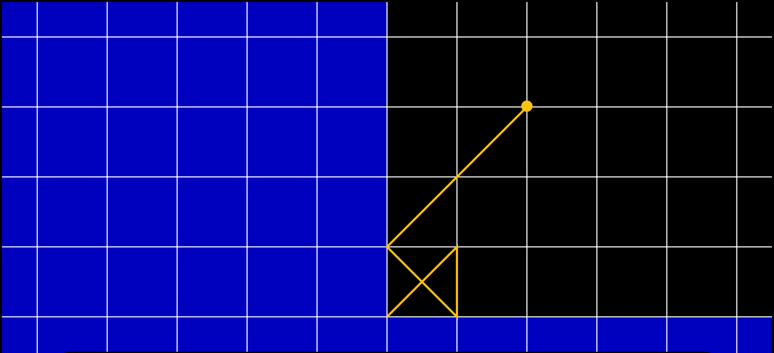

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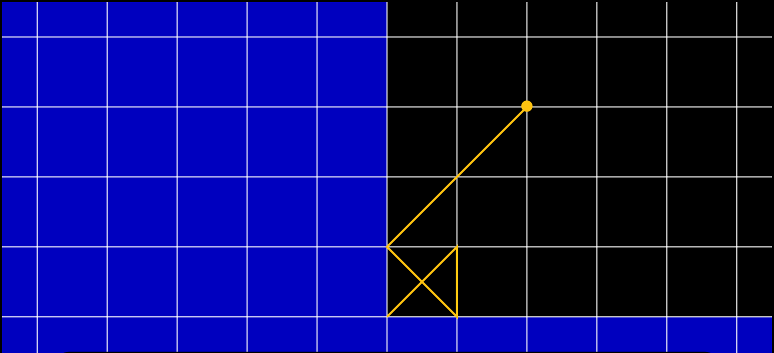
- make n steps (e.g. $n = 5$)



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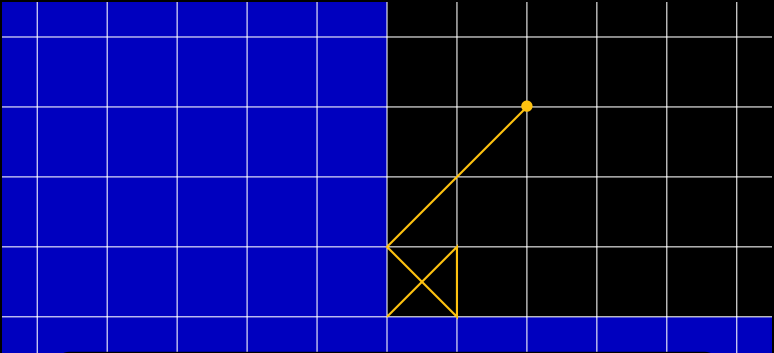

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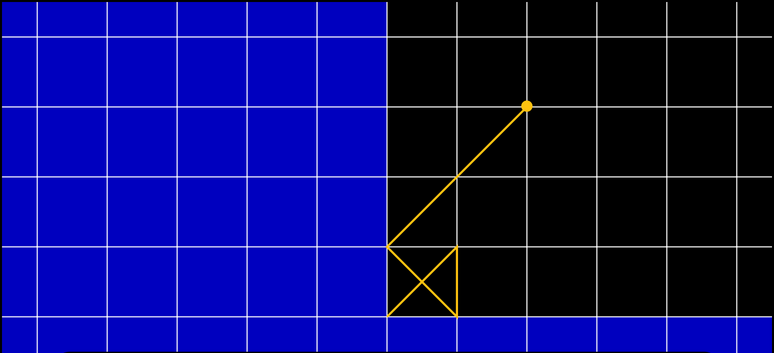

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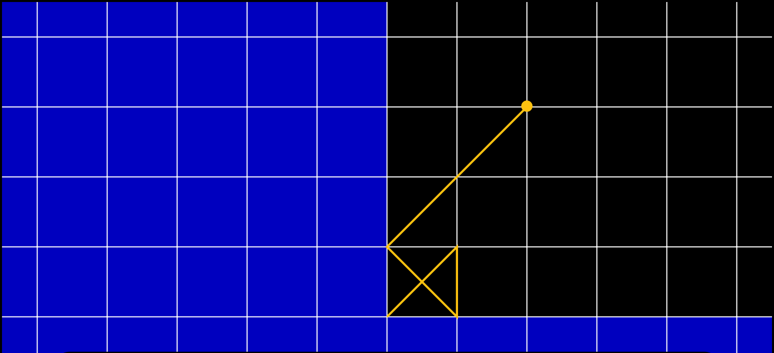

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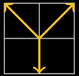

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$$a(x,y,t) = \sum_{n=0}^{\infty} \sum_{i,j} a_{n,i,j}\, x^i y^j t^n$$

be the corresponding generating function.

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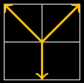
Example: For the step set  we have

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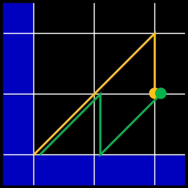
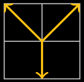


Example: For the step set  we have

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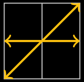
Example: For the step set  we have

$$\begin{aligned} a(x,y,t) = 1 &+ (x + xy)\, t \\ &+ (2 + x^2 + y + 2x^2y + x^2y^2)t^2 \\ &+ (5x + x^3 + 6xy + 3x^3y + 2xy^2 + 3x^3y^2 + x^3y^3)t^3 \\ &+ \cdots \end{aligned}$$

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Note:

- $a(1,1,t)$ counts the number of walks with arbitrary endpoint.
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Question:

$$\boxed{\text{How does the nature of } a(x, y, t) \text{ depend on the step set?}}$$

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More precisely: For which step sets is $a(x,y,t)$ D-finite (or even algebraic), and for which step sets is it not D-finite?

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Recall: a is D-finite $:\Longleftrightarrow$

$$p_0\, a + p_1 \frac{d}{dt} a + \cdots + p_r \frac{d^r}{dt^r} a = 0$$

for some polynomials $p_0, \ldots, p_r$ in x, y, t, not all zero.

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What does it depend on?

