A Mathematica q-Analogue of Zeilberger's Algorithm for Proving q-Hypergeometric Identities

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Diploma Thesis

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Abstract

Besides an elementary introduction to q-identities and basic hypergeometric series, a newly developed Mathematica implementation of a q-analogue of Zeilberger's fast algorithm for proving terminating q-hypergeometric identities together with its theoretical background is described. To illustrate the usage of the package and its range of applicability, non-trivial examples are presented as well as additional features like the computation of companion and dual identities.

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0 Introduction

The study of q-identities essentially took its origin in 1748 when Euler found the generating function for $p(n)$, the number of partitions of a positive integer n into positive integers, to be

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{k=1}^{\infty} (1-q^k)^{-1}.$$

In 1812, Gauss considered the infinite series

$$F(a,b;c,z) = \\ 1 + \frac{ab}{1\cdot c}\, z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}\, z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}\, z^3 + \ldots$$

and derived the famous summation formula

$$F(a,b;c,1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

Gauss' series is a special instance of the so-called *generalized hypergeometric series* defined as

$${}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k}\, z^k,$$

where $(a)_k$ denotes the *shifted factorial* of a given by $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

Thirty years later, Heine introduced the series

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}\, z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}\, z^2 + \ldots,$$

which tends (at least termwise) to Gauss' series for $q \to 1$, since $\lim_{q\to 1}[(1-q^a)/(1-q)] = a$.

Based on the *q-shifted factorial* defined as

$$(a;q)_k = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & k > 0, \\ 1, & k = 0, \end{cases}$$

a *basic hypergeometric series* (also called q-hypergeometric series) is given by

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k.$$

In this notation, Heine's q-analogue of Gauss' summation formula reads as

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a;q)_\infty\,(c/b;q)_\infty}{(c;q)_\infty\,(c/ab;q)_\infty}, $$

where

$$ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). $$

If we define the *q-binomial coefficient* to be

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & 0 \le k \le n, \\ 0, & \text{otherwise}, \end{cases} $$

then for $q = 1$ (resp. $q \to 1$), $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ turns into $\binom{n}{k}$, the ordinary binomial coefficient. Since

$$ \begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1 \quad \text{and} \quad \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, $$

the q-binomial coefficient is a polynomial in q and therefore also called Gaussian polynomial. But what about an algebraic interpretation of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$?

There are several ones and the most natural one, for instance, described in Andrews [1] is related to finite vector spaces. Let V_n be a finite-dimensional vector space of dimension n over $GF(q)$, the finite field of q elements. Then we could ask for the number of subspaces of V_n of dimension k. First we observe that the number of k-tuples of linearly independent vectors in V_n is

$$ (q^n-1)(q^n-q)\cdots(q^n-q^{k-1}). $$

Each such k-tuple spans a k-dimensional subspace. However, two different k-tuples may span the same subspace. But the number of k-tuples spanning the same subspace is just the number of linearly independent k-tuples that exist in a k-dimensional subspace, and therefore equals

$$ (q^k-1)(q^k-q)\cdots(q^k-q^{k-1}). $$

Hence, the number of k-dimensional subspaces of V_n is

$$ \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} = \begin{bmatrix} n \\ k \end{bmatrix}_q. $$

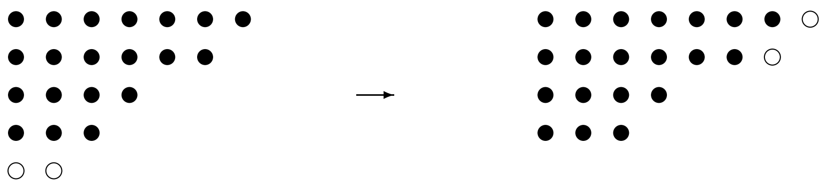
Another field of mathematics being strongly related with q-identities is the theory of number partitions. For example, let $p(N, M, n)$ denote the number of partitions of n into at most M parts each not exceeding N. Then it can be shown that

$$\sum_{n=0}^{M(N-M)} p(N-M, M, n)\, q^n = \begin{bmatrix} N \\ M \end{bmatrix}_q .$$

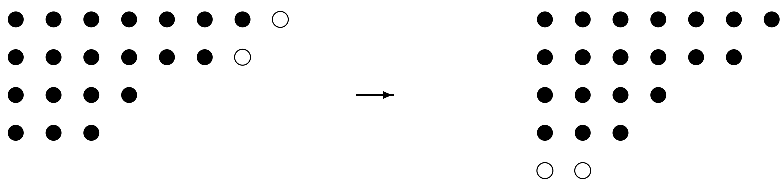
Franklin (cf. Andrews [2]) combinatorially proved the following result. Let $D_e(n)$ and $D_o(n)$ denote the set of partitions of n into an even, respectively odd number of distinct parts. Then

$$|D_e(n)| - |D_o(n)| = \begin{cases} (-1)^m, & \text{if } n = m(3m \pm 1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

The idea of the proof is as follows. We establish a one-to-one correspondence between the sets $D_e(n)$ and $D_o(n)$ whenever n is not one of the so-called pentagonal numbers $m(3m \pm 1)/2$. First, for instance, $22 = 7 + 6 + 4 + 3 + 2$ is mapped to $22 = 8 + 7 + 4 + 3$ as shown below.



Conversely, $22 = 8 + 7 + 4 + 3$ is mapped to $22 = 7 + 6 + 4 + 3 + 2$ as following.



Clearly, only one of the mappings can be applied to a partition. However, there are certain partitions for which none of the mappings works. In the first case this happens e.g. for $12 = 5 + 4 + 3$ and in the second case for $15 = 6 + 5 + 4$ as it is seen from the pictures below.

But these exceptional cases only arise, if n is partitioned into m parts of the form

$$n = m + (m+1) + \ldots + (2m-1) = m(3m-1)/2$$

or

$$n = (m+1) + (m+2) + \ldots + 2m = m(3m+1)/2,$$

respectively, which completes the proof.

From this result, the generating function version of Euler's pentagonal number theorem stating that

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m-1)/2}$$

can be deduced immediately.

Finally, for many binomial coefficient identities we can derive q-analogues. Let us consider three classical identities, a special case of the binomial theorem

$$\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k} = \delta_{n,0},$$

of an alternating-sign version of Vandermonde's identity

$$\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^2 = \binom{2n}{n},$$

and of Dixon's identity

$$\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^3 = \frac{(3n)!}{(n!)^3}.$$

There are several techniques for proving or finding these identities. First we could try to apply some standard methods like manipulating generating functions, comparing coefficients of formal power series or introducing suitable operators. On the other hand we could transform the identities into hypergeometric form and then compare it with yet known results in the so-called hypergeometric database containing basic summation and transformation formulas for ${}_rF_s$ series. Or we could make use of Gosper's and Zeilberger's algorithms to come up with a closed form or at least a recurrence for the sum.

For the examples above, the corresponding q-identities, which for $q = 1$ (or $q \to 1$) specialize to the identities above, become special cases of the q-binomial theorem

$$\sum_{k=-n}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}2n\\n+k\end{bmatrix}_q = \delta_{n,0},$$

of the q-Vandermonde identity

$$\sum_{k=-n}^{n} (-1)^k q^{k^2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2},$$

and of the q-Dixon identity

$$\sum_{k=-n}^{n} (-1)^k q^{(3k^2+k)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 = \frac{(q;q)_{3n}}{(q;q)_n^3},$$

which can be proven by adapting the tools listed above to the q-case. We want to emphasize that a given binomial coefficient identity might find several, substantially different, q-generalizations.

The thesis is organized as follows. In Section 1 we shall give an elementary introduction to q-identities first by presenting the q-differentiation operator and the q-exponential function. Then we shall derive q-analogues of the polynomials $(x-a)^n$ and $(x+a)^n$, and prove the q-binomial theorem for operators. Finally we shall turn to the notion of basic hypergeometric series and deduce some fundamental summation and transformation formulas. Besides serving as an introduction, the goal of this section is to illustrate how q-identities could be handled by classic *non-algorithmic* (operator) methods.

In Section 2 we change to an *algorithmic* point of view. We shall outline how Gosper's and Zeilberger's algorithms can be carried over to the q-case. We shall give a precise description of the author's Mathematica implementation and compare it with an already existing Maple package.

In Section 3 we shall give non-trivial applications of the program to illustrate certain proof-strategies and describe additional features like the computation of companion and dual identities.

Acknowledgments: I thank my supervisor Peter Paule for successfully infecting me with the q-disease. His extensive cooperation enabled me to fill a lot of theoretical gaps, as well as his inexhaustible source of examples repeatedly provoked me to improve the abilities of the program, especially the run-time behavior. The equation solver used in my package was kindly made available by Erhard Aichinger.

1 A Guide to q-Identities

The objective of this section is to give an overview of some well-known elementary q-identities by presenting two different proof-strategies. We shall first follow the approach given in Cigler [3], [4] by introducing the q-differentiation operator, which will lead us to q-analogues of the binomial coefficients, the exponential function and the polynomials $(x-a)^n$ and $(x+a)^n$. Finally, we shall derive the q-binomial theorem for operators. In Subsection 1.6 we shall turn to the theory of basic hypergeometric series by following Gasper and Rahman [6] and Andrews [2]. We shall derive several fundamental summation and transformation formulas of the q-hypergeometric database. Since basic hypergeometric series can be regarded as some kind of normal form for q-binomial coefficient identities, the highly non-trivial task of proving boils down to a table lookup in this frame.

To avoid questions concerning convergence and other analytical problems, which may arise in the treatment of infinite series, we will view all identities strictly in the sense of formal power series or Laurent-series, but *not* analytically over the real or complex numbers.

We will state the results in a way such that they immediately reduce to the classical ones by setting $q = 1$.

1.1 The q-Differentiation Operator

In the following let q be an indeterminate, which could be specialized to a non-zero complex number (probably subject to further conditions depending on the context). Let P be the ring, resp. vector space, of the polynomials over the complex numbers $\mathbf{C}$, and Q the ring, resp. vector space, of the formal power series over $\mathbf{C}$. The symbols $a(x)$, $b(x)$ and $f(x)$ denote formal power series of x, whereas k, l, m and n are integers.

Definition 1.1.1. The operator D_q on Q given by

$$(D_q f)(x) := \frac{f(qx) - f(x)}{qx - x}$$

is called the *q-differentiation operator*.

The q-differentiation operator is like the difference operator

$$(\Lambda_h f)(x) = \frac{f(x+h) - f(x)}{h}$$

a discrete analogue of the ordinary differentiation operator D_1 (D_q with $q = 1$ or Λ_h with $h = 0$). We will shortly write $f'^{q}(x)$ for $(D_q f)(x)$ and $f^{(n)_q}(x)$ for $(D_q^n f)(x)$.

For formal power series $a(x) = \sum_{n=0}^{\infty} a_n x^n$, q-differentiation yields

$$\begin{aligned} a'^{q}(x) &= \frac{a(qx) - a(x)}{(q-1)x} = \frac{\sum_{n=0}^{\infty} a_n (qx)^n - \sum_{n=0}^{\infty} a_n x^n}{(q-1)x} \\ &= \frac{\sum_{n=1}^{\infty} a_n x^{n-1}(q^n - 1)}{q-1} = \sum_{n=1}^{\infty} [n]_q \, a_n x^{n-1}, \end{aligned}$$

where $[n]_q := 1 + q + q^2 + \ldots + q^{n-1} = \dfrac{q^n - 1}{q - 1}$ for $n \geq 1$ and $[0]_q := 0$.

