The Positive Part of Multivariate Infinite Series

Manuel Kauers

based on joint work with Alin Bostan, Frédéric Chyzak, Lucien Pech and Mark van Hoeij

Task: "Given" an infinite series

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"compute" its positive part

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Problems:

- How are such series supposed to be "given"?
- Bilateral formal infinite series cannot be multiplied in general.

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Thus the positive part of a univariate rational function is a univariate rational function.

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So

$$[x^{\geq}y^{\geq}]\frac{xy}{x-y} = y \quad \text{or} \quad [x^{\geq}y^{\geq}]\frac{xy}{x-y} = -x \quad ?$$

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- Indeed, the formal Laurent series $K((x))$ form a field.

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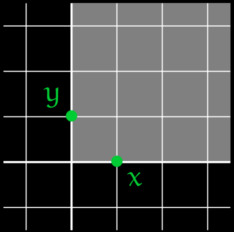

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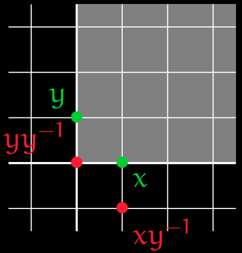

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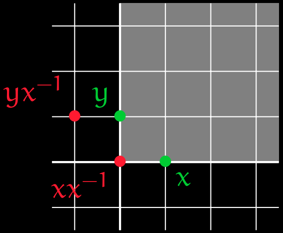

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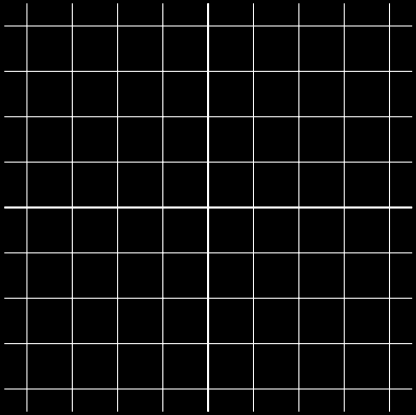
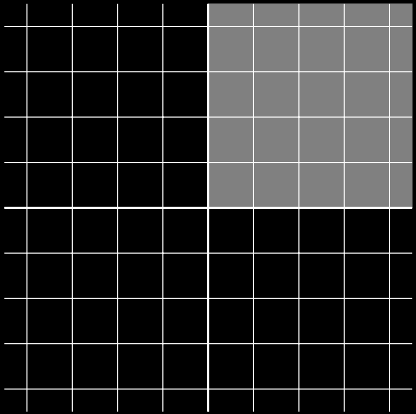
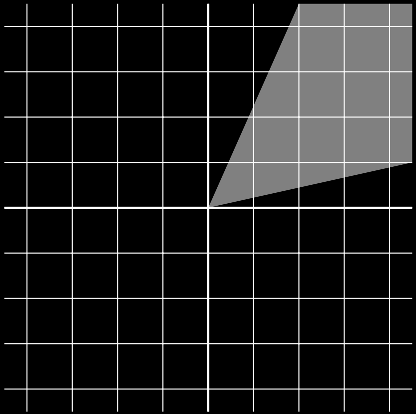
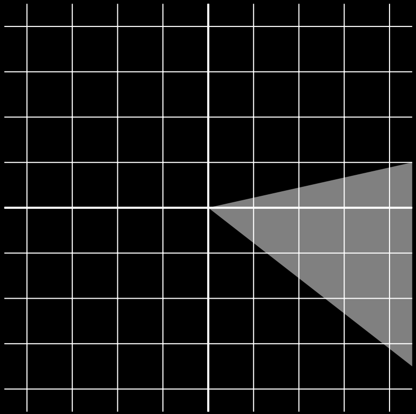
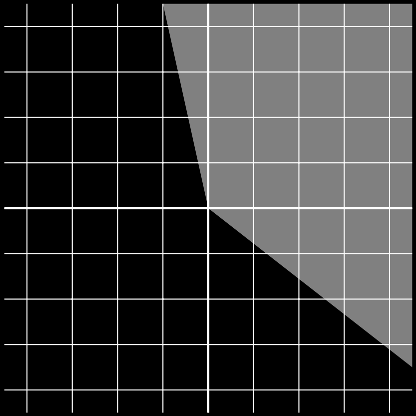
Special case: The cone C generated by the unit vectors gives the usual formal power series ring.