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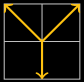
Example: For the step set  we obtain

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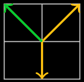
Example: For the step set  we obtain

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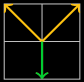
Example: For the step set  we obtain

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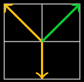
Example: For the step set  we obtain

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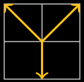
Example: For the step set  we obtain

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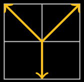
Example: For the step set  we obtain

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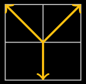
Example: For the step set  we obtain

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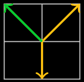
Example: For the step set  we obtain

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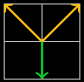
Example: For the step set  we obtain

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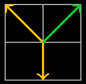
Example: For the step set  we obtain

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Together with $a_{0,0,0} = 1$ and the boundary conditions $a_{-1,j,n} = a_{i,-1,n} = 0$, this recurrence gives rise to a functional equation for the generating function $a(x, y, t)$.

Example: For the step set above we obtain

$$\left(1 - \left(\tfrac{y}{x} + \tfrac{1}{y} + xy\right)t\right) a(x, y, t) = 1 - \tfrac{t}{y}\, a(x, 0, t) - \tfrac{yt}{x}\, a(0, y, t)$$

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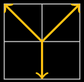
Example: For the step set  we obtain

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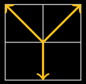
Example: For the step set  we obtain

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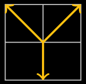
Example: For the step set  we obtain

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All properties of $a(x, y, t)$ must somehow follow from it.

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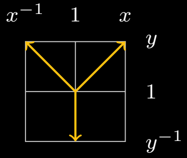

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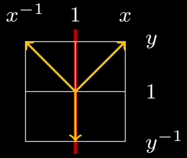

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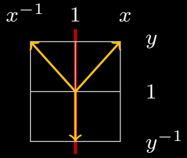

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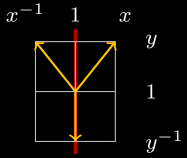

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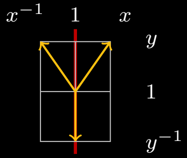

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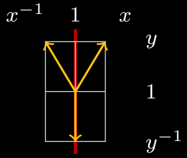

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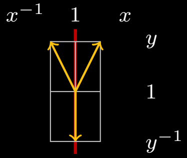

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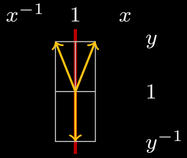

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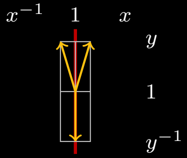

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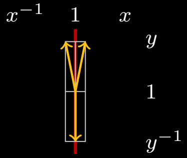

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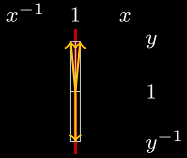

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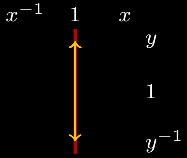

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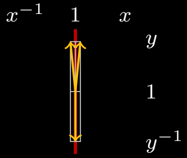

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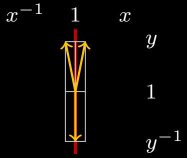

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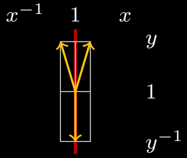

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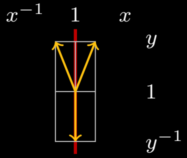

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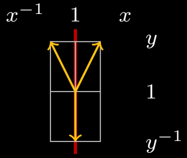

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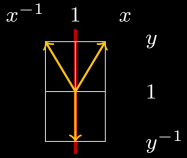

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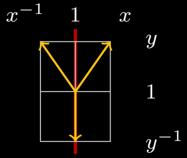

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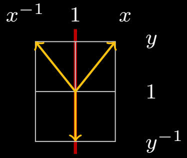

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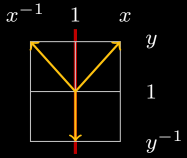

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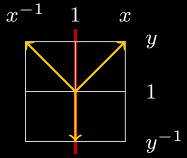

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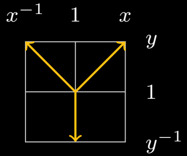

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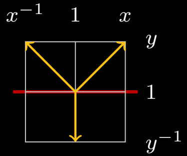

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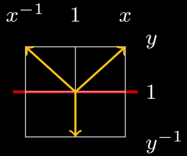

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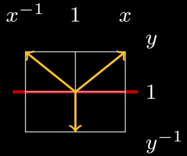

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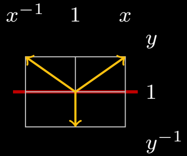

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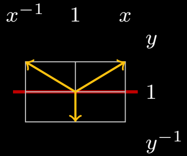

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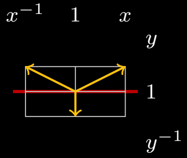

$$\phi:\ \ (x,\ y)\mapsto\left(\frac{1}{x},\ y\right)$$

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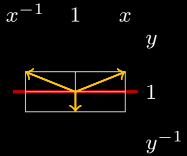

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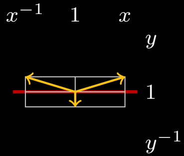

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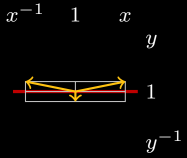

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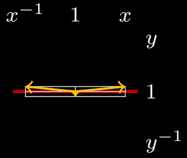

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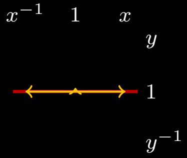

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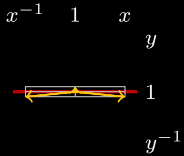

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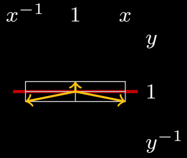

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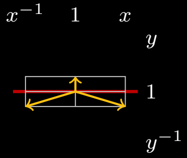

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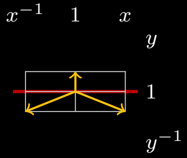

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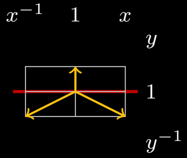

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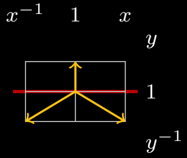

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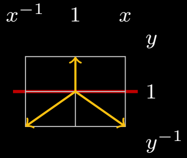

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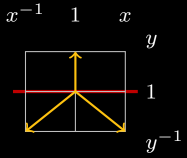

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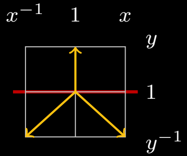