Since $a(qx) - a(x) = \sum_{n=1}^{\infty} a_n x^n (q^n - 1)$ is a multiple of x, we can always divide by x without leaving Q. Clearly, D_q is a linear operator on Q, i.e., for $\lambda, \mu \in \mathbf{C}$ and $a, b \in Q$, we have

$$(\lambda a(x) + \mu b(x))'^{q} = \lambda a'^{q}(x) + \mu b'^{q}(x).$$

Definition 1.1.2. Let ϵ denote the *q-shift operator* on Q given by

$$(\epsilon f)(x) := f(qx).$$

Now we can reformulate Definition 1.1.1 in operator notation as

$$D_q = \frac{1}{(q-1)x} \, (\epsilon - Id).$$

Theorem 1.1.1 (product formulas for D_q).

$$\begin{aligned} (a(x)\, b(x))'^{q} &= a(qx)\, b'^{q}(x) + a'^{q}(x)\, b(x) && (1.1.1) \\ &= a(x)\, b'^{q}(x) + a'^{q}(x)\, b(qx). && (1.1.2) \end{aligned}$$

Proof.

$$\begin{aligned} (a(x)\, b(x))'^{q} &= \frac{a(qx)\, b(qx) - a(x)\, b(x)}{(q-1)x} \\ &= \frac{a(qx)\,(b(qx) - b(x))}{(q-1)x} + \frac{(a(qx) - a(x))\, b(x)}{(q-1)x} \\ &= a(qx)\, b'^{q}(x) + a'^{q}(x)\, b(x). \end{aligned}$$

The second equation is immediately obtained by exchanging a and b. ■

In operator notation, (1.1.1) and (1.1.2) read as

$$\begin{aligned} D_q(ab) &= (\epsilon a)(D_q b) + (D_q a)b \\ &= a(D_q b) + (D_q a)(\epsilon b). \end{aligned}$$

Choosing $a(x) = x^k$ $(k \geq 1)$ and $b(x) = x^n$ $(n \geq 1)$ in the product formulas (1.1.1) and (1.1.2), we get

$$[n+k]_q \, x^{n+k-1} = q^k x^k [n]_q \, x^{n-1} + [k]_q \, x^{n+k-1}$$

and

$$[n+k]_q \, x^{n+k-1} = x^k [n]_q \, x^{n-1} + [k]_q \, x^{k-1} q^n x^n$$

respectively, proving that $[n]_q$ satisfies the recurrence relations $(n \geq 0, k \geq 0)$

$$[n+k]_q \;=\; q^k [n]_q + [k]_q \;=\; [n]_q + q^n [k]_q. \qquad (1.1.3)$$

Definition 1.1.3. For $n \in \mathbf{N}, k \in \mathbf{Z}$,

$$[n]_q! := \begin{cases} [1]_q \, [2]_q \cdots [n]_q, & n > 0, \\ 1, & n = 0 \end{cases}$$

is called *q-factorial* of n.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{[n]_q!}{[k]_q! \, [n-k]_q!}, & 0 \leq k \leq n, \\ 0, & \text{otherwise} \end{cases}$$

is called the *q-binomial coefficient* of n and k.

Remark 1.1.4. $[n]_q \Big|_{q=1} = n, \quad [n]_q! \Big|_{q=1} = n! \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=1} = \binom{n}{k}.$

Theorem 1.1.2. For $0 \leq k \leq n$, the q-binomial coefficients satisfy the recurrence relations

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \qquad (1.1.4) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \qquad (1.1.5) \end{aligned}$$

Proof.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q!\,[n+1-k]_q!}\left([n+1]_q - [k]_q\right) \\
&\overset{(1.1.3)}{=} \frac{[n]_q!}{[k]_q!\,[n+1-k]_q!}\, q^k [n+1-k]_q \;=\; q^k \begin{bmatrix} n \\ k \end{bmatrix}_q .
\end{aligned}$$

Equation (1.1.5) is obtained by replacing k by $n+1-k$ in (1.1.4). ■

From the recurrence relations (1.1.4), (1.1.5) and the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$$

it follows that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is a polynomial of degree $k(n-k)$ in q. Therefore, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is also called *Gaussian polynomial*.

With the tools provided so far we are able to prove a q-analogue of Leibniz' formula, which states that

$$(f(x)\,g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x).$$

Theorem 1.1.3 (q-Leibniz).

$$(a(x)\,b(x))^{(n)_q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{(n-k)_q}(q^k x)\, b^{(k)_q}(x).$$

Proof. Induction base: $(a(x)\,b(x))'^{q} \overset{(1.1.1)}{=} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_q a'^{q}(x)\, b(x) + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]_q a(qx)\, b'^{q}(x)$.

Induction step: Since $(f(cx))'^{q} = c\, f'^{q}(cx)$, we have

$$\begin{aligned}
(a(x)\,b(x))^{(n+1)_q} &\overset{(1.1.1)}{=} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{(n-k)_q}(q^{k+1}x)\, b^{(k+1)_q}(x) \\
&\qquad + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k a^{(n+1-k)_q}(q^k x)\, b^{(k)_q}(x) \\
&= \sum_{k=0}^{n+1} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) a^{(n+1-k)_q}(q^k x)\, b^{(k)_q}(x) \\
&\overset{(1.1.4)}{=} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q a^{(n+1-k)_q}(q^k x)\, b^{(k)_q}(x).
\end{aligned}$$

■

Sometimes we will turn from base q to base q^{-1}, where the following inversion formulas will turn out to be very helpful.

Lemma 1.1.4. For $n \geq 0$,

$$[n]_q! = [n]_{\frac{1}{q}}!\, q^{\binom{n}{2}}. \tag{1.1.6}$$

Proof.

$$[n]_q! = \prod_{k=1}^{n} \frac{q^k - 1}{q - 1} = \frac{q^{1+2+\ldots+n}}{q^n} \prod_{k=1}^{n} \frac{1 - q^{-k}}{1 - q^{-1}} = [n]_{\frac{1}{q}}!\, q^{\binom{n}{2}}. \qquad \blacksquare$$

Lemma 1.1.5.

$$\epsilon^{-1} D_q = D_{\frac{1}{q}}. \tag{1.1.7}$$

Proof.

$$(\epsilon^{-1} D_q)\, x^k = \frac{[k]_q}{q^{k-1}}\, x^{k-1} = \frac{q^{-k} - 1}{q^{-1} - 1}\, x^{k-1} = [k]_{\frac{1}{q}}\, x^{k-1} = D_{\frac{1}{q}}\, x^k. \qquad \blacksquare$$

1.2 The *q*-Exponential Function

Definition 1.2.1.

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

is called the *q-exponential function*.

There are several other possibilities to introduce a q-analogue of the exponential function. See, for instance, Subsection 1.7.

Theorem 1.2.1. $f(x) = e_q(ax)$ is the uniquely determined solution of the differential equation

$$f'^{q}(x) = a f(x) \quad \text{with} \quad f(0) = 1,$$

where by $f(0)$ we mean the constant term, i.e., the coefficient of x^0 in the power series $f(x)$.

Proof. Because $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we have $f(0) = a_0 = 1$. Since $f'^{q}(x) = af(x)$ is equivalent to

$$\sum_{n=1}^{\infty} [n]_q \, a_n x^{n-1} = a \sum_{n=0}^{\infty} a_n x^n,$$

it follows that

$$a_{n+1} = \frac{a\, a_n}{[n+1]_q} = \frac{a^2 a_{n-1}}{[n+1]_q\, [n]_q} = \ldots = \frac{a^{n+1} a_0}{[n+1]_q!} = \frac{a^{n+1}}{[n+1]_q!}.$$

Hence we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{a^n}{[n]_q!}\, x^n = e_q(ax).$$ ∎

We use this result to write

$$e_q(ax)'^{q} = \frac{e_q(aqx) - e_q(ax)}{(q-1)x} = a\, e_q(ax),$$

giving an alternative description of the q-exponential function.

Corollary 1.2.2. $e_q(ax)$ is also characterized by

$$e_q(aqx) = (1 + (q-1)ax)\, e_q(ax) \quad \text{and} \quad e_q(0) = 1. \tag{1.2.1}$$

A q-analogue of the inversion formula $1/e(x) = e(-x)$ for the ordinary exponential function reads as following.

Theorem 1.2.3.

$$\frac{1}{e_q(x)} = e_{\frac{1}{q}}(-x) \qquad \left(= \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}}\, \frac{x^n}{[n]_q!} \right). \tag{1.2.2}$$

Proof.

$$\begin{aligned}
D_q\big(e_q(x)\, e_{\frac{1}{q}}(-x)\big) &\overset{(1.1.2)}{=} e_q(x)\, D_q\big(e_{\frac{1}{q}}(-x)\big) + D_q\big(e_q(x)\big)\, \epsilon\big(e_{\frac{1}{q}}(-x)\big) \\
&\overset{(1.1.7)}{=} e_q(x)\, \big(\epsilon D_{\frac{1}{q}}\big)\big(e_{\frac{1}{q}}(-x)\big) + e_q(x)\, e_{\frac{1}{q}}(-qx) \\
&= e_q(x)\, (-e_{\frac{1}{q}}(-qx)) + e_q(x)\, e_{\frac{1}{q}}(-qx) \\
&= 0.
\end{aligned}$$

Because $e_q(0) = e_{\frac{1}{q}}(0) = 1$, we get $e_q(x)\, e_{\frac{1}{q}}(-x) = 1$. The parenthesized assertion is an immediate consequence of (1.1.6). ∎

Theorem 1.2.4.

$$e_q(x) = e_{q^2}\left(\frac{x}{[2]_q}\right)e_{q^2}\left(\frac{qx}{[2]_q}\right). \tag{1.2.3}$$

Proof. By (1.2.1) we have

$$e_{q^2}\left(\frac{qx}{[2]_q}\right)e_{q^2}\left(\frac{q^2x}{[2]_q}\right) = e_{q^2}\left(\frac{qx}{[2]_q}\right)\left(1+\frac{q^2-1}{[2]_q}\,x\right)e_{q^2}\left(\frac{x}{[2]_q}\right).$$

Thus, denoting the right hand side of equation (1.2.3) by $f(x)$ we find that $f(x)$ satisfies

$$f(qx) = (1+(q-1)x)f(x)$$

with $f(0) = 1$. Hence, (1.2.1) proves the assertion. ∎

To handle the frequently occurring products of formal power series let us consider the following special case of Cauchy's product formula for infinite series stating that

$$\sum_{k=0}^{\infty}\frac{a_k}{[k]_q!}\,x^k\cdot\sum_{l=0}^{\infty}\frac{b_l}{[l]_q!}\,x^l = \sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\left[{n \atop k}\right]_q a_k b_{n-k}}{[n]_q!}\,x^n, \tag{1.2.4}$$

which provides a powerful tool for deriving new identities in combination with the comparison of coefficients.

Theorem 1.2.5.

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k = (1+q)(1+q^2)\cdots(1+q^n). \tag{1.2.5}$$

Proof. By (1.2.3) and (1.2.4) we have

$$e_{q^2}\left(\frac{x}{[2]_q}\right)e_{q^2}\left(\frac{qx}{[2]_q}\right) = \sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\left[{n \atop k}\right]_{q^2} q^{n-k}}{[2]_q^n\,[n]_{q^2}!}\,x^n = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}.$$

Comparing the coefficients leads to

$$\begin{aligned}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{n-k} &= \frac{[n]_{q^2}!\,[2]_q^n}{[n]_q!} = \prod_{k=1}^{n}\frac{q^{2k}-1}{q^2-1}\,\frac{q^2-1}{q-1}\,\frac{q-1}{q^k-1}\\ &= (1+q)(1+q^2)\cdots(1+q^n).\end{aligned}$$ ∎

For $q = 1$, (1.2.5) reduces to the well-known formula

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

1.3 A q-Analogue of the Polynomials $(x-a)^n$

To derive a q-analogue of the polynomials $\overline{p}_n(x,a) = (x-a)^n$ of degree n we first observe that the $\overline{p}_n(x,a)$ are characterized by

$$\overline{p}_n(a,a) = \delta_{n,0} \quad \text{and} \quad \overline{p}_n'(x,a) = n\,\overline{p}_{n-1}(x,a).$$

Hence, we are looking for nth degree polynomials $p_n(x,a)$ satisfying

$$p_n(a,a) = \delta_{n,0} \quad \text{and} \quad p_n^{\prime_q}(x,a) = [n]_q\, p_{n-1}(x,a).$$

If such polynomials do exist, then we must have

$$p_n^{\prime_q}(x,a) = \frac{p_n(qx,a) - p_n(x,a)}{(q-1)x} = [n]_q\, p_{n-1}(x,a),$$

or in other words

$$p_n(qx,a) = p_n(x,a) + (q^n - 1)x\, p_{n-1}(x,a).$$

Thus the p_n must satisfy

$$p_n(qa,a) = 0 \text{ for } n > 1$$

and, more generally,

$$p_n(q^i a,a) = 0 \text{ for } n > i,$$

which leads to the following theorem.