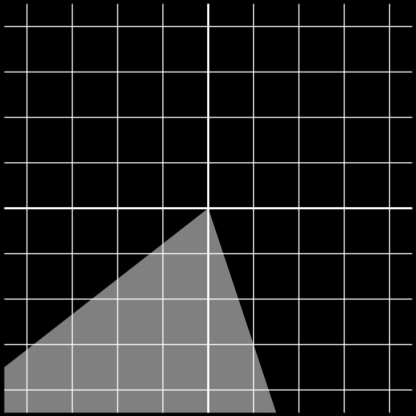


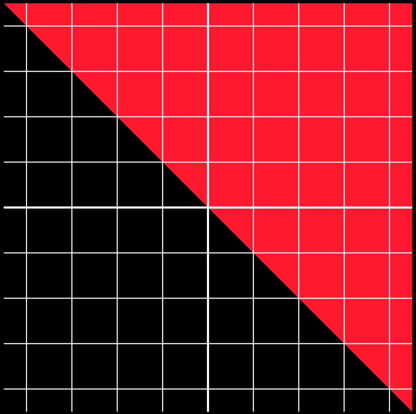












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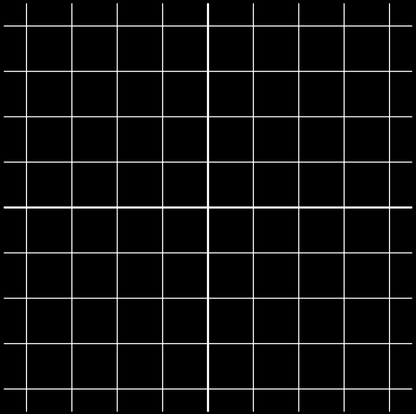