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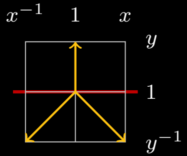

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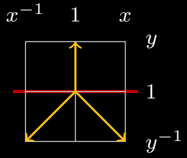

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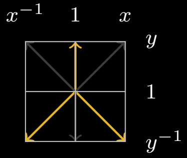

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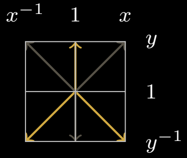

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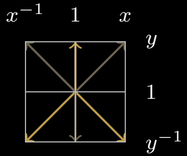

$$\phi\colon\;\; (x,\; y) \mapsto \big(\tfrac{1}{x},\; y\big)$$

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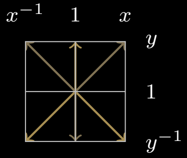

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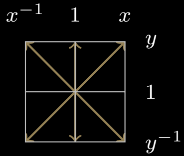

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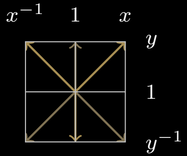

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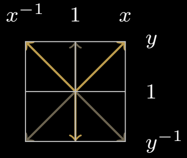

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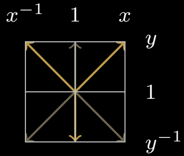

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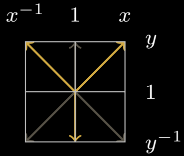

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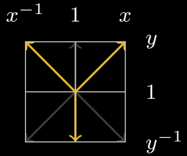

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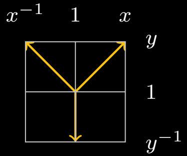

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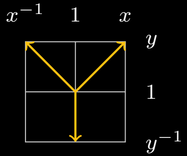

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These two maps together with composition generate a group, the so-called group of the model.

$$\left(1 - \left(\frac{y}{x} + \frac{1}{y} + xy\right)t\right) a(x,y,t) = 1 - \frac{t}{y}\, a(x,0,t) - \frac{yt}{x}\, a(0,y,t)$$

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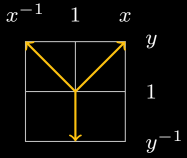

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These two maps together with composition generate a group, the so-called group of the model.

For some step sets this group is finite, for others it is infinite.

$$\left(1 - \left(\frac{y}{x} + \frac{1}{y} + xy\right)t\right) a(x,y,t) = 1 - \frac{t}{y}\, a(x,0,t) - \frac{yt}{x}\, a(0,y,t)$$

Here, $G = \{1, \phi, \psi, \phi\psi\}$ is finite.

$$\Big(1-\left(\tfrac{y}{x}+\tfrac{1}{y}+xy\right)t\Big)\,a(x,y,t)=1-\tfrac{t}{y}\,a(x,0,t)-\tfrac{yt}{x}\,a(0,y,t)$$

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$$\begin{aligned} K\, xy\, a(x,y,t) &= xy - xt\, a(x,0,t) - y^2 t\, a(0,y,t) \\ -\,\phi\big(\; K\, xy\, a(x,y,t) &= xy - xt\, a(x,0,t) - y^2 t\, a(0,y,t)\;\big) \end{aligned}$$

$$\Big(1 - \big(\tfrac{y}{x} + \tfrac{1}{y} + xy\big)t\Big)\, a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

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only monomials x^iy^j with $i<0$ or $j<0$

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\end{aligned}$$

$$\Rightarrow xy\,a(x,y,t)=[x^>][y^>]\,\frac{xy-\frac{1}{x}y-x\frac{1}{1+\frac{1}{x}}\frac{1}{y}+\frac{1}{x}\frac{1}{1+\frac{1}{x}}\frac{1}{y}}{1-(\frac{y}{x}+\frac{1}{y}+xy)t}$$

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$$\Big(1 - \big(\tfrac{y}{x} + \tfrac{1}{y} + xy\big)t\Big)\, a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

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\end{aligned}$$

$$\Rightarrow xy\, a(x,y,t) = \underbrace{[x^>][y^>]\, \underbrace{\frac{xy - \frac{1}{x}y - x\frac{1}{1+\frac{1}{x}}\frac{1}{y} + \frac{1}{x}\frac{1}{1+\frac{1}{x}}\frac{1}{y}}{1 - (\frac{y}{x} + \frac{1}{y} + xy)t}}_{\text{rational}}}_{\text{D-finite}}$$

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There are three possible reasons why this approach can fail:

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What to do then? Try using computer algebra, as follows.

$$\left(1 - \left(\tfrac{y}{x} + \tfrac{1}{y} + xy\right)t\right)\, a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

$$\underbrace{\Big(1 - \big(\tfrac{y}{x} + \tfrac{1}{y} + xy\big)t\Big)}_{=0 \text{ for } y=Y(x,t):=\frac{x-\sqrt{x(x-4t^2(1+x^2))}}{2t(1+x^2)}=t+\left(x+\frac{1}{x}\right)t^3+\cdots} a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

$$\underbrace{\left(1-\left(\tfrac{y}{x}+\tfrac{1}{y}+xy\right)t\right)}_{=0 \text{ for } y=Y(x,t):=\frac{x-\sqrt{x(x-4t^2(1+x^2))}}{2t(1+x^2)}=t+\left(x+\frac{1}{x}\right)t^3+\cdots} a(x,y,t) = 1-\tfrac{t}{y}\,a(x,0,t)-\tfrac{yt}{x}\,a(0,y,t)$$

For this choice of $Y(x,t)$ we find

$$0 = 1-\tfrac{t}{Y(x,t)}a(x,0,t)-\tfrac{Y(x,t)\,t}{x}\,a(0,Y(x,t),t)$$

$$\underbrace{\left(1-\left(\tfrac{y}{x}+\tfrac{1}{y}+xy\right)t\right)}_{=0 \text{ for } y=Y(x,t):=\frac{x-\sqrt{x(x-4t^2(1+x^2))}}{2t(1+x^2)}=t+\left(x+\frac{1}{x}\right)t^3+\cdots} a(x,y,t) = 1 - \tfrac{t}{y}\, a(x,0,t) - \tfrac{yt}{x}\, a(0,y,t)$$

For this choice of $Y(x,t)$ we find

$$a(x,0,t) = \tfrac{Y(x,t)}{t} - x\, Y(x,t)^2\, a(0, Y(x,t), t)$$

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$$a(x,0,t)=\tfrac{Y(x,t)}{t}-x\,Y(x,t)^2\,a(0,Y(x,t),t)$$

Setting $x \rightsquigarrow Y^{-1}(x,t)$ in this equation and rearranging terms gives

$$a(0,x,t)=\tfrac{1}{txY^{-1}(x,t)}-\tfrac{1}{Y^{-1}(x,t)\,x^2}a(Y^{-1}(x,t),0,t)$$

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Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

$$U(x,t) = \frac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t)$$
$$V(x,t) = \frac{1}{txY^{-1}(x,t)} - \frac{1}{Y^{-1}(x,t)\,x^2}U(Y^{-1}(x,t),t)$$

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Observe:

- This system has a unique solution.