Theorem 1.3.1. The uniquely determined polynomials $p_n(x,a)$ of degree n with $p_n(a,a) = \delta_{n,0}$ and $p_n^{\prime_q}(x,a) = [n]_q\, p_{n-1}(x,a)$ are given by

$$p_n(x,a) = \begin{cases} (x-a)(x-qa)\cdots(x-q^{n-1}a), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Next we are going to derive the expanded representation of the p_n, i.e., we try to determine coefficients a_{nk} such that $p_n(x,a) = \sum_{k=0}^{n} a_{nk} x^k$.

Definition 1.3.1. Let L denote the operator on Q defined by

$$(Lf)(x) := f(0).$$

Lemma 1.3.2. The coefficients of a formal power series $a(x) = \sum_{n=0}^{\infty} a_n x^n$ are given by

$$a_k = L\left(\frac{D_q^k}{[k]_q!}\, a(x)\right). \tag{1.3.1}$$

Proof.

$$L\left(\frac{D_q^k}{[k]_q!}\sum_{n=0}^{\infty}a_nx^n\right)=L\left(\sum_{n=k}^{\infty}\frac{[n]_q\,[n-1]_q\cdots[n-k+1]_q}{[k]_q!}\;a_nx^{n-k}\right)=a_k. \quad \blacksquare$$

If we apply this result to $p_n(x,a)=\sum_k a_{nk}x^k$, we obtain

$$a_{nk}=L\left(\frac{D_q^k}{[k]_q!}\;p_n(x,a)\right)=L\left(\begin{bmatrix}n\\k\end{bmatrix}_q p_{n-k}(x,a)\right)=(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}a^{n-k},$$

giving the coefficients of the p_n.

Theorem 1.3.3.

$$p_n(x,a)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}a^kx^{n-k}. \tag{1.3.2}$$

Choosing $a=1$ and replacing x by $-1/x$ in (1.3.2), we get

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}x^k=(1+x)(1+qx)\cdots(1+q^{n-1}x). \tag{1.3.3}$$

The polynomials p_n occur in various problems and are strongly related with the q-exponential function. For instance, from (1.3.2) it follows that

$$p_n(x,a)=e_{\frac{1}{q}}(-aD_q)x^n=\frac{1}{e_q(aD_q)}\;x^n.$$

Hence, for the generating function of the p_n we obtain

$$\sum_{n=0}^{\infty}\frac{p_n(x,a)}{[n]_q!}\;t^n=\sum_{n=0}^{\infty}\frac{1}{e_q(aD_q)}\;x^n\;\frac{t^n}{[n]_q!}=\frac{1}{e_q(aD_q)}\;e_q(xt).$$

Since

$$D_q^n\,e_q(xt)=t^n\,e_q(xt),$$

we proved the following result.

Theorem 1.3.4.

$$\sum_{n=0}^{\infty}\frac{p_n(x,a)}{[n]_q!}\;t^n=\frac{e_q(xt)}{e_q(at)}. \tag{1.3.4}$$

Theorem 1.3.5.

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x,a)\, p_{n-k}(a,y) = p_n(x,y).$$

Proof. From

$$\frac{e_q(xt)}{e_q(at)}\,\frac{e_q(at)}{e_q(yt)} = \frac{e_q(xt)}{e_q(yt)}$$

we get by (1.3.4)

$$\sum_{n=0}^{\infty} \frac{p_n(x,a)}{[n]_q!}\, t^n \cdot \sum_{n=0}^{\infty} \frac{p_n(a,y)}{[n]_q!}\, t^n = \sum_{n=0}^{\infty} \frac{p_n(x,y)}{[n]_q!}\, t^n$$

and by (1.2.4)

$$\sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x,a)\, p_{n-k}(a,y)}{[n]_q!}\, t^n = \sum_{n=0}^{\infty} \frac{p_n(x,y)}{[n]_q!}\, t^n.$$

Comparing the coefficients proves the assertion. ∎

1.4 A q-Analogue of the Polynomials $(x + a)^n$

Similar to the previous subsection we are now looking for a q-analogue of the polynomials $\overline{r}_n(x,a) = (x+a)^n$ of degree n, which are characterized by

$$L\,\overline{r}_n(x,a) = a^n \quad \text{and} \quad \overline{r}_n'(x,a) = n\,\overline{r}_{n-1}(x,a).$$

This leads to the following question. Do there exist polynomials $r_n(x,a)$ of degree n satisfying

$$L\,r_n(x,a) = a^n \quad \text{and} \quad r_n'^{q}(x,a) = [n]_q\, r_{n-1}(x,a)?$$

The answer is given in the following theorem.

Theorem 1.4.1. The uniquely determined polynomials $r_n(x,a)$ of degree n with $L\,r_n(x,a) = a^n$ and $r_n'^{q}(x,a) = [n]_q\, r_{n-1}(x,a)$ (called the *Rogers-Szegö polynomials*) are given by

$$r_n(x,a) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k x^{n-k}. \tag{1.4.1}$$

Proof. Let $r_n(x,a) = \sum_{k=0}^{n} a_{nk} x^k$. By (1.3.1) we have

$$a_{nk} = L\left(\frac{D_q^k}{[k]_q!}\, r_n(x,a)\right) = L\left(\begin{bmatrix} n \\ k \end{bmatrix}_q r_{n-k}(x,a)\right) = \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k}. \qquad \blacksquare$$

Similar to (1.3.4), for the generating function of the r_n we write (1.4.1) as

$$r_n(x,a) = e_q(aD_q)x^n$$

to obtain the following result.

Theorem 1.4.2.

$$\sum_{n=0}^{\infty} \frac{r_n(x,a)}{[n]_q!}\, t^n = e_q(at)\, e_q(xt). \tag{1.4.2}$$

Finally, to come up with a recurrence for the r_n we use (1.4.1) and (1.1.4) to find that

$$\begin{aligned} r_{n+1}(x,a) &= \sum_{k=0}^{n+1} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q\right) a^{n+1-k} x^k \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} x^{k+1} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n+1-k} q^k x^k \\ &= (x + a\epsilon)\, r_n(x,a), \end{aligned} \tag{1.4.3}$$

or in other words

$$r_{n+1}(x,a) = (x+a)\, r_n(x,a) + a(\epsilon - Id)\, r_n(x,a).$$

Since

$$\begin{aligned} (\epsilon - Id)\, r_n(x,a) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (q^k - 1) a^{n-k} x^k \\ &= (q-1)x\, r_n'^{q}(x,a) \\ &= (q^n - 1)x\, r_{n-1}(x,a), \end{aligned}$$

we proved the following theorem.

Theorem 1.4.3.

$$r_{n+1}(x,a) = (x+a)\, r_n(x,a) + a(q^n - 1)x\, r_{n-1}(x,a). \tag{1.4.4}$$

For $a = 1$ and $x = 1$, (1.4.4) reduces to

$$\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = 2\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q + (q^n - 1)\sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q .$$

Choosing $a = -1$ and $x = 1$ in (1.4.4) gives, for even indices, Gauss' identity

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q = (1-q)(1-q^3)\cdots(1-q^{2n-1}).$$

1.5 The q-Binomial Theorem for Operators

In the literature many formulas are called "q-binomial theorem". In this subsection we shall present two equivalent versions for non-commutative linear operators acting on polynomials, which will allow us to deduce some already proven results in a very elegant way.

Theorem 1.5.1 (q-binomial theorem for operators I). Let A_0 and A_1 be linear operators on P with $A_1 A_0 = q A_0 A_1$. Then

$$(A_0 + A_1)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_0^k A_1^{n-k}. \tag{1.5.1}$$

Proof. Induction base: $A_0 + A_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]_q A_1 + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]_q A_0$.

Induction step:

$$\begin{aligned} (A_0 + A_1)^{n+1} &= (A_0 + A_1)\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_0^k A_1^{n-k} \\ &= \sum_{k=0}^{n+1}\left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q\right) A_0^k A_1^{n+1-k} \\ &\overset{(1.1.4)}{=} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q A_0^k A_1^{n+1-k}. \end{aligned}$$ ■

For A_0 and A_1 satisfying $A_1 A_0 = q A_0 A_1$, it follows from (1.2.4) and (1.5.1) that

$$e_q(A_0 t)\, e_q(A_1 t) = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q A_0^k A_1^{n-k}}{[n]_q!}\, t^n = \sum_{n=0}^{\infty} \frac{(A_0 + A_1)^n}{[n]_q!}\, t^n,$$

which proves the following version of the q-binomial theorem.

Theorem 1.5.2 (q-binomial theorem for operators II). Let A_0 and A_1 be linear operators on P with $A_1A_0 = qA_0A_1$. Then

$$e_q(A_0t)\, e_q(A_1t) = e_q((A_0 + A_1)t). \tag{1.5.2}$$

Without proof (induction on s) we state the corresponding q-multinomial theorem for operators.

Definition 1.5.1. Let $k_1, \ldots, k_s$ be non-negative integers such that $\sum_{j=1}^{s} k_j = n$. Then

$$\begin{bmatrix} n \\ k_1, \ldots, k_s \end{bmatrix}_q := \frac{[n]_q!}{[k_1]_q! \cdots [k_s]_q!}$$

is called the *q-multinomial coefficient* of n and $k_1, \ldots, k_s$.

Theorem 1.5.3 (q-multinomial theorem for operators). Let $A_1, \ldots, A_s$ be linear operators on P with $A_jA_i = qA_iA_j$ for $i < j$. Then

$$(A_1 + \ldots + A_s)^n = \sum_{k_1+\ldots+k_s=n} \begin{bmatrix} n \\ k_1, \ldots, k_s \end{bmatrix}_q A_1^{k_1} \cdots A_s^{k_s}.$$

Theorem 1.5.4. If $ad - bc = 1$, then

$$(A_0, A_1) := (x^a \epsilon^b, x^c \epsilon^d)$$

satisfy $A_1A_0 = qA_0A_1$, where in this context x is meant to be the multiplication operator, i.e., $(xf)(x) = xf(x)$.

Proof. Since

$$(A_1A_0)x^k = A_1(x^{a+k}q^{bk}) = x^{a+c+k}q^{bk+d(a+k)}$$

and

$$(A_0A_1)x^k = A_0(x^{c+k}q^{dk}) = x^{a+c+k}q^{dk+b(c+k)},$$

we have

$$\frac{(A_1A_0x^k)}{(A_0A_1x^k)} = q \iff bk + d(a+k) - dk - b(c+k) = 1 \iff ad - bc = 1. \quad \blacksquare$$

Therefore, examples for operators (A_0, A_1) satisfying $A_1A_0 = qA_0A_1$ are, for instance, (x, ϵ), $(x, x\epsilon)$ or $(x\epsilon, \epsilon)$. The following examples shall illustrate the effectiveness of the q-binomial theorem.

1. We put $(A_0, A_1) = (x, -x\epsilon)$. Then by (1.5.2) we have
$$e_q(x)\, e_q(-x\epsilon) = e_q(x(Id - \epsilon)).$$
Since $(Id - \epsilon)1 = 0$, we can conclude that
$$(e_q(x)\, e_q(-x\epsilon))1 = 1.$$
Observing that
$$(-x\epsilon)^n 1 = (-x\epsilon)^{n-1}(-x) = (-x\epsilon)^{n-2}(qx^2) = \ldots = (-1)^n q^{\binom{n}{2}} x^n,$$
we obtain
$$\frac{1}{e_q(x)} = e_q(-x\epsilon)1 = e_{\frac{1}{q}}(-x),$$
which again proves (1.2.2).

2. For $(A_0, A_1) = (x\epsilon, \epsilon)$ we know from (1.5.1) that
$$(x\epsilon + \epsilon)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x\epsilon)^k \epsilon^{n-k}.$$
Applying both sides to 1 we obtain (1.3.3), since
$$\begin{aligned} (x\epsilon + \epsilon)^n 1 &= (x\epsilon + \epsilon)^{n-1}\,(x + 1) \\ &= (x\epsilon + \epsilon)^{n-2}\,(x + 1)(xq + 1) \\ &\;\vdots \\ &= (1 + x)(1 + xq)\cdots(1 + xq^{n-1}) \end{aligned}$$
and
$$\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x\epsilon)^k \epsilon^{n-k}\right) 1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (x\epsilon)^k 1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k.$$

3. For $(A_0, A_1) = (-x\epsilon, a\epsilon)$ we use (1.5.1) to find that
$$(-x\epsilon + a\epsilon)^n 1 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x\epsilon)^k (a\epsilon)^{n-k} 1 = p_n(a, x).$$
Hence, we obtain once more (1.3.4), because by (1.5.2)
$$\frac{e_q(at)}{e_q(xt)} = (e_q(-x\epsilon t)\, e_q(a\epsilon t))1 = \sum_{n=0}^{\infty} \frac{(-x\epsilon + a\epsilon)^n 1}{[n]_q!}\, t^n = \sum_{n=0}^{\infty} \frac{p_n(a, x)}{[n]_q!}\, t^n.$$

4. Finally, for $(A_0, A_1) = (x, a\epsilon)$ we find by (1.5.2) and (1.4.3) that
$$e_q(xt)\, e_q(at) = (e_q(xt)\, e_q(a\epsilon t))1 = \sum_{n=0}^{\infty} \frac{(x + a\epsilon)^n 1}{[n]_q!}\, t^n = \sum_{n=0}^{\infty} \frac{r_n(x, a)}{[n]_q!}\, t^n,$$
which is a single-line proof for (1.4.2).