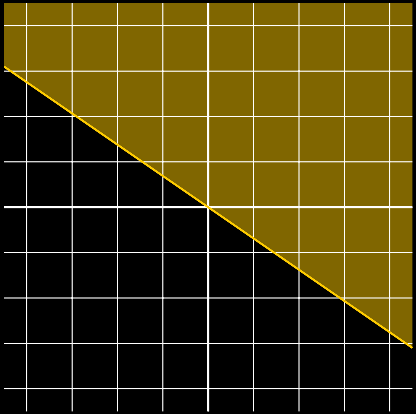

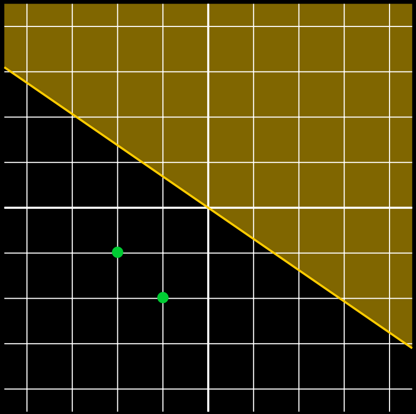

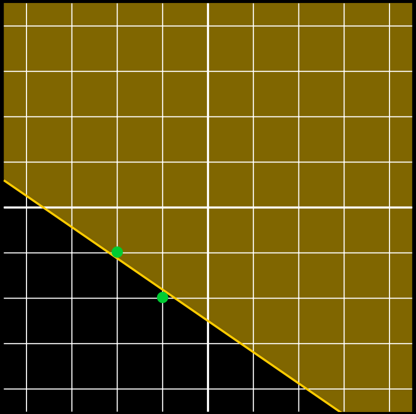

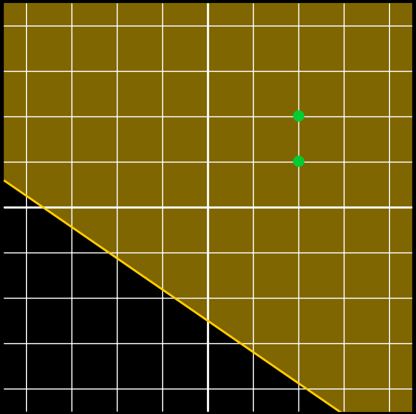

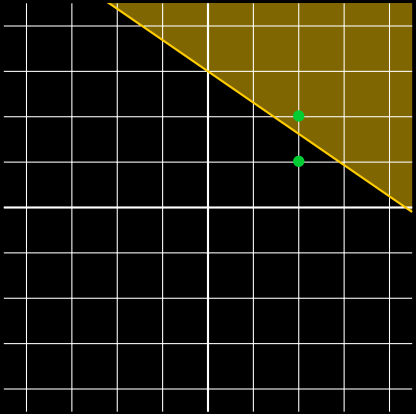

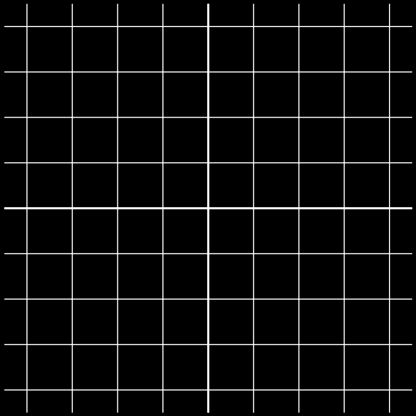

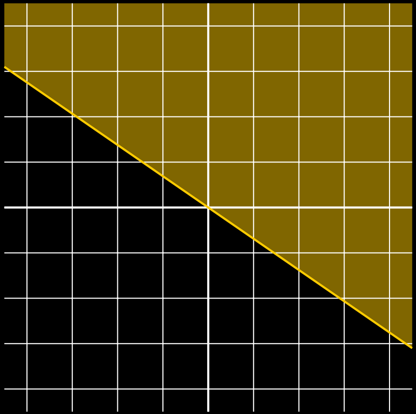

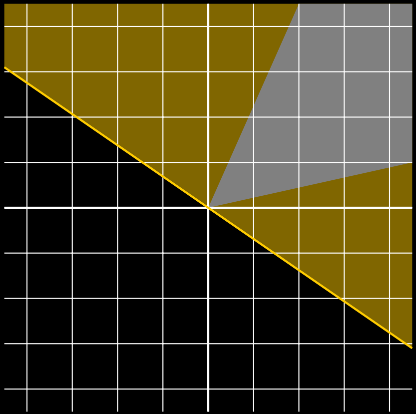

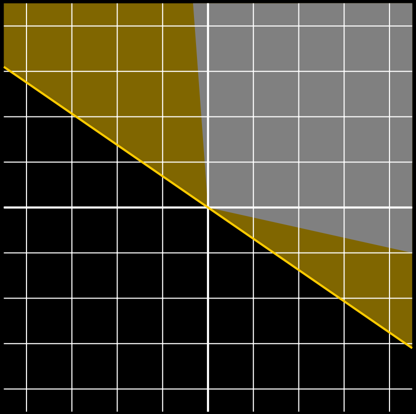

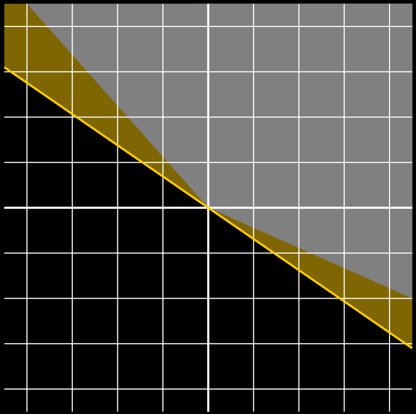

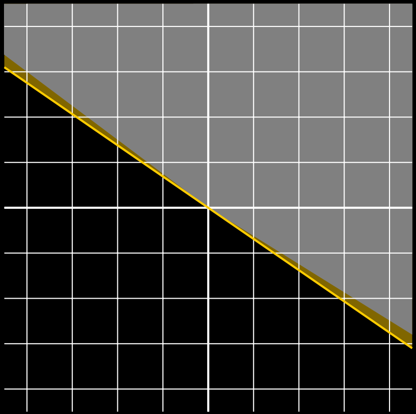

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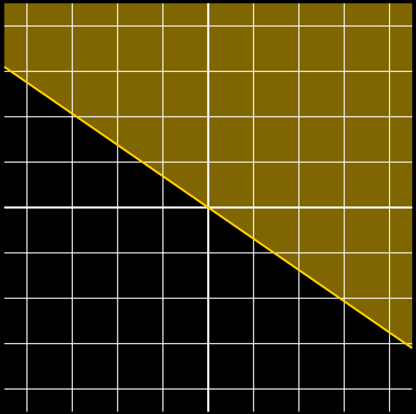
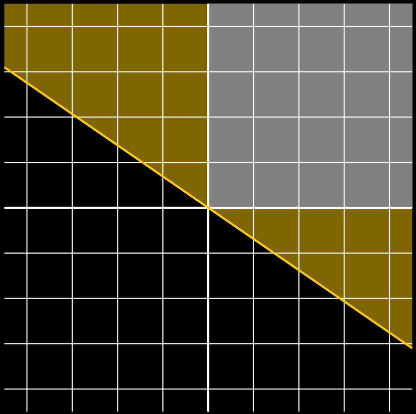