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Observe:

- This system has a unique solution.
- By construction, the solution must be

$$U = a(x,0,t) \text{ and } V = a(0,x,t).$$

Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

$$U(x,t) = \frac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t)$$
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Now turn on the computer. . .

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Now turn on the computer...

- generate lots of coefficients of $a(x,0,t)$, and $a(0,x,t)$.

Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

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- guess a system of D-finite differential equations possibly satisfied by these series.

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Now turn on the computer...

- generate lots of coefficients of $a(x,0,t)$, and $a(0,x,t)$.
- guess a system of D-finite differential equations possibly satisfied by these series.
- prove that the series solutions of the guessed D-finite system solve the functional equations.

Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

$$U(x,t) = \frac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t)$$
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Conclude:

Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

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Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.

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Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.
- Because of

$$\Big(1 - \big(\tfrac{y}{x} + \tfrac{1}{y} + xy\big)t\Big)\,a(x,y,t) = 1 - \tfrac{t}{y}\,a(x,0,t) - \tfrac{yt}{x}\,a(0,y,t)$$

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\begin{aligned}
U(x,t) &= \tfrac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t) \\
V(x,t) &= \tfrac{1}{txY^{-1}(x,t)} - \tfrac{1}{Y^{-1}(x,t)\,x^2}U(Y^{-1}(x,t),t)
\end{aligned}
$$

Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.
- Because of

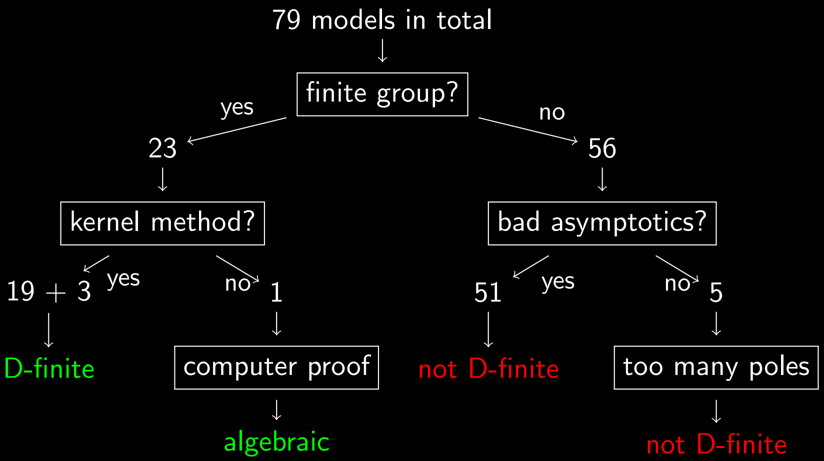
$$a(x,y,t) = \frac{1 - \frac{t}{y}\,a(x,0,t) - \frac{yt}{x}\,a(0,y,t)}{1 - \left(\frac{y}{x} + \frac{1}{y} + xy\right)t}$$

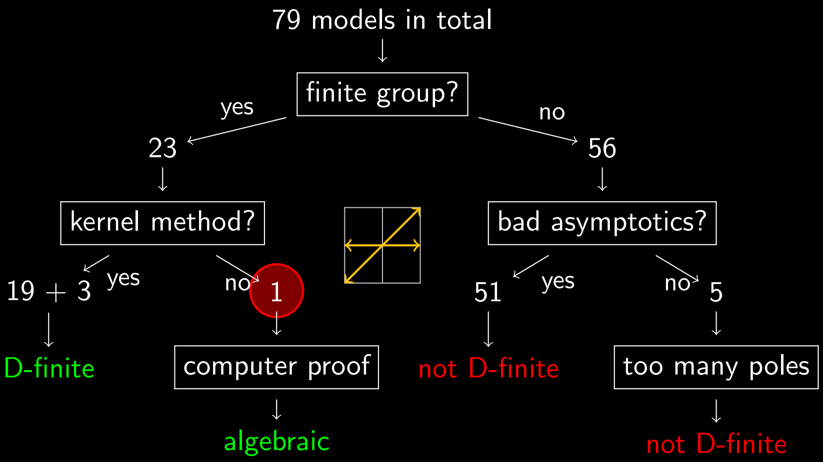
Now consider the following system of functional equations for two unknown power series $U(x,t), V(x,t)$:

$$U(x,t) = \frac{Y(x,t)}{t} - x\,Y(x,t)^2\,V(Y(x,t),t)$$
$$V(x,t) = \frac{1}{txY^{-1}(x,t)} - \frac{1}{Y^{-1}(x,t)\,x^2}U(Y^{-1}(x,t),t)$$

Conclude:

- $a(x,0,t)$ and $a(0,x,t)$ are D-finite.
- Because of
$$a(x,y,t) = \frac{1 - \frac{t}{y}\,a(x,0,t) - \frac{yt}{x}\,a(0,y,t)}{1 - \left(\frac{y}{x} + \frac{1}{y} + xy\right)t}$$
  it follows that also $a(x,y,t)$ is D-finite.