1.6 Basic Hypergeometric Series

In 1812, Gauss considered the infinite series

$$F(a,b;c,z) = \\ 1 + \frac{ab}{1\cdot c}\,z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}\,z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}\,z^3 + \dots$$

as a function of a, b, c, z for $c \neq 0, -1, -2, \dots$, and derived the closed form for the sum when $z = 1$, namely

$$F(a,b;c,1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

Gauss' series is an instance of the so-called (generalized) hypergeometric series. First we introduce the notion of hypergeometric sequences.

Definition 1.6.1. A sequence $(u_k)_{k\in\mathbf{Z}}$ is called *hypergeometric* if u_k/u_{k-1} is a rational function in k for all k where the quotient is well-defined. The rational function coefficients are taken from $\mathbf{C}$ (or from a suitable ground field containing the rational numbers).

We define a (generalized) hypergeometric series as follows.

Definition 1.6.2. An ${}_rF_s$ *(generalized) hypergeometric series* is given by

$$\begin{aligned} {}_rF_s(a_1, a_2, \dots, a_r; b_1, \dots, b_s; z) &:= {}_rF_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z\right] \\ &:= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{k!\,(b_1)_k \cdots (b_s)_k}\, z^k, \end{aligned} \tag{1.6.1}$$

where $(a)_k$ denotes the *shifted factorial* of a defined by $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$, and where the b_i $(1 \leq i \leq s)$ are assumed to be such that none of the denominator factors evaluates to zero for all $k \geq 0$.

Thirty years later, Heine generalized Gauss' series by

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}\,z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}\,z^2 + \dots \tag{1.6.2}$$

for $c \neq 0, -1, -2, \dots$. This series equals Gauss' series for $q = 1$, since

$$\frac{1-q^a}{1-q}\bigg|_{q=1} = [a]\Big|_{q=1} = a,$$

as we already saw in Subsection 1.1.

The series (1.6.2) is usually called Heine's series. Originally, Heine denoted it by $\phi(a, b, c, q, z)$. However, we do not want to restrict ourselves to handling only powers of q, and we would also like to consider the case when q^a, q^b or q^c is replaced by zero, which leads to the definition of the so called q-shifted factorial.

Definition 1.6.3. The *q-shifted factorial* of a is defined by

$$(a;q)_k := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & k>0, \\ 1, & k=0, \\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{k})\right]^{-1}, & k<0 \end{cases}$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

With this notation, Heine's series (1.6.2) can be written as

$$\sum_{k=0}^{\infty} \frac{(q^a;q)_k\,(q^b;q)_k}{(q;q)_k\,(q^c;q)_k}\ z^k.$$

For manipulating formulas involving q-shifted factorials the following easily verified transformations will be frequently used.

Rules 1.6.4. For $n \in \mathbf{Z}$ and $k \in \mathbf{Z}$,

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty};$$

$$(a;q)_{n+k} = (a;q)_n\,(aq^n;q)_k;$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk};$$

$$(a;q)_{2n} = (a;q^2)_n\,(aq;q^2)_n;$$

$$(a^2;q^2)_n = (a;q)_n\,(-a;q)_n;$$

$$(a^k;q^k)_n = (a;q)_n\,(a\omega_k;q)_n\cdots(a\omega_k^{k-1};q)_n, \quad \omega_k = e^{2\pi i/k}.$$

Since products of q-shifted factorials arise so often, we will use the following compact abbreviations.

Definition 1.6.5.

$$(a_1, a_2, \ldots, a_n; q)_k := (a_1; q)_k \, (a_2; q)_k \cdots (a_n; q)_k;$$
$$(a_1, a_2, \ldots, a_n; q)_\infty := (a_1; q)_\infty \, (a_2; q)_\infty \cdots (a_n; q)_\infty.$$

Remark 1.6.6. For $n \in \mathbf{N}$ and $k \in \mathbf{Z}$,

$$\begin{aligned} [n]_q &= \frac{(q^2; q)_{n-1}}{(q; q)_{n-1}}; \\ [n]_q! &= \frac{(q; q)_n}{(1-q)^n}; \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{(q; q)_n}{(q; q)_k \, (q; q)_{n-k}}. \end{aligned}$$

Now we are able to expand the notions of hypergeometric sequences and hypergeometric series to the corresponding q-terms.

Definition 1.6.7. A sequence $(u_k)_{k \in \mathbf{Z}}$ is said to be *q-hypergeometric* if u_k/u_{k-1} is a rational function in q^k for all k where the quotient is well-defined. The rational function coefficients are taken from $\mathbf{C}(q)$ (or from a suitable ground field containing q and the rational numbers).

Definition 1.6.8. An ${}_r\phi_s$ *basic hypergeometric series* is given by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) &:= {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &:= \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} z^k, \end{aligned} \tag{1.6.3}$$

where the b_i $(1 \leq i \leq s)$ are assumed to be such that none of the denominator factors evaluates to zero for all $k \geq 0$.

An ${}_rF_s$ series terminates if one of its numerator parameter is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-n} $(n \geq 0)$, since

$$(-n)_k = (q^{-n}; q)_k = 0 \quad \text{for } k > n.$$

If we denote the terms of the series (1.6.1) and (1.6.3) by u_k and v_k, respectively, we immediately see that for $k \geq 1$,

$$\frac{u_k}{u_{k-1}} = \frac{(a_1 + k - 1)(a_2 + k - 1) \cdots (a_r + k - 1)}{k(b_1 + k - 1) \cdots (b_s + k - 1)} \, z$$

is a rational function of k, and

$$\frac{v_k}{v_{k-1}} = \frac{(1-a_1q^{k-1})(1-a_2q^{k-1})\cdots(1-a_rq^{k-1})}{(1-q^k)(1-b_1q^{k-1})\cdots(1-b_sq^{k-1})}\,(-q^{k-1})^{1+s-r}z$$

is a rational function of q^k, confirming that u_k and v_k are in fact hypergeometric and q-hypergeometric, respectively.

We define the additional factor $\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}$ in (1.6.3), since when studying these series from an analytic point of view one sometimes wants to replace z by z/a_r and let $a_r \to \infty$. In this case, because

$$\begin{aligned}
\lim_{a_r\to\infty}(a_r;q)_k\left(\frac{z}{a_r}\right)^k &= \lim_{a_r\to\infty}(1-a_r)(1-a_rq)\cdots(1-a_rq^{k-1})\left(\frac{z}{a_r}\right)^k \\
&= \lim_{a_r\to\infty}(a_r^{-1}-1)(a_r^{-1}-q)\cdots(a_r^{-1}-q^{k-1})\,z^k \\
&= (-1)^k q^{\binom{k}{2}} z^k,
\end{aligned}$$

the resulting series is again of form (1.6.3) with r replaced by $r-1$. Note that there is no loss of generality, because we can always choose s sufficiently large by adding parameters equal to zero.

Finally, for sake of completeness we extend the notion of basic hypergeometric series to series which are infinite in both directions as following.

Definition 1.6.9. An ${}_r\psi_s$ *basic bilateral hypergeometric series* is given by

$${}_r\psi_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right] := \sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_k}{(b_1,b_2,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k,$$

where the parameters are such that each term of the series is well-defined.

Since for $n \geq 0$ we have

$$(a;q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \tag{1.6.4}$$

it is easily seen that

$${}_r\psi_r\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right] = {}_r\psi_r\left[\begin{matrix}q/b_1,q/b_2,\ldots,q/b_r\\ q/a_1,q/a_2,\ldots,q/a_r\end{matrix};q,\frac{b_1b_2\cdots b_r}{a_1a_2\cdots a_rz}\right].$$

1.7 The q-Binomial Theorem for ${}_1\phi_0$ Series

As we mentioned at the beginning of Subsection 1.5, many formulas are known as the q-binomial theorem. The one we will study now is an important summation formula for basic hypergeometric series due to Cauchy.

Let us first consider another q-analogue of the exponential function. We define

$$\widetilde{e}_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}.$$

Clearly, we have $\widetilde{e}_q(x) = e_q(x/(1-q))$ and therefore $\widetilde{e}_q(x(1-q))\big|_{q=1} = e(x)$.

Theorem 1.7.1.

$$\widetilde{e}_q(x) = \frac{1}{(x;q)_\infty}. \tag{1.7.1}$$

Proof. We try the "Ansatz"

$$F(x) = \frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} a_n x^n.$$

From

$$(1-x)F(x) = \frac{1-x}{(x;q)_\infty} = \frac{1}{(qx;q)_\infty} = F(qx)$$

we obtain the condition that

$$(1-x)\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n q^n x^n,$$

which is equivalent to

$$a_n - a_{n-1} = a_n q^n.$$

Since $a_0 = F(0) = 1$, we have

$$a_n = \frac{a_{n-1}}{1-q^n} = \frac{a_{n-2}}{(1-q^n)(1-q^{n-1})} = \ldots = \frac{1}{(q;q)_n}$$

proving that

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \widetilde{e}_q(x). \qquad \blacksquare$$

An alternative proof of Theorem 1.7.1 would follow immediately from Corollary 1.2.2.

Theorem 1.7.2 (q-binomial theorem for ${}_1\phi_0$ series).

$$ {}_1\phi_0(a,—;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.7.2}$$

Proof. Since $p_n(1,a) = (a;q)_n$, we use (1.3.4) to write

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{[n]_q!}\, z^n = \frac{e_q(z)}{e_q(az)},$$

or equivalently, replacing z by $z/(1-q)$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, z^n = \frac{\widetilde{e}_q(z)}{\widetilde{e}_q(az)}.$$

Applying (1.7.1) we obtain the theorem. ∎

For the terminating case $a = q^{-n}$, $n \geq 0$, (1.7.2) reduces to

$$ {}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n. \tag{1.7.3}$$

The analogy of (1.7.2) to the binomial theorem becomes evident for $a = q^\alpha$. Then we have

$$ {}_1F_0(\alpha;—;z) = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} z^k = (1-z)^{-\alpha}$$

and

$$ {}_1\phi_0(q^\alpha;—;q,z) = \sum_{k=0}^{\infty} \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q z^k = \frac{1}{(z;q)_\alpha}.$$

As a consequence of the q-binomial theorem we get the following product formula giving a q-analogue of the trivial formula

$$(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}.$$

Corollary 1.7.3 (product formula for ${}_1\phi_0$ series).

$$ {}_1\phi_0(a;—;q,z)\, {}_1\phi_0(b;—;q,az) = {}_1\phi_0(ab;—;q,z). \tag{1.7.4}$$

Finally, we state another version of the q-binomial theorem, where the analogy to the $q=1$ case cannot fail to be noticed.

Corollary 1.7.4 (q-binomial theorem).

$$(ab;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k \, (b;q)_{n-k}.$$

Proof. From the product formula for ${}_1\phi_0$ series (1.7.4) we know that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(ab;q)_n}{(q;q)_n} z^n &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} b^n z^n \cdot \sum_{n=0}^{\infty} \frac{(b;q)_n}{(q;q)_n} z^n \\
&\overset{(1.2.4)}{=} \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k \, (b;q)_{n-k}}{(q;q)_n} z^n.
\end{aligned}$$

Comparing the coefficients proves the assertion. ■

1.8 Fundamental Summation and Transformation Formulas

We shall finish our introductory guide to q-identities with the presentation of some of the most important fundamental summation and transformation formulas for basic hypergeometric series, which follow more or less directly from the q-binomial theorem discussed above.