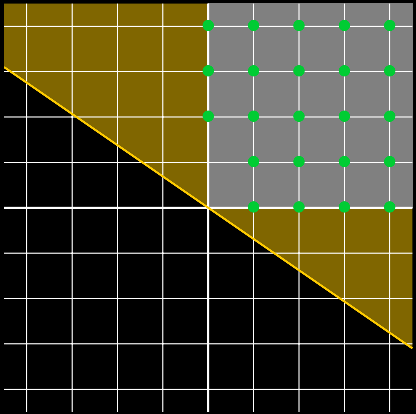
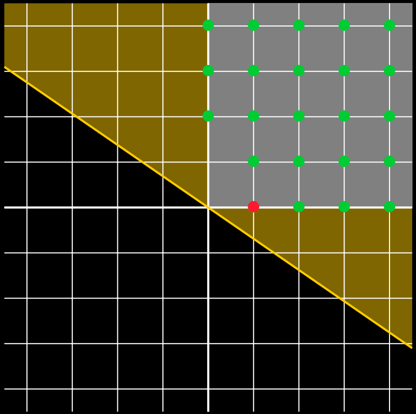
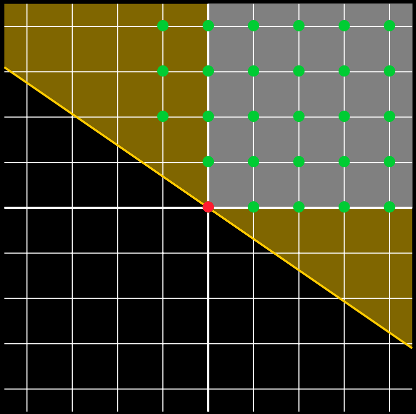
Fact: This is a field.

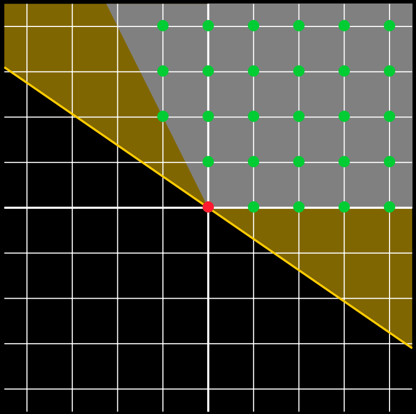












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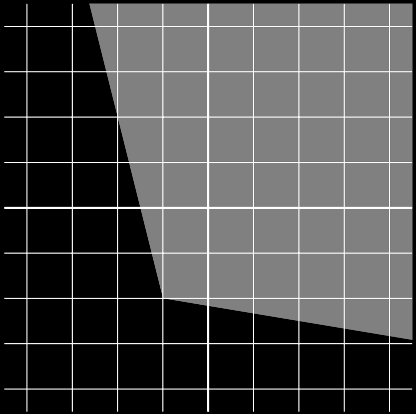
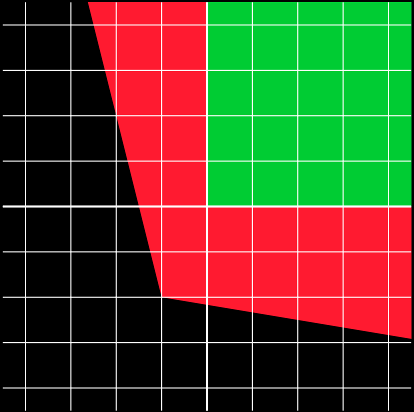
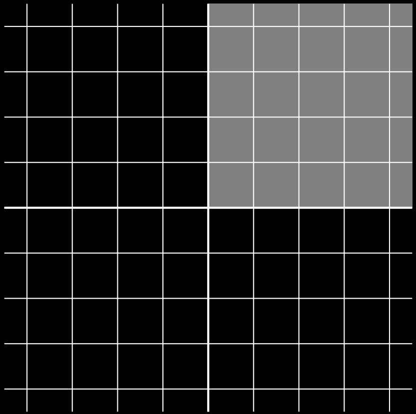
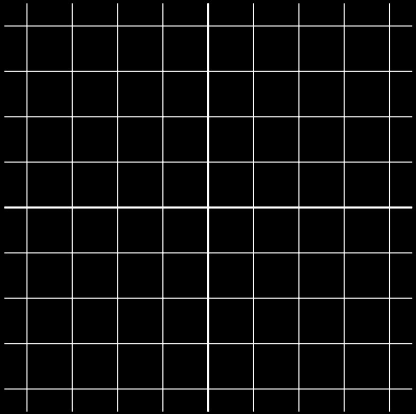
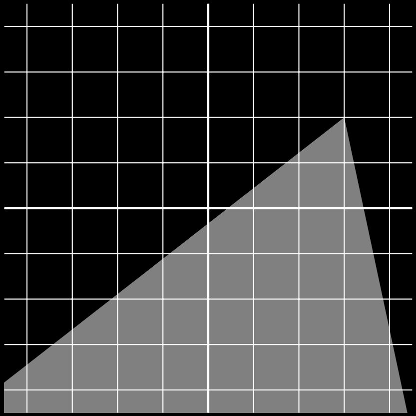
But it is still D-finite. In fact, when $f \in K_{\leq}((x_1, \dots, x_k))$ is D-finite, then so is its positive part. (Lipshitz)