79 models in total
finite group?
yes
no
23
56
kernel method?
bad asymptotics?
19 + 3
yes
no
1
51
yes
no
5
D-finite
computer proof
not D-finite
too many poles
algebraic
not D-finite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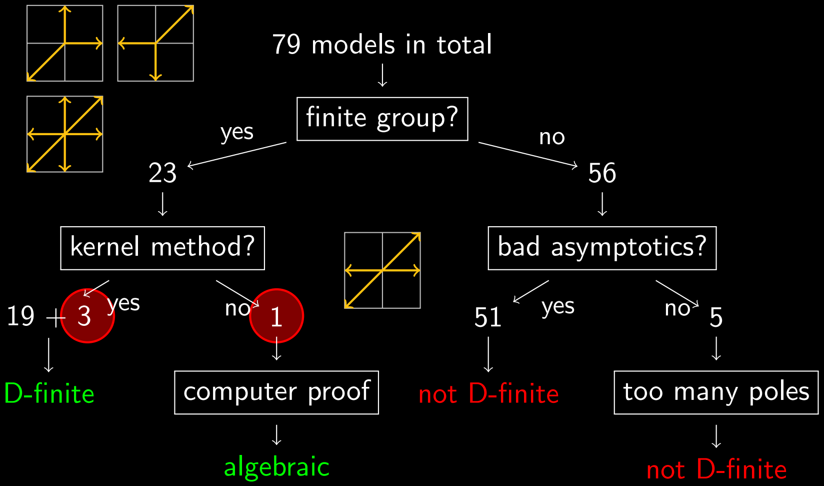

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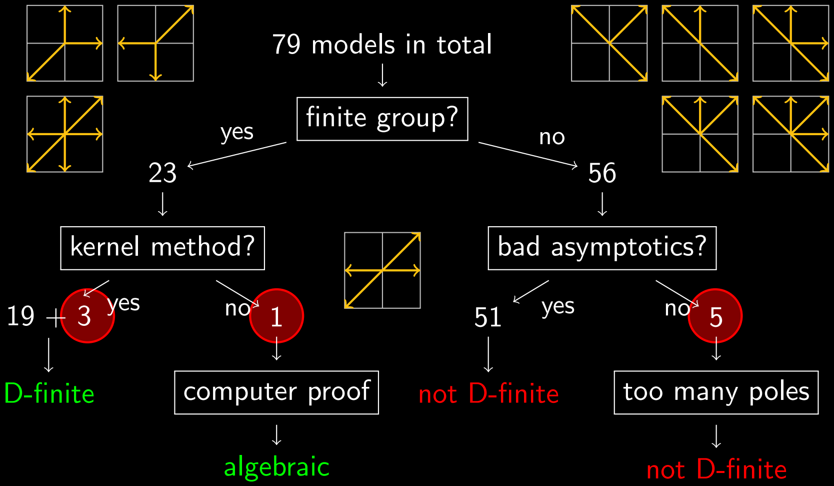
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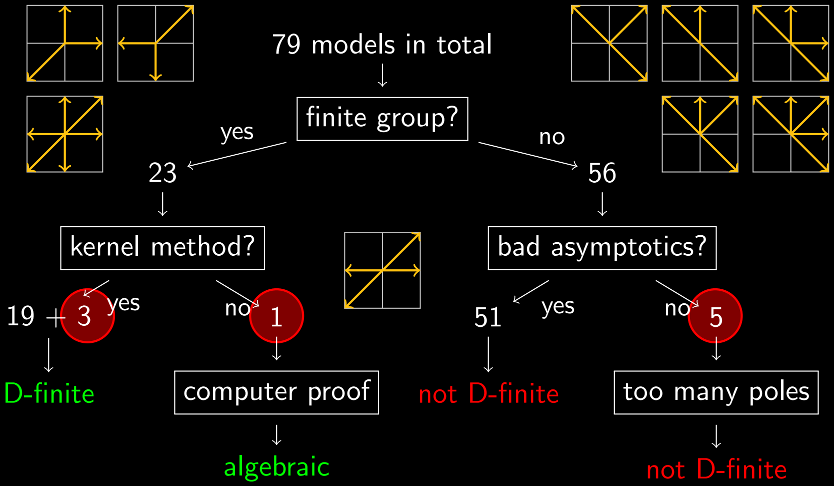
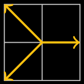
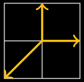
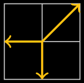
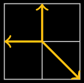
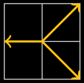
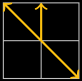
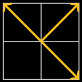
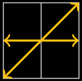
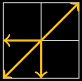
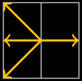
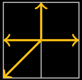
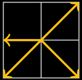
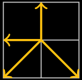
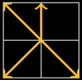
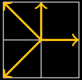


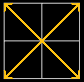
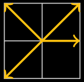
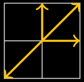
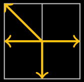
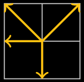
A posteriori observation:

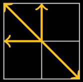
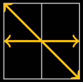
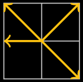
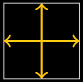
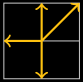
D-finite generating function $\Longleftrightarrow$ finite group.

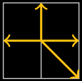
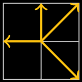
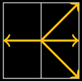
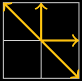
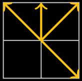
step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	D-finite	D-finite	D-finite
	algebraic	algebraic	algebraic
	algebraic	algebraic	algebraic
	D-finite	algebraic	D-finite
	D-finite	D-finite	D-finite

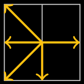
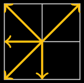
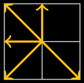
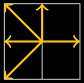
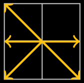
step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	algebraic	not D-finite	not D-finite
	algebraic	not D-finite	not D-finite
	algebraic	algebraic	algebraic
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite

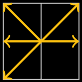
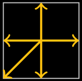
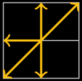
step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite
	algebraic	not D-finite	not D-finite
	algebraic	not D-finite	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
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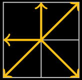
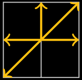
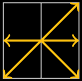
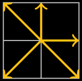
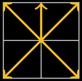
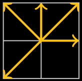
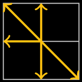
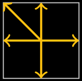
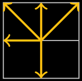
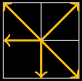
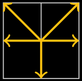
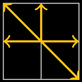
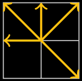
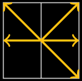
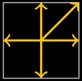
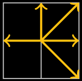
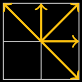
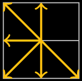
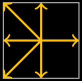
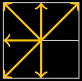
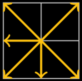
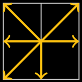
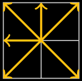
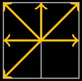
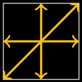
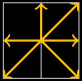
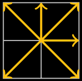
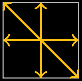
step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite

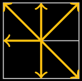
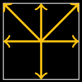
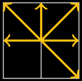
step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	algebraic	not D-finite	not D-finite
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	D-finite	D-finite	D-finite
	algebraic	algebraic	algebraic
	not D-finite	not D-finite?	not D-finite
	not D-finite	not D-finite?	not D-finite
	D-finite	algebraic	D-finite

step set	$a(0,0,t)$	$a(1,1,t)$	$a(x,y,t)$
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite
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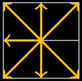	D-finite	D-finite	D-finite