Theorem 1.8.1 (Heine's transformation formula for ${}_2\phi_1$ series).

$${}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \, {}_2\phi_1(c/b,z;az;q,b). \tag{1.8.1}$$

Proof.

$$\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k \, (cq^k;q)_\infty}{(q;q)_k \, (bq^k;q)_\infty} z^k \\
&\overset{(1.7.2)}{=} \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k \sum_{l=0}^{\infty} \frac{(c/b;q)_l}{(q;q)_l} (bq^k)^l \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{l=0}^{\infty} \frac{(c/b;q)_l}{(q;q)_l} b^l \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (zq^l)^k \\
&\overset{(1.7.2)}{=} \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{l=0}^{\infty} \frac{(c/b;q)_l}{(q;q)_l} b^l \frac{(azq^l;q)_\infty}{(zq^l;q)_\infty} \\
&= \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \, {}_2\phi_1(c/b,z;az;q,b).
\end{aligned}$$

■

Heine also showed that Euler's transformation formula

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z) $$

has the q-analogue

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1(c/a,c/b;c;q,abz/c), \tag{1.8.2} $$

which follows by iterated application of (1.8.1).

As mentioned in Subsection 1.6, Gauss proved the summation formula

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. $$

Heine derived the following q-analogue.

Corollary 1.8.2 (Heine's q-analogue of Gauss' summation formula).

$$ {}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}. \tag{1.8.3} $$

Proof. The theorem can be obtained directly from (1.8.1) and (1.7.2), since

$$ {}_2\phi_1(a,b;c;q,c/ab) = \frac{(b,c/b;q)_\infty}{(c,c/ab;q)_\infty}\,{}_1\phi_0(c/ab;—;q,b) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}. \qquad \blacksquare $$

For the terminating case when $a = q^{-n}, n \geq 0$, (1.8.3) reduces to

$$ {}_2\phi_1(q^{-n},b;c;q,cq^n/b) = \frac{(c/b;q)_n}{(c;q)_n}, $$

giving a q-analogue of Vandermonde's formula

$$ {}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}. $$

A q-analogue of Kummer's formula

$$ {}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\,\Gamma(1+a/2)}{\Gamma(1+a)\,\Gamma(1+a/2-b)} $$

was discovered independently by Bailey and Daum.

Corollary 1.8.3 (Bailey-Daum's summation formula).

$$ {}_2\phi_1(a,b;aq/b;q,-q/b) = \frac{(-q;q)_\infty (aq, aq^2/b^2;q^2)_\infty}{(aq/b,-q/b;q)_\infty}. $$

Proof.

$$ \begin{aligned} {}_2\phi_1(a,b;aq/b;q,-q/b) &\overset{(1.8.1)}{=} \frac{(a,-q;q)_\infty}{(aq/b,-q/b;q)_\infty}\ {}_2\phi_1(q/b,-q/b;-q;q,a) \\ &= \frac{(a,-q;q)_\infty}{(aq/b,-q/b;q)_\infty}\ {}_1\phi_0(q^2/b^2;—;q^2,a) \\ &\overset{(1.7.2)}{=} \frac{(a,-q;q)_\infty}{(aq/b,-q/b;q)_\infty}\ \frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty} \\ &= \frac{(-q;q)_\infty (aq, aq^2/b^2;q^2)_\infty}{(aq/b,-q/b;q)_\infty}. \end{aligned} $$

■

The so-called Pfaff-Saalschütz formula

$$ {}_3F_2(-n,a,b;c,1+a+b-c-n;1) = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n} $$

has the following q-analogue discovered by Jackson.

Corollary 1.8.4 (Jackson's q-analogue of Pfaff-Saalschütz' formula).

$$ {}_3\phi_2(q^{-n},a,b;c,abc^{-1}q^{1-n};q,q) = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{1.8.4} $$

Proof. Since by the q-binomial theorem (1.7.2) we have

$$ \frac{(abz/c;q)_\infty}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{(ab/c;q)_k}{(q;q)_k}\ z^k, $$

the right hand side of Heine's formula (1.8.2) equals

$$ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(ab/c;q)_k\,(c/a,c/b;q)_m}{(q;q)_k\,(q,c;q)_m}\ (ab/c)^m z^{k+m}. $$

Comparing the coefficients of z^n on both sides of (1.8.2) leads to

$$ \sum_{k=0}^{n} \frac{(ab/c;q)_{n-k}\,(c/a,c/b;q)_k}{(q;q)_{n-k}\,(q,c;q)_k}\ (ab/c)^k = \frac{(a,b;q)_n}{(c,q;q)_n}. $$

Because

$$ \frac{(ab/c;q)_{n-k}}{(q;q)_{n-k}} = \frac{(ab/c;q)_n\,(q^{-n};q)_k}{(q;q)_n\,(cq^{1-n}/ab;q)_k}\ (cq/ab)^k, $$

we obtain

$$\sum_{k=0}^{n} \frac{(q^{-n}, c/a, c/b; q)_k}{(q, c, cq^{1-n}/ab; q)_k}\, q^k = \frac{(a, b; q)_n}{(c, ab/c; q)_n}.$$

Replacing a and b by c/a and c/b, respectively, proves (1.8.4). ■

Finally, the following proof of Jacobi's triple product identity shall illustrate how to link up q-binomial coefficient identities with basic hypergeometric series.

Theorem 1.8.5 (Jacobi's triple product identity).

$$\sum_{k=-\infty}^{\infty} q^{k^2} z^k = (q^2, -zq, -q/z; q^2)_\infty. \tag{1.8.5}$$

Proof. We will show that identity (1.8.5) is a limiting case of the q-binomial coefficient identity

$$\sum_{k=-n}^{n} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q x^k = (-x, -q/x; q)_n. \tag{1.8.6}$$

Transforming the left hand side of equation (1.8.6) into basic hypergeometric form we find that

$$\sum_{k=-n}^{n} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q x^k = \frac{q^{\binom{n+1}{2}}}{x^n}\, {}_1\phi_0(q^{-2n}; —; q, -q^n x),$$

which can be summed by the terminating version of the q-binomial theorem (1.7.3) giving

$$\sum_{k=-n}^{n} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q x^k = \frac{q^{\binom{n+1}{2}}}{x^n}\, (-xq^{-n}; q)_{2n}.$$

As a consequence we get (1.8.6), because, by (1.6.4),

$$\frac{q^{\binom{n+1}{2}}}{x^n}\, (-xq^{-n}; q)_{2n} = \frac{q^{\binom{n+1}{2}}}{x^n}\, \frac{(-x; q)_n}{(-x; q)_{-n}} = (-x, -q/x; q)_n.$$

Since

$$\lim_{n\to\infty} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q = \lim_{n\to\infty} \frac{(q; q)_{2n}}{(q; q)_{n+k}\,(q; q)_{n-k}} = \frac{1}{(q; q)_\infty},$$

it follows that for $n \to \infty$, (1.8.6) turns into

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} x^k = (q, -x, -q/x; q)_\infty.$$

Replacing q by q^2 and substituting qz for x completes the proof. ■

2 The q-Analogue of Zeilberger's Algorithm

In this section we shall discuss a completely different approach for dealing with q-hypergeometric summation. So far we have seen how to prove identities either by applying operators or by transforming identities into basic hypergeometric notation (this can be done algorithmically, e.g., using the Mathematica package `HYPQ` written by Krattenthaler [12]) and then looking up standard results in the q-hypergeometric database containing summation and transformation formulas of ${}_r\phi_s$ basic hypergeometric series (see e.g. the appendices in Gasper and Rahman [6] or Slater [19]). We will exploit the fact that the algorithms presented by Gosper [9] and Zeilberger [23] for indefinite and definite hypergeometric summation, respectively, can be — after appropriate adaptations — also applied in the q-case.

We will first investigate the underlying theoretical background of q-analogues of these algorithms, then describe the author's Mathematica implementation and compare it with the already existing Maple version written by Koornwinder [11].

2.1 Theoretical Background

The q-Gosper Algorithm

Based on recent work of Paule [15], [16] (cf. also Paule and Strehl [18]) we shall outline how Gosper's algorithm for definite hypergeometric summation can be carried over to the q-case.

Let $\mathcal{K} := \mathcal{F}(\kappa_1, \ldots, \kappa_n)$ denote the field of rational functions in some indeterminates $\kappa_1, \ldots \kappa_n$, $n \geq 0$, where $\kappa_i \neq q$ and $\kappa_i \neq q^k$, $1 \leq i \leq n$, over some computable field $\mathcal{F}$ (for sake of simplicity with regard to the implementation we will restrict ourselves to the case where $\mathcal{F}$ is the field of the rational numbers). Assume we are given a q-hypergeometric function $f(k)$ over $\mathcal{K}(q)$, i.e., a function for which the sequence $(f(k))_{k\in\mathbf{Z}}$ is q-hypergeometric. Then Gosper's algorithm decides whether there exists a q-hypergeometric function $g(k)$, such that

$$g(k+1) - g(k) = f(k), \tag{2.1.1}$$

and if so, determines $g(k)$ with the motive that

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a) \quad (a \leq b),$$

which solves the indefinite summation problem.

Following the notation of basic hypergeometric series we will first consider functions with r (≥ 0) numerator and s (≥ 0) denominator parameters $a_i, b_j \in \mathcal{K}(q)$, $(1 \leq i \leq r, 1 \leq j \leq s)$ of the form

$$f(k) = \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k} q^{\alpha\binom{k}{2}+\beta k},$$

where the argument z is a rational function in $\mathcal{K}(q)$ with $z(0) \neq 0$, and α and β are integers.

Using x as an abbreviation for q^k, we call $A(x)/B(x) = f(k+1)/f(k)$ the *rational representation* of a q-hypergeometric function $f(k)$, which in the case described above becomes

$$\frac{A(x)}{B(x)} = \frac{(1-a_1x)\cdots(1-a_rx)}{(1-b_1x)\cdots(1-b_sx)(1-qx)} x^\alpha q^\beta z,$$

If additionally $\gcd(A, B) = 1$ holds, then $A(x)/B(x)$ is called the *reduced rational representation* of $f(k)$.

Now Gosper's classical algorithm adapted to the q-case consists of the following steps:

1. Given a q-hypergeometric function $f(k)$ specified by its rational representation $A/B \in \mathcal{K}(q)(x)$, compute the q-Gosper-Petkovšek form of $f(k)$, i.e., determine polynomials P, Q, $R \in \mathcal{K}(q)[x]$ such that

$$\frac{A}{B} = \frac{\epsilon P}{P} \frac{Q}{\epsilon R}, \tag{2.1.2}$$

   where the conditions of Theorem 2.1.1 below are satisfied.

2. Try to solve the equation

$$P = Q\,(\epsilon Y) - R\,Y \tag{2.1.3}$$

   for a polynomial $Y \in \mathcal{K}(q)[x]$.

3. If such a polynomial solution Y exists, then

$$g(k) = \frac{Y(q^k)\,R(q^k)}{P(q^k)}\, f(k) \tag{2.1.4}$$

   is a q-hypergeometric solution of (2.1.1), otherwise no q-hypergeometric solution $g(k)$ exists.

We want to point out that not only the Gosper form but also the degree setting for solving (2.1.3) are slightly different from the $q = 1$ case.

Based on the concept of greatest factorial factorization, Paule developed an alternative approach to Gosper's algorithm. The main point is taking a normal form point of view, for instance, using instead of the ordinary q-Gosper form representation the so-called q-Gosper-Petkovšek (q-GP) representation (2.1.2) for rational functions, which is unique.

Definition 2.1.1. A polynomial $P \in \mathcal{K}(q)[x]$ is called *q-monic* if $P(0) = 1$.

The property of being q-monic is clearly invariant with respect to the q-shift operator ϵ. Any non-zero rational function $r = A/B$ with $A, B \in \mathcal{K}(q)[x]$ can be brought into the form

$$r(x) = \frac{A(x)}{B(x)} = \frac{A_1(x)}{B_1(x)}\, x^\alpha q^\beta z,$$

where $A_1, B_1 \in \mathcal{K}(q)[x]$ are q-monic, α and β are integers, and z is a rational function in $\mathcal{K}(q)$ with $z(0) \neq 0$.

If we denote the numerator and denominator of a reduced rational function s by $\mathrm{num}(s)$ and $\mathrm{den}(s)$, respectively, and let $\mu(x) := x^\alpha \in \mathcal{K}(x)$ and $\pi(q) := q^\beta \in \mathcal{K}(q)$, then Paule showed the following result.