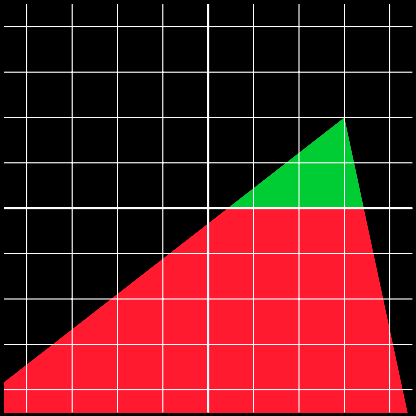


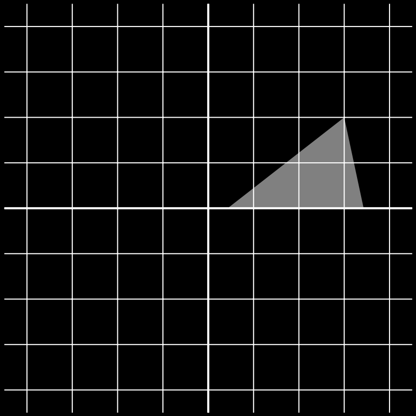












Observe: The positive part can be expressed as Hadamard product.

$$[x_1^{\geq} \cdots x_k^{\geq}]f = f \odot \underbrace{\sum_{n_1,\dots,n_k=0}^{\infty} 1\, x_1^{n_1} \cdots x_k^{n_k}}_{=\frac{1}{(1-x_1)\cdots(1-x_k)}}$$

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Observe also: For any two cones $A, B \subseteq \mathbb{R}^k$ and any two series $f \in K_A((x_1, \ldots, x_k))$ and $g \in K_B((x_1, \ldots, x_k))$ the Hadamard product $f \odot g$ is well-defined.

Theorem. Let $A, B \subseteq \mathbb{R}^k$ be two closed line-free cones, and let $f \in K_A((x_1, \ldots, x_k))$ and $g \in K_B((x_1, \ldots, x_k))$. Then

$$f \odot g = \operatorname{res}_{y_1,\ldots,y_k} y_1^{-1} \cdots y_k^{-1} f\left(\frac{x_1}{y_1}, \ldots, \frac{x_k}{y_k}\right) g(y_1, \ldots, y_k)$$

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and the expression on the right is meaningful.

Corollary: For all $f \in K_{\leq}((x_1, \ldots, x_k))$ we have

$$[x_1^{\geq} \cdots x_k^{\geq}] f = \operatorname{res}_{y_1,\ldots,y_k} f\left(\frac{x_1}{y_1}, \ldots, \frac{x_k}{y_k}\right) \frac{y_1^{-1} \cdots y_k^{-1}}{(1 - y_1) \cdots (1 - y_k)}$$
$$= \operatorname{res}_{y_1,\ldots,y_k} f(y_1, \ldots, y_k) \frac{y_1^{-2} \cdots y_k^{-2}}{(y_1 - x_1) \cdots (y_k - x_k)}$$

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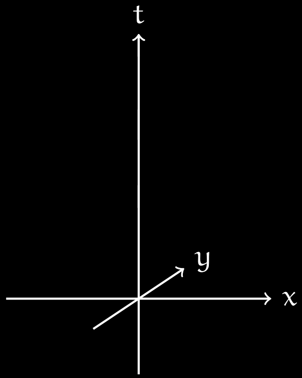
\end{aligned}$$

The multivariate version of this calculation gives rise to a new proof that taking positive parts preserves D-finiteness.