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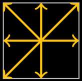
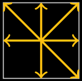
	not D-finite	not D-finite?	not D-finite
	D-finite	D-finite	D-finite
	not D-finite	not D-finite?	not D-finite
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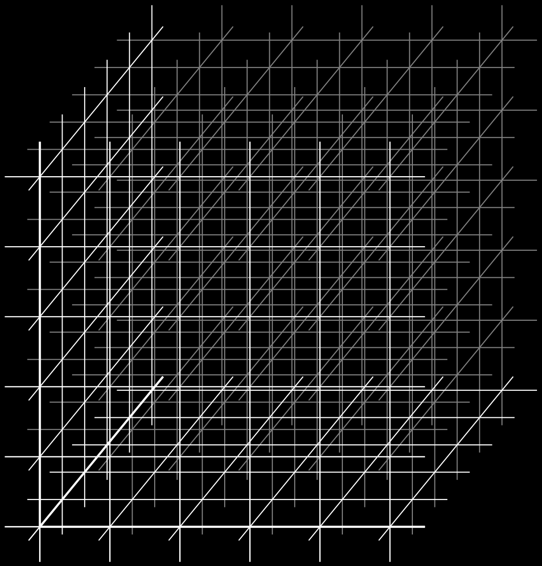

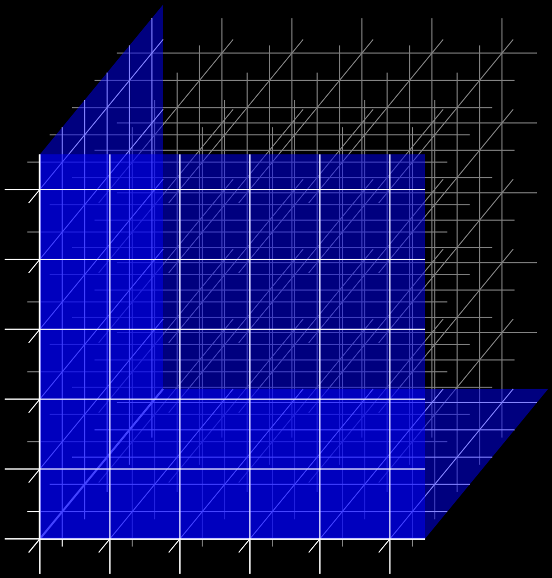

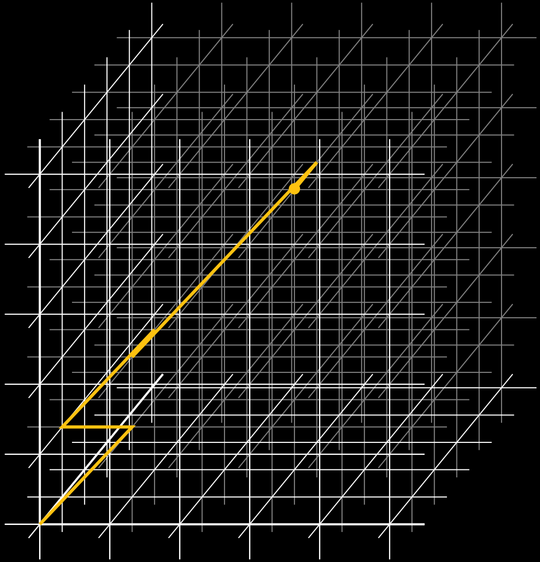

- start at $(0, 0, 0)$
- make n steps (e.g., $n = 7$)
- end at (i, j, k) (e.g., $(i, j, k) = (3, 4, 2)$)
- never step out of the first octant
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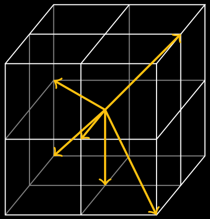

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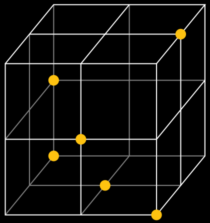

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$\mathrm{dim} = 3$

$$2^{3^{3}-1} = 67108864$$

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$$\underbrace{2}_{\in \text{ or } \notin}^{\overbrace{3}^{|\{-1,0,1\}|}{}^{\overbrace{3}^{\dim = 3}} \underbrace{-1}_{\text{except } (0,0,0)}} = 67108864$$

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35548 models to be investigated

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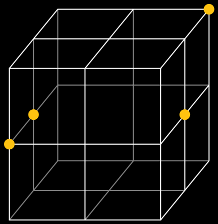
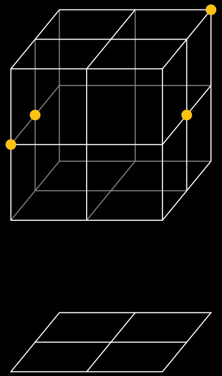
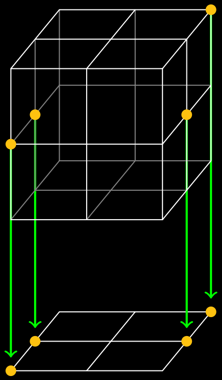
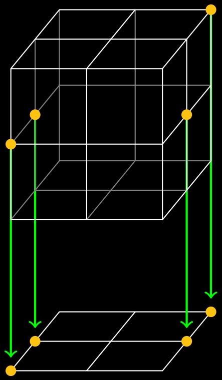
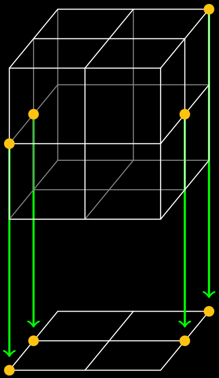


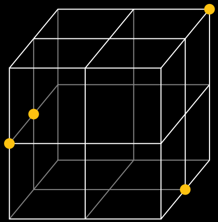






Models are in bijection!