Theorem 2.1.1 (q-Gosper-Petkovšek representation). For any non-zero rational function $r \in \mathcal{K}(q)(x)$, as above, there exist unique q-monic polynomials $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{R} \in \mathcal{K}(q)[x]$ such that

$$\frac{A_1}{B_1} = \frac{\epsilon\widetilde{P}}{\widetilde{P}}\,\frac{\widetilde{Q}}{\epsilon\widetilde{R}},$$

with $\gcd(\widetilde{P}, \widetilde{Q}) = \gcd(\widetilde{P}, \widetilde{R}) = 1$ and $\gcd(\widetilde{Q}, \epsilon^j \widetilde{R}) = 1$ for all $j \geq 1$, and

$$r = \frac{\epsilon P}{P}\,\frac{Q}{\epsilon R},$$

where

$$\begin{aligned} P &= \widetilde{P}\,\mathrm{num}(\pi(x)), \\ Q &= \widetilde{Q}\, z\, \mathrm{num}(\mu(x))/\,\mathrm{den}(\pi(q)), \\ \epsilon R &= (\epsilon\widetilde{R})\,\mathrm{den}(\mu(x)). \end{aligned}$$

Now let $A(x)/B(x)$ be the reduced rational representation of $f(k)$ with q-GP representation

$$\frac{A}{B} = \frac{\epsilon P}{P}\,\frac{Q}{\epsilon R} \tag{2.1.5}$$

for polynomials P, Q and R, say, and suppose that a q-hypergeometric solution $g(k)$ of

$$g(k+1) - g(k) = f(k) \tag{2.1.6}$$

exists, where the reduced rational representation of $g(k)$ is given by $C(x)/D(x)$. Then (2.1.6) is equivalent to

$$g(k) = \frac{D(q^k)}{C(q^k) - D(q^k)} \, f(k),$$

showing that $g(k)$ is a rational function multiple of the input.

Using this representation for $g(k)$ in (2.1.6), we find that

$$\frac{\epsilon D}{\epsilon(C-D)} \, \frac{A}{B} - \frac{D}{C-D} = 1,$$

or equivalently

$$\frac{A}{B} = \frac{\epsilon(C-D)}{C-D} \, \frac{C}{\epsilon D}. \tag{2.1.7}$$

This is very close to a q-GP representation, but in general we have no guarantee that $\gcd(C, \epsilon^j D) = 1$ for all $j \geq 1$. To overcome this problem let us consider the q-GP representation for $C/(\epsilon D)$,

$$\frac{C}{\epsilon D} = \frac{\epsilon \widetilde{P}}{\widetilde{P}} \, \frac{\widetilde{Q}}{\epsilon \widetilde{R}} \tag{2.1.8}$$

for polynomials $\widetilde{P}, \widetilde{Q}$ and $\widetilde{R}$, say. Then (2.1.7) turns into a true q-GP representation, namely

$$\frac{A}{B} = \frac{\epsilon((C-D)\widetilde{P})}{(C-D)\widetilde{P}} \, \frac{\widetilde{Q}}{\epsilon \widetilde{R}}. \tag{2.1.9}$$

Since the q-GP representation is unique, after comparing (2.1.5) and (2.1.9) we may conclude that

$$Q = \widetilde{Q}, \quad R = \widetilde{R}$$

and

$$P = (C-D)\widetilde{P}. \tag{2.1.10}$$

Using (2.1.8) to rewrite equation (2.1.10) as

$$P = Q \, \epsilon\!\left(\frac{D}{R} \, \widetilde{P}\right) - R \left(\frac{D}{R} \, \widetilde{P}\right)$$

shows that $Y = \widetilde{P}D/R$ is a solution of the q-Gosper equation (2.1.3). Note that since R divides D by the properties of the q-GP representation (2.1.8), Y is in fact a polynomial.

Finally, from

$$\begin{aligned} g(k) &= \frac{D(q^k)}{C(q^k) - D(q^k)}\, f(k) \overset{(2.1.10)}{=} D(q^k)\, \frac{\widetilde{P}(q^k)}{P(q^k)}\, f(k) \\ &= \left(\frac{D(q^k)}{R(q^k)}\, \widetilde{P}(q^k)\right) \frac{R(q^k)}{P(q^k)}\, f(k) \end{aligned}$$

we obtain that

$$g(k) = \frac{Y(q^k)\, R(q^k)}{P(q^k)}\, f(k),$$

which is equation (2.1.4).

In the present implementation we allow as summand for the q-Gosper algorithm any q-hypergeometric function of the form

$$f(k) = \frac{\prod_{r=1}^{rr} (A_r\, q^{(c_r i_r)k + d_r}; q^{i_r})_{a_r k + b_r}}{\prod_{s=1}^{ss} (B_s\, q^{(v_s j_s)k + w_s}; q^{j_s})_{t_s k + u_s}}\, L(q^k)\, q^{\alpha \binom{k}{2} + \beta k}\, z^k,$$

with

A_r, B_s	power products in $\mathcal{K}$,
a_r, t_s	specific integers (i.e. integers free of any parameters),
b_r, u_s	integers, which may depend on parameters free of k,
c_r, d_r, v_s, w_s	specific integers,
i_r, j_s	specific non-zero integers,
L	a Laurent-polynomial in q^k with coefficients in $\mathcal{K}(q)$,
α, β	specific integers and
z	a rational function in $\mathcal{K}(q)$.

For the actual computation of the q-GP representation we proceed as following. Let $A(x)/B(x)$ denote the possibly non-reduced rational representation of the summand $f(k)$. Observing that

1. any Laurent-polynomial $L(x)$ as above can be written as

   $$L(x) = \widetilde{P}(x)\, x^{\tilde{\alpha}}\, \tilde{z},$$

   where $\widetilde{P}$ is a q-monic polynomial in $\mathcal{K}(q)[x]$, $\tilde{\alpha}$ is an integer and $\tilde{z} \in \mathcal{K}(q)$, and

2. any rational function $z \in \mathcal{K}(q)$ satisfying $z(0) = 0$ can be transformed into

   $$z = q^{\tilde{\beta}}\, \tilde{z},$$

   where $\tilde{\beta}$ is an integer and $\tilde{z}(0) \neq 0$, it follows that A/B can always be converted into the form

$$\begin{aligned}\frac{A(x)}{B(x)} &= \frac{(\epsilon\bar{P})(x)}{\bar{P}(x)}\,\frac{(1-\alpha_1 q^{e_1}x^{f_1})\cdots(1-\alpha_m q^{e_m}x^{f_m})}{(1-\beta_1 q^{g_1}x^{h_1})\cdots(1-\beta_n q^{g_n}x^{h_n})}\,x^{\bar{\alpha}}q^{\bar{\beta}}\bar{z}\\ &= \frac{(\epsilon\bar{P})(x)}{\bar{P}(x)}\,\frac{A_1(x)}{B_1(x)}\,x^{\bar{\alpha}}q^{\bar{\beta}}\bar{z}\end{aligned}$$

for certain integers $m \geq 0$, $n \geq 0$, where for $1 \leq i \leq m$ and $1 \leq j \leq n$ the α_i, β_j are power products in $\mathcal{K}$, the e_i, g_j are integers, the f_i, h_j are non-negative integers, $\bar{\alpha}, \bar{\beta}$ are integers, $\bar{P}$ is a q-monic polynomial in $\mathcal{K}(q)[x]$ and $\bar{z}$ is a rational function in $\mathcal{K}(q)$ with $z(0) \neq 0$.

Now, the computation of the q-GP representation, as in Theorem 2.1.1,

$$\frac{A}{B} = \frac{\epsilon P}{P}\,\frac{Q}{\epsilon R}$$

is straightforward.

Let $P_0 := \bar{P}$, $Q_0 := A_1$ and $R_0 := \epsilon^{-1}B_1$. Due to the input restrictions listed above it is possible to compute the maximal positive integer l such that $\gcd(Q_0, \epsilon^l R_0) \neq 1$ simply by comparing all factors in Q_0 and R_0. Now we successively rewrite these polynomials in the following way.

For i from 1 to l, let

$$g := \gcd(Q_{i-1}, \epsilon^i R_{i-1})$$

and put

$$P_i := P_{i-1}\,(\epsilon^{-1}g)\,(\epsilon^{-2}g)\cdots(\epsilon^{-i+1}g),$$
$$Q_i := \frac{Q_{i-1}}{g} \quad \text{and} \quad R_i := \frac{R_{i-1}}{(\epsilon^{-i}g)}.$$

Finally, we end up with q-monic polynomials $\widetilde{P} := P_l$, $\widetilde{Q} := Q_l$ and $\widetilde{R} := R_l$ satisfying

$$\frac{A_1}{B_1} = \frac{\epsilon\widetilde{P}}{\widetilde{P}}\,\frac{\widetilde{Q}}{\epsilon\widetilde{R}},$$

such that $\gcd(\widetilde{Q}, \epsilon^j\widetilde{R}) = 1$ for all $j \geq 1$. For the remaining terms $\bar{\mu}(x) := x^{\bar{\alpha}}$, $\bar{\pi}(q) := q^{\bar{\beta}}$ and $\bar{z}$ we proceed exactly as described in Theorem 2.1.1, i.e., we put

$$\begin{aligned}P &:= \widetilde{P}\,\mathrm{num}(\bar{\pi}(x)),\\ Q &:= \widetilde{Q}\,\bar{z}\,\mathrm{num}(\bar{\mu}(x))/\,\mathrm{den}(\bar{\pi}(q)),\\ \epsilon R &:= (\epsilon\widetilde{R})\,\mathrm{den}(\bar{\mu}(x)).\end{aligned}$$

This is the desired q-GP representation for $f(k)$.

The q-Zeilberger Algorithm

The basic idea of the q-analogue of Zeilberger's algorithm is as follows. From now on, unless stated otherwise, n denotes a non-negative integer and k an arbitrary integer. Assume we are given a function $F(n,k)$ being q-hypergeometric in n and k, i.e., the quotients $F(n,k)/F(n-1,k)$ and $F(n,k)/F(n,k-1)$ are rational functions in q^n and q^k for all n and k where the quotients are well-defined. Then we can prove, under some mild side-conditions, that for a certain integer $d \geq 0$ and $n \geq d$ there exists a linear recurrence

$$\sigma_0(n)F(n,k)+\sigma_1(n)F(n-1,k)+\ldots+\sigma_d(n)F(n-d,k) = G(n,k)-G(n,k-1),$$

where the coefficients are polynomials in q^n not depending on k, and where $G(n,k)$ is q-hypergeometric in n and k. Given the order d, which is in general not a priori known, Gosper's algorithm will determine the coefficient polynomials and the solution function $G(n,k)$. If we now sum over both sides of the recurrence above, for instance, for k running from a to b, with $a, b \in \mathbf{Z}$ and $a \leq b$, the right hand side telescopes and we obtain

$$(\sigma_0(n)Id + \sigma_1(n)N + \ldots + \sigma_d(n)N^d)\sum_{k=a}^{b} F(n,k) = G(n,b) - G(n,a-1),$$

where N denotes the backward shift operator in n, i.e., $NF(n,k) = F(n-1,k)$. In most applications a and b also depend on n. In this case we have to introduce corresponding correction terms for the inhomogeneous part of the recurrence to achieve a shift in the bounds, too.

Let us now turn to the question why such a recurrence always exists. We will follow the proof given in Wilf and Zeilberger [22] and extend it to a more general input form for the algorithm.

Definition 2.1.2. A function $F(n,k)$ is called *simple q-proper-hypergeometric*, if it is of the form

$$F(n,k) = \frac{\prod_{r=1}^{rr}(A_r;q)_{a_r n+b_r k+c_r}}{\prod_{s=1}^{ss}(B_s;q)_{u_s n+v_s k+w_s}}\; P(q^n,q^k)\, q^{\alpha\binom{k}{2}+(\beta n+\gamma)k} z^k, \qquad (2.1.11)$$

with

A_r, B_s	rational functions in $\mathcal{K}(q)$,
a_r, b_r, u_s, v_s	specific integers free of any parameters,
c_r, w_s	integers, which may depend on parameters,
P	a polynomial in q^n and q^k with coefficients in $\mathcal{K}(q)$,
α, β, γ	specific integers and
z	a rational function in $\mathcal{K}(q)$.