Example If $f_{n,i,j}$ is the number of lattice walks in $\mathbb{N}^2$ starting at $(0,0)$, ending at (i,j), and consisting of n steps, where each step is one of $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$, then

$$f(x,y,t) = \sum_{n=0}^{\infty} \sum_{i,j} f_{n,i,j} x^i y^j t^n = \frac{1}{xy}[x^> y^>] \frac{(x - \frac{1}{x})(y - \frac{1}{y})}{1 - (y + x + \frac{1}{x} + \frac{1}{y})t}$$

$$\frac{1}{xy}[x^{>}y^{>}]\,\frac{(x-\frac{1}{x})(y-\frac{1}{y})}{1-(y+x+\frac{1}{x}+\frac{1}{y})t}$$



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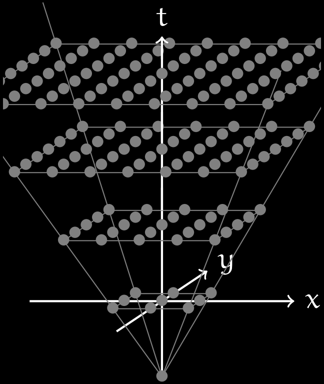

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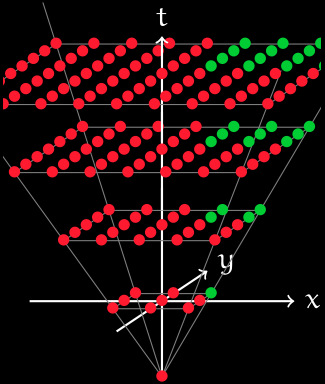

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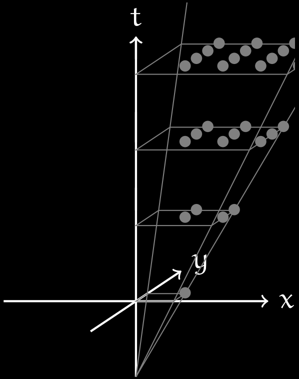

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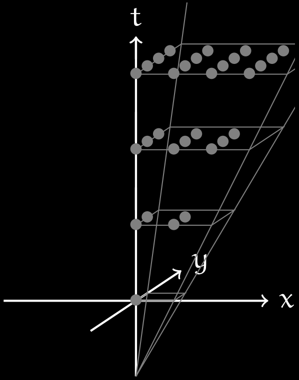

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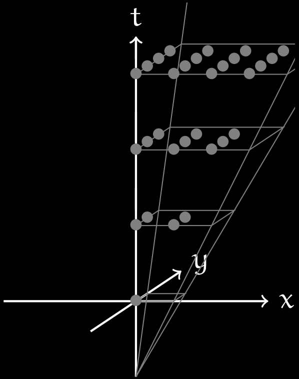
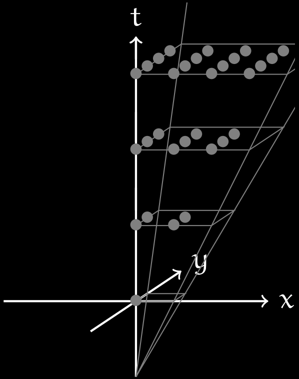


$$\frac{1}{xy}[x^{>}y^{>}]\,\underbrace{\frac{(x-\frac{1}{x})(y-\frac{1}{y})}{1-(y+x+\frac{1}{x}+\frac{1}{y})t}}_{\text{rational}}$$



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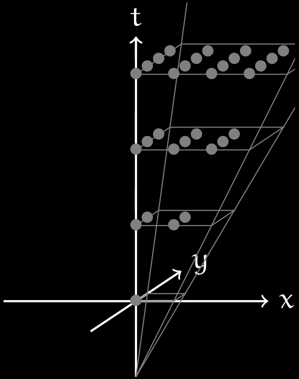

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not rational, but still D-finite

$\Downarrow$

asymptotics,

closed forms,

etc.

References

References

- For formal Laurent series in several variables:

  Ainhoa Aparicio Monforte and MK, *Expositiones Mathematicae* 31(4):350–367, Dec. 2013

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- For positive part extraction via creative telescoping and applications to counting lattice walks:

  Alin Bostan, Frédéric Chyzak, MK, Lucien Pech, Mark van Hoeij, *in preparation*