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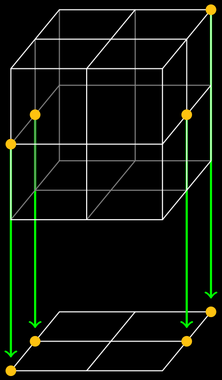

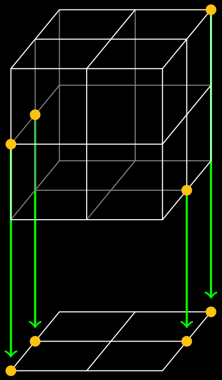

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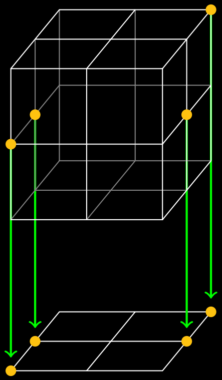

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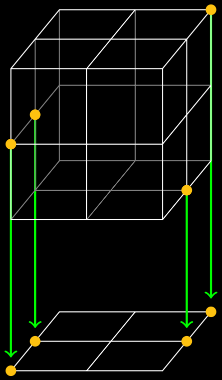
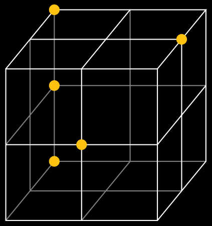
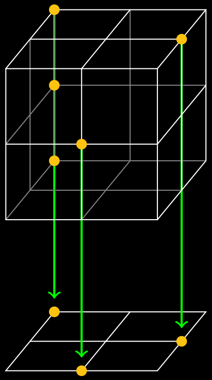
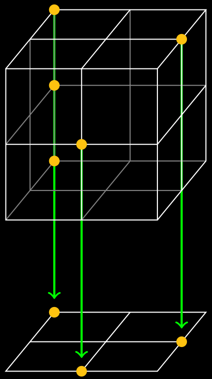


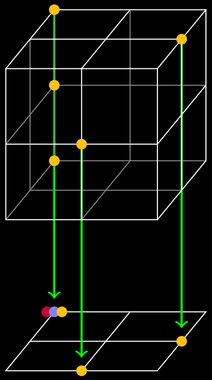
Not a valid bijection!







Bijection?



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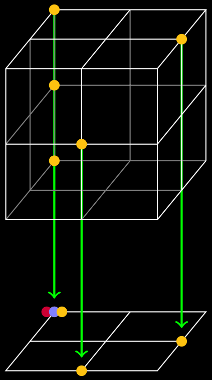

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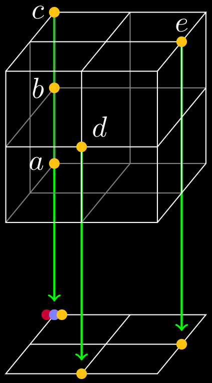

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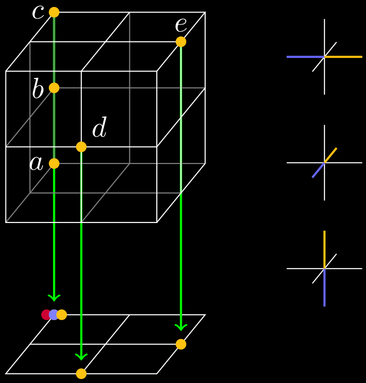

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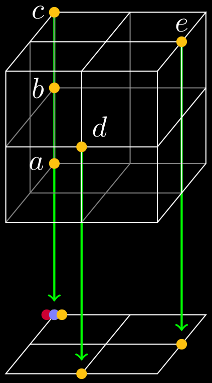

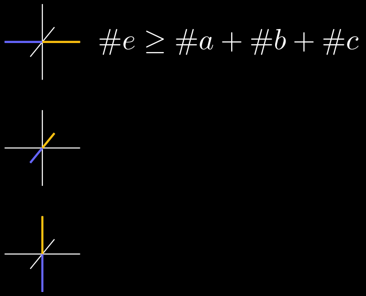

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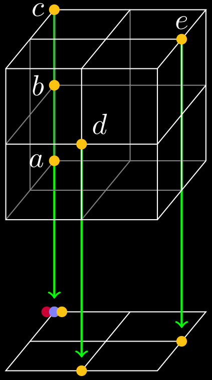
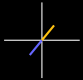


 $\#e \geq \#a + \#b + \#c$

 $\#a + \#b + \#c \geq \#d$

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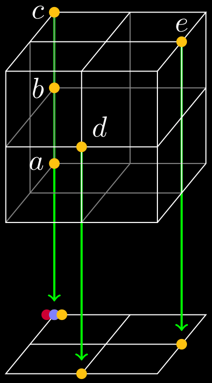

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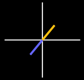

$$\#a + \#b + \#c \geq \#d$$

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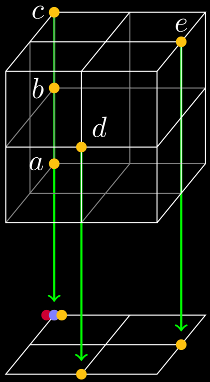

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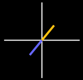

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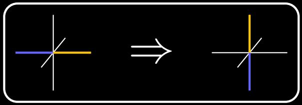

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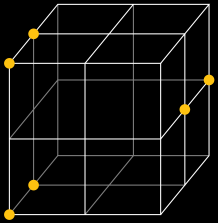