So far we assume the A_r and B_s to be free of both n and k. This is done for technical reasons in the proof of the crucial theorem below.

Definition 2.1.3. A function $F(n,k)$ satisfies a *k-free recurrence* at a point (n_0, k_0), if there exist non-negative integers I and J and polynomials $\sigma_{ij}(n)$ not depending on k and not all zero, such that

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\sigma_{ij}(n)F(n-j,k-i)=0 \tag{2.1.12}$$

holds for (n_0, k_0) in the sense that F is well-defined at all of the arguments that occur.

Theorem 2.1.2. Let $F(n,k)$ be a simple q-proper-hypergeometric function. Then $F(n,k)$ satisfies a k-free recurrence at every point (n_0,k_0) for which $F(n_0,k_0)\neq 0$.

Proof. For $F(n,k)\neq 0$ we form the linear combination

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\sigma_{ij}(n)\,\frac{F(n-j,k-i)}{F(n,k)}, \tag{2.1.13}$$

where the σ_{ij} are to be determined, if possible, so as to make the double sum vanish. We will construct a common denominator for the $(I+1)(J+1)$ quotients $F(n-j,k-i)/F(n,k)$ appearing in (2.1.13). Then each of these ratios will be expressible as a certain numerator divided by that common denominator. We will show that the sum of all numerators vanishes identically in k by equating to zero all coefficients of each power of q^k that appears in the common numerator. This corresponds to solving a linear system of homogeneous equations. To guarantee the existence of a non-trivial solution we will prove that I and J can be always chosen in such a way that the number of variables exceeds the number of equations.

For the quotient of q-shifted factorials, according to Definition 1.6.3, we define

$$\Lambda_d(p;c) := \frac{(p;q)_{c+d}}{(p;q)_c} = \begin{cases} (1-pq^c)\cdots(1-pq^{c+d-1}), & d>0,\\ 1, & d=0,\\ \big[(1-pq^{c+d})\cdots(1-pq^{c-1})\big]^{-1}, & d<0.\end{cases}$$

Then we have

$$\begin{aligned}\frac{F(n-j,k-i)}{F(n,k)} \quad &= \quad \frac{\prod_{r=1}^{rr}\Lambda_{-a_rj-b_ri}(A_r;a_rn+b_rk+c_r)}{\prod_{s=1}^{ss}\Lambda_{-u_sj-v_si}(B_s;u_sn+v_sk+w_s)}\,\frac{P(q^{n-j},q^{k-i})}{P(q^n,q^k)}\cdot\\ &\quad\; q^{\alpha(-ik+(i^2+i)/2)+\beta(-jk-in+ij)-\gamma i}z^{-i}.\end{aligned} \tag{2.1.14}$$

Clearly, (2.1.14) is a rational function of q^k. But we have to check whether a Λ-entry actually contributes to the numerator or to the denominator. This leads to the following four cases.

Case 1: [Contribution of the numerator of (2.1.14) to the actual denominator] Consider a factor of the product in the numerator of (2.1.14),

$$\Lambda_{-aj-bi}(A; an+bk+c) = (1 - Aq^{an+bk+c}) \cdots (1 - Aq^{an+bk+c+(-aj-bi)-1}),$$

in which $aj + bi < 0$. Let $x = q^k$ and $t^+ = \max(t, 0)$. Then we obtain a polynomial of degree $|aj + bi|$ in x^b. If $b \geq 0$ this factor does not contribute to the actual denominator. For $b < 0$ this factor is a polynomial of degree $|aj + bi|$ in $x^{-|b|}$. Hence, after multiplying top and bottom by $x^{|b(aj+bi)|}$ we have, in case of $aj + bi < 0$ and $b \geq 0$ or $b < 0$, a contribution of $|(-b)^+(aj + bi)|$ to the actual denominator.

If, on the other hand $aj+bi > 0$, then the factor is the reciprocal of a polynomial in x^b of degree $aj+bi$. For $b \geq 0$ we have a contribution of $b(aj+bi)$ and for $b < 0$, again after multiplying top and bottom by $x^{|b|(aj+bi)}$ we obtain a contribution of $|b|(aj + bi)$. So the overall contribution to the actual denominator in the case of $aj + bi > 0$ and $b \geq 0$ or $b < 0$ is of degree $|b|(aj + bi)$.

Summarizing Case 1, a factor in the numerator of (2.1.14) contributes a polynomial in x of degree $|b|(aj + bi)^+ + (-b)^+(-aj - bi)^+$ to the actual denominator.

Case 2: [Contribution of the numerator of (2.1.14) to the actual numerator] Again, if $aj + bi > 0$ the same factor is the reciprocal of a polynomial of degree $aj + bi$ in x^b. If $b \geq 0$ we have no contribution to the actual numerator. If $b < 0$ we would multiply top and bottom by $x^{|b|(aj+bi)}$ obtaining a contribution of $|b|(aj + bi)$ in the actual numerator.

Similarly, for $aj + bi < 0$ we get a contribution of $|b(aj + bi)|$ to the actual numerator.

Summarizing Case 2, a factor in the numerator of (2.1.14) contributes a polynomial in x of degree $(-b)^+(aj + bi)^+ + |b|(-aj - bi)^+$ to the actual numerator.

Case 3: [Contribution of the denominator of (2.1.14) to the actual numerator] Similar to Case 1 we get a contribution of $|v|(uj + vi)^+ + (-v)^+(-uj - vi)^+$.

Case 4: [Contribution of the denominator of (2.1.14) to the actual denominator] Similar to Case 2 we get a contribution of $(-v)^+(uj + vi)^+ + |v|(-uj - vi)^+$.

The factor $q^{\alpha(-ik)+\beta(-jk)}$ contributes a factor of degree $(\alpha i + \beta j)^+$ to the actual denominator and a factor of degree $(-\alpha i - \beta j)^+$ to the actual numerator.

Combining these results we obtain for the degrees of the actual numerator and denominator

$$\begin{aligned}
\nu_{ij} \quad = \quad & \sum_{r=1}^{rr}(-b_r)^+(a_r j + b_r i)^+ + |b_r|(-a_r j - b_r i)^+ + \\
& \sum_{s=1}^{ss}(-v_s)^+(-u_s j - v_s i)^+ + |v_s|(u_s j + v_s i)^+ + (-\alpha i - \beta j)^+ + \deg_x P
\end{aligned} \tag{2.1.15}$$

and

$$\begin{aligned}
\delta_{ij} \quad = \quad & \sum_{r=1}^{rr}|b_r|(a_r j + b_r i)^+ + (-b_r)^+(-a_r j - b_r i)^+ + \\
& \sum_{s=1}^{ss}|v_s|(-u_s j - v_s i)^+ + (-v_s)^+(u_s j + v_s i)^+ + (\alpha i + \beta j)^+ + \deg_x P,
\end{aligned} \tag{2.1.16}$$

respectively.

By maximizing every term in (2.1.16) over all $0 \leq i \leq I$ and $0 \leq j \leq J$ we obtain an upper bound, say Δ, for the degree of the common denominator. Next we put all terms in (2.1.13) over this common denominator and observe that every term contributes a factor whose degree is at most $\nu_{ij} + \Delta - \delta_{ij}$ to the common numerator. Since all terms in (2.1.15) and (2.1.16) are linear in i and j, there exist non-negative integers ξ, η and ρ such that $\nu_{ij} + \Delta - \delta_{ij} \leq I\xi + J\eta + \rho$. Now we are able to equate to zero all coefficients of each power of q^k in the common numerator. The result will be at most $I\xi + J\eta + \rho + 1$ homogeneous equations in $(I+1)(J+1)$ unknowns $\sigma_{ij}(n)$. Hence, if we choose $(I+1)(J+1) > I\xi + J\eta + \rho + 1$, which is always possible for sufficiently large I and J, we will have a non-trivial solution. ■

In [22], Wilf and Zeilberger give an upper bound for the order J, which is, however, far from being optimal and therefore of not too much use from an algorithmic point of view. We will not go further into the details here.

Since the notion of simple q-proper-hypergeometric functions defined in (2.1.11) is too restrictive in practice, we now want to extend the definition to so-called q-proper-hypergeometric functions for which the theorem presented above remains valid.

Definition 2.1.4. A function $F(n,k)$ is called *q-proper-hypergeometric*, if it is of the form

$$F(n,k) = \frac{\prod_{r=1}^{rr} (A_r q^{(d_r i_r)n+(e_r i_r)k}; q^{i_r})_{a_r n+b_r k+c_r}}{\prod_{s=1}^{ss} (B_s q^{(f_s j_s)n+(g_s j_s)k}; q^{j_s})_{u_s n+v_s k+w_s}} \, P(q^n, q^k) \, q^{\alpha\binom{k}{2}+(\beta n+\gamma)k} z^k, \tag{2.1.17}$$

where in addition to the restrictions of (2.1.11) the $d_r, e_r, f_s, g_s, i_r, j_s$ are specific integers with $i_r, j_s \neq 0$.

Corollary 2.1.3. Let $F(n,k)$ be a q-proper-hypergeometric function. Then $F(n,k)$ satisfies a k-free recurrence at every point (n_0, k_0) where $F(n_0,k_0) \neq 0$.

Proof. Since for $i > 0$

$$(Aq^{(di)n+(ei)k}; q^{-i})_{an+bk+c} = \left[(Aq^{(di)n+(ei)k+i}; q^i)_{-an-bk-c}\right]^{-1},$$

and

$$(Aq^{(di)n+(ei)k}; q^i)_{an+bk+c} = (A^{(1)}q^{dn+ek}, \ldots, A^{(i)}q^{dn+ek}; q)_{an+bk+c},$$

where the $A^{(j)}, 1 \leq j \leq i$, are the complex roots of A, it suffices to prove the validity of Theorem 2.1.2 for A_r and B_s being replaced by $A_r q^{d_r n+e_r k}$ and $B_s q^{f_s n+g_s k}$, respectively. Proceeding like in the proof of Theorem 2.1.2 we observe that the degree bounds obtained there only change by values being linear in i and j with coefficients depending on d_r, e_r, f_s and g_s. As a consequence there exist non-negative integers $\bar{\xi}, \bar{\eta}$ and $\bar{\rho}$ such that the number of equations is at most $I\bar{\xi} + J\bar{\eta} + \bar{\rho} + 1$. ∎

The existence of a k-free recurrence finally leads us to the desired recurrence in one variable.

Definition 2.1.5. We say that a function $F(n,k)$ has *finite support* w.r.t. k, if for all n there exists a finite integer interval I_n such that $F(n,k) \neq 0$ for $k \in I_n$ and $F(n,k) = 0$ for $k \notin I_n$.

Theorem 2.1.4. Let F be a q-proper-hypergeometric function, and let (n,k) be a point at which $F(n,k) \neq 0$ and such that $F(n-j,k-i)$ is well-defined for all $0 \leq i \leq I$ and $0 \leq j \leq J$. Then there exist polynomials $\sigma_0(n), \ldots, \sigma_J(n)$, not all zero, and a function $G(n,k)$ such that $G(n,k) = R(n,k)F(n,k)$ for some rational function R (the *certificate* of F) and such that

$$\sigma_0(n)F(n,k) + \sigma_1(n)F(n-1,k) + \ldots + \sigma_J(n)F(n-J,k) = G(n,k) - G(n,k-1). \tag{2.1.18}$$

Proof. From Corollary 2.1.3 we know that for the given q-proper-hypergeometric function F there exists a k-free recurrence

$$\sum_{i=0}^{I}\sum_{j=0}^{J}\sigma_{ij}(n)F(n-j,k-i)=0. \tag{2.1.19}$$

In operator notation we can write (2.1.19) in the form $H(N,K,n)F(n,k)=0$, where N and K are the backward shift operators in n and k, respectively, and where H is a polynomial operator, which can be expanded as

$$H(N,K,n)=H(N,Id,n)+(K-Id)\,V(N,K,n).$$

Note that N and n are non-commuting variables. Thus we have for $G(n,k)=V(N,K,n)F(n,k)$,

$$\begin{aligned}
0 &= H(N,K,n)F(n,k)\\
&= H(N,Id,n)F(n,k)+(K-Id)\,V(N,K,n)F(n,k)\\
&= H(N,Id,n)F(n,k)+(K-Id)\,G(n,k)\\
&= H(N,Id,n)F(n,k)+G(n,k-1)-G(n,k).
\end{aligned}$$

Clearly, $G(n,k)=V(N,K,n)F(n,k)$ is a rational function multiple of $F(n,k)$, because any linear combination of shift operators N^iK^j applied to a q-hypergeometric function amounts to the same as multiplication by a rational function.