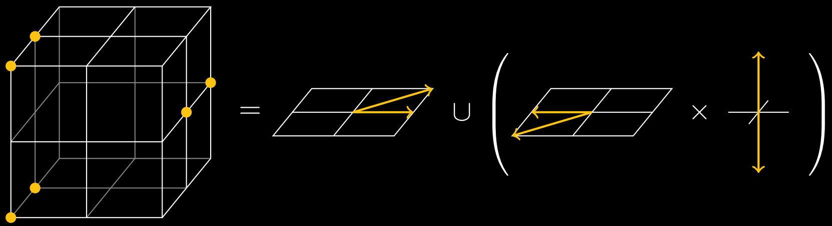

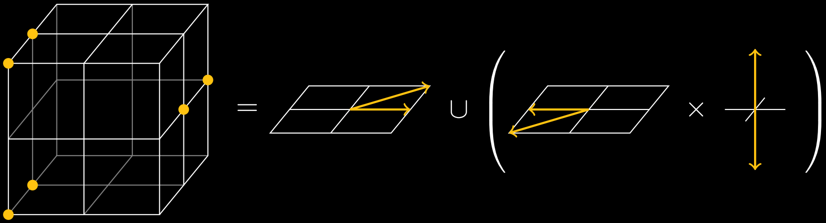

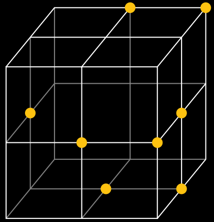

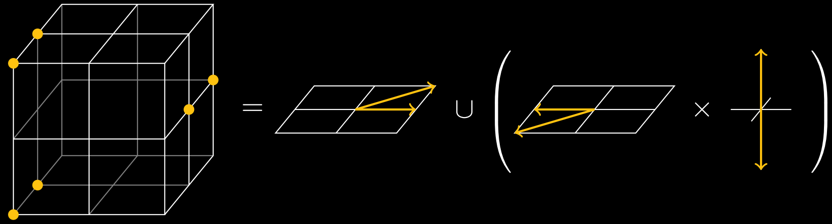

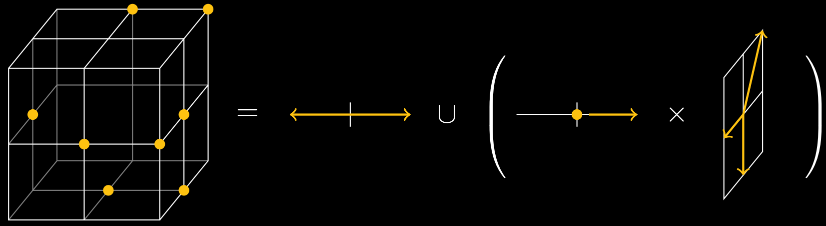

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reducible to 2D	decomposable	finite group	3	4	5	6	Σ
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					
		Σ					

reducible to 2D
decomposable
finite group

			3	4	5	6	Σ
T	T	T	8	47	110	175	340
T	T	F	46	437	1864	4821	7168
T	F	T	0	0	0	0	0
T	F	F	18	275	1599	5344	7236
F	T	T	0	18	47	82	147
F	T	F	0	9	125	411	545
F	F	T	0	8	0	15	23
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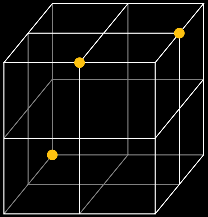

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F	T	T	0	18	47	82	147	D-finite!
F	T	F	0	9	125	411	545	not D-finite?
F	F	T	0	8	0	15	23	not so clear...
F	F	F	1	185	2680	17223	20089	not D-finite?
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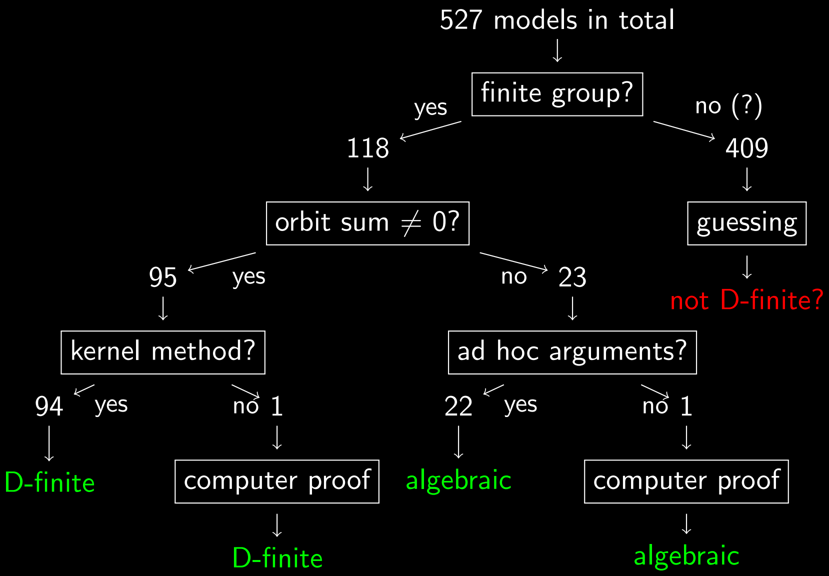
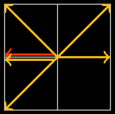

527 models in total
finite group?
yes
no (?)
118
409
orbit sum $\neq 0$?
guessing
95
yes
no
23
not D-finite?
kernel method?
ad hoc arguments?
94
yes
no
1
22
yes
no
1
D-finite
computer proof
algebraic
computer proof
D-finite
algebraic



527 models in total

↓

finite group?

yes → 118 | no (?) → 409

118 ↓ orbit sum $\neq 0$?

yes → 95 | no → 23

95 ↓ kernel method?

yes → 94 ↓ D-finite

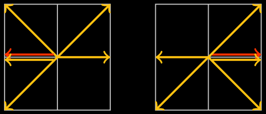
no → 1 ↓ computer proof ↓ D-finite

23 ↓ ad hoc arguments?

yes → 22 ↓ algebraic

no → 1 ↓ computer proof ↓ algebraic

409 ↓ guessing ↓ not D-finite?



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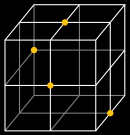
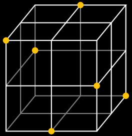
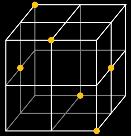
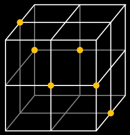
There are 23 models in 3D which are not reducible to 2D, which are not decomposable, and which have a finite group. For 4 of them, the orbit sum is nonzero and the kernel method implies that they are D-Finite.









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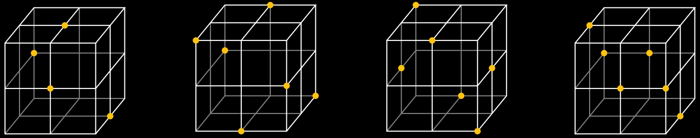

The remaining 19 models are mysterious. Even on a super-computer we were not able to find any evidence for possible differential equations. Can it be that they are not D-finite?

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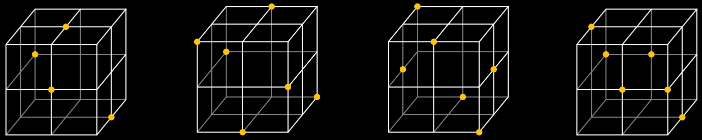

The remaining 19 models are mysterious. Even on a super-computer we were not able to find any evidence for possible differential equations. Can it be that they are not D-finite?

This would imply that the equivalence between D-finiteness and a finite group does not carry over to walks in three dimensions.