It remains to show that $H(N,Id,n)\not\equiv 0$. Since, by definition $H(N,Id,n)=\sum_{j=0}^{J}N^j\sum_{i=0}^{I}\sigma_{ij}(n)$, it suffices to prove that not all of the sums $\sum_{i=0}^{I}\sigma_{ij}(n)$, $0\le j\le J$, can vanish. Suppose that F has finite support. If we multiply (2.1.19) by y^k, where y is an indeterminate not occurring in F, and sum over all k, we obtain

$$\sum_{j=0}^{J}\psi_{n-j}(y)\,\phi_j(y)=0, \tag{2.1.20}$$

where $\psi_n(y)=\sum_k F(n,k)y^k$ and $\phi_j(y)=\sum_{i=0}^{I}\sigma_{ij}(n)y^i$. Now suppose that all of the ϕ_j's vanish at $y=1$. Then they are all divisible by the factor $(1-y)$, moreover there exists a positive integer l, such that l is the highest power of $(1-y)$ which divides all $\phi_j(y)$. Now we take the recurrence (2.1.20) and divide the ϕ_j's by $(1-y)^l$. The result is a new recurrence in which it is no longer true that all of the coefficient polynomials vanish at $y=1$. Therefore we can conclude that not all of the ϕ_j's can vanish simultaneously, implying that $H(N,Id,n)\not\equiv 0$. In the case of a non-finite support we have to modify (2.1.20) by introducing the corresponding correction terms. The arguments remain the same. ∎

Once we have found the recurrence (2.1.18) for the summand $F(n, k)$ we can immediately extend it to a recurrence for the sum itself by distinguishing two cases concerning the bounds. Let

$$SUM(n) := \sum_{k=ln+m}^{on+p} F(n, k)$$

for fixed integers l, m, o and p, such that the summand is well-defined throughout the range. Now, if the limits of the sum include the finite support — in this case we speak of *naturally induced bounds* — we simply sum over both sides of (2.1.18) from $k = -\infty$ to $k = \infty$ and obtain the homogeneous recurrence

$$\sigma_0(n)SUM(n) + \sigma_1(n)SUM(n-1) + \ldots + \sigma_J(n)SUM(n-J) = 0.$$

Otherwise, if the bounds are not naturally induced, the recurrence we are looking for is given by

$$\begin{aligned}\sigma_0(n)SUM(n) + \sigma_1(n)SUM(n-1) + \ldots + \sigma_J(n)SUM(n-J) &= \\ G(n, on+p) - G(n, ln+m-1) + CT(n), &\end{aligned}$$

where the corresponding correction term $CT(n)$ is defined as

$$CT(n) := \sum_{j=1}^{J} \sigma_j(n)(CT_1(j, n) - CT_2(j, n)),$$

with

$$CT_1(j, n) := \begin{cases} \displaystyle\sum_{k=l(n-j)+m}^{ln+m-1} F(n-j, k), & l > 0, \\ 0, & l = 0, \\ \displaystyle -\sum_{k=ln+m}^{l(n-j)+m-1} F(n-j, k), & l < 0 \end{cases}$$

and

$$CT_2(j, n) := \begin{cases} \displaystyle\sum_{k=o(n-j)+p+1}^{on+p} F(n-j, k), & o > 0, \\ 0, & o = 0, \\ \displaystyle -\sum_{k=on+p+1}^{o(n-j)+p} F(n-j, k), & o < 0. \end{cases}$$

2.2 The Mathematica Implementation

In this subsection we shall introduce the author's Mathematica implementation of the q-analogue of Zeilberger's algorithm. Nowadays Gosper's algorithm for indefinite hypergeometric summation (see, e.g., Gosper [9] or Graham, Knuth and Patashnik [10]) is implemented in most computer algebra systems. Extensions to Zeilberger's algorithm have been done by Zeilberger [24] and Koornwinder [11] in Maple. A very powerful Mathematica version of Zeilberger's algorithm has been written by Paule and Schorn [17]. However, implementations of q-analogues are anything but widespread. Up to now there exist only two Maple versions: one, being on a quite rudimentary level, written by Zeilberger, and one by Koornwinder [11]. We will come back to the latter one in the following subsection.

Installation

The package consists of five files named `qZeil.m`, `qGosper.m`, `qInput.m`, `qSimplify.m` and `LinSolve.m`, which have to be copied into one directory. After starting a Mathematica session from this directory and typing `<<qZeil.m` all files are loaded automatically. In addition to these files containing the code for the algorithm, the ASCII-file `qZeilExamples.txt`, consisting of about 200 identities at the moment, can be used as a source of examples.

Interfaces

The package has two interfaces. You can run Gosper's algorithm to find a closed form for a sum or Zeilberger's algorithm to come up with a recurrence for a sum. The corresponding commands are given by

```
qGosper[SUMMAND, RANGE, <INTCONST>, <POLYDEG>]
```

and

```
qZeil[SUMMAND, RANGE, RECVAR, ORDER, <INTCONST>, <POLYDEG>],
```

where <PARAMETER> denotes an optional argument. Before we will give a detailed description of the parameters, let us first present some illustrating examples.

Warm-up Examples

1. Load the package:

```
In[1]:= <<qZeil.m

Out[1]= Axel Riese's q-Zeilberger implementation version 1.3 loaded
```

2. Compute the closed form for a special case of the q-Chu-Vandermonde summation formula

$$\sum_{k=0}^{n} \frac{(b;q)_k}{(q;q)_k}\, q^k = \frac{(bq;q)_n}{(q;q)_n}.$$

```
In[2]:= qGosper[qfac[b,q,k] q^k / qfac[q,q,k], {k, 0, n}]

         qfac[b q, q, n]
Out[2]= ---------------
          qfac[q, q, n]
```

3. Find a recurrence for the left hand side of identity (1.3.3) involving q-binomial coefficients:

```
In[3]:= qGosper[qBinomial[n,k,q] q^Binomial[k,2] x^k, {k, 0, n}]

Out[3]= No Solution !!

In[4]:= qZeil[qBinomial[n,k,q] q^Binomial[k,2] x^k, {k, 0, n}, n, 1]

                              -1 + n
Out[4]= SUM[n] == (1 + q         x) SUM[-1 + n]
```

4. As we know from (1.4.4), the polynomials $r_n(x,a)$ satisfy a recurrence of order 2:

```
In[5]:= qZeil[qBinomial[n,k,q] a^(n-k) x^k, {k, 0, n}, n, 2]

                                 -1 + n
Out[5]= SUM[n] == a (-1 + q       ) x SUM[-2 + n] +

          (a + x) SUM[-1 + n]
```

5. For Jackson's q-analogue of the Pfaff-Saalschütz formula (1.8.4) we obtain:

```
In[6]:= qZeil[qfac[q^(-n),q,k] qfac[a,q,k] qfac[b,q,k] q^k /
              (qfac[c,q,k] qfac[a b / c q^(1-n),q,k] qfac[q,q,k]),
               {k, 0, n}, n, 1]
```

```
                           -1 + n           -1 + n
                        c q              c q
                   (1 - ---------) (1 - ---------) SUM[-1 + n]
                            a                b
Out[6]= SUM[n] == -------------------------------------------
                                                  -1 + n
                                -1 + n         c q
                       (1 - c q       ) (1 - ---------)
                                                  a b
```

The Summand

As summand we allow every q-proper-hypergeometric function, cf. (2.1.17), of the form

$$F(n,k) = \frac{\prod_{r=1}^{rr} (A_r q^{(d_r i_r)n+(e_r i_r)k+l_r}; q^{i_r})_{a_r n+b_r k+c_r}}{\prod_{s=1}^{ss} (B_s q^{(f_s j_s)n+(g_s j_s)k+m_s}; q^{j_s})_{u_s n+v_s k+w_s}} \cdot P(q^n, q^k)\, q^{\alpha\binom{k}{2}+(\beta n+\gamma)k} z^k, \qquad (2.2.1)$$

with

A_r, B_s	power products in $\mathcal{K}$,
a_r, b_r, u_s, v_s	specific integers (i.e. integers free of any parameters),
c_r, w_s	integers, which may depend on parameters free of n and k,
d_r, e_r, f_s, g_s	specific integers,
l_r, m_s	integers free of n and k,
i_r, j_s	specific non-zero integers,
P	a Laurent-polynomial in q^n and q^k with coefficients in $\mathcal{K}(q)$,
α, β, γ	specific integers and
z	a rational function in $\mathcal{K}(q)$.

The q-shifted factorial $(a;q)_c$ has to be typed as `qfac[a,q,c]`. In addition we allow terms of the form `qBrackets[a,q]` for $[a]_q$, `qFactorial[a,q]` for $[a]_q!$ and `qBinomial[a,b,q]` for $\begin{bmatrix} a \\ b \end{bmatrix}_q$, provided that those expressions can be translated correctly — with respect to (2.2.1) — into terms of q-shifted factorials.

The Summation Range

The range of summation has to be specified in the form

`RANGE := {SUMVAR, LOW, UPP}.`

In qGosper, LOW and UPP may be arbitrary integers free of SUMVAR satisfying LOW $\leq$ UPP. In qZeil, LOW and UPP are linear integer functions in RECVAR being free of SUMVAR such that LOW $\leq$ UPP.

In Zeilberger's algorithm the user may specify the bounds to be -Infinity and Infinity. In this case, the bounds corresponding to the finite support bounds are assumed to be naturally induced. The algorithm runs moderately faster in this *Turbo*-mode, since no inhomogeneous part and no correction terms of the recurrence have to be computed.

The Optional Arguments

Since Mathematica is not able to handle typed variables, it is necessary to simulate them by telling the system explicitly which indeterminates should be treated as non-negative integer constants. If the optional argument INTCONST is assigned a list of Mathematica symbols representing those indeterminates, the program will assume them to be non-negative integers. This also improves the simplification abilities of the program.

Consider the following example. Suppose we want to find a closed form for the indefinite sum

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix}_q q^k.$$

```
In[7]:= qGosper[qBinomial[m+k,k,q] q^k, {k, 0, n}]

qfac[q, q, k + m]: third argument is not a linear integer function in k
qfac[q, q, m]: third argument is not a linear integer function in k

Out[7]= Fatal Error: Input error
```

The messages show that without any knowledge about m the program is not able to recognize m and $m+k$ as integers. The problem disappears if we make the assignment INTCONST := {m}.

```
In[8]:= qGosper[qBinomial[m+k,k,q] q^k, {k, 0, n}, {m}]

Out[8]= qBinomial[1 + m + n, 1 + m, q]
```

Note that all indeterminates appearing in the bounds as well as the recursion variable RECVAR in Zeilberger's algorithm are assumed to be elements of INTCONST automatically.

The second optional argument POLYDEG may be set to a non-negative integer, if one knows in advance the degree of the solution polynomial $Y(q^k)$ in Gosper's algorithm. This turned out to be useful for quite a few applications, since it might happen that the algorithm computes more than one possible value for the degree. Now, if the program does not find a solution for the least of these values, one can save a lot of "trial and error" run-time by setting POLYDEG to the actual degree.

Global Variables

The output behavior of the program can be influenced by the global boolean variables Talk and Output.

If Talk is set to True, the user can easily observe which step of the algorithm is executed at the moment. This is mainly thought for time-consuming examples. Default value for Talk is False. The protocol looks like as follows.

```
In[9]:= Talk = True; qGosper[q^k / qfac[q,q,k], {k, 0, n}]

p, r1, r2 ready...
Degree bound(s) {0} ready...
Gosper-equation ready...
Starting LinSolve...
LinSolve ready...
Simplifying result...

                1
Out[10]= -------------
         qfac[q, q, n]
```

If Output is set to True, then running Gosper's or Zeilberger's algorithm generates the file GoOut, where some intermediate results of the computation are written to. Since the default value for Output is True, GoOut refers now to the last example.

```
In[11]:= !!GoOut

Out[11]=
0. summation variable: k
----------------------

1. Gosper summand F(k) =
----------------------
```

```
      k
     q
-------------
qfac[q, q, k]

2. P-factor P_fac(T) =
--------------------
T

3. Q-numerator Q_num(T) =
-----------------------
1

4. R-denominator R_den(T) =
-------------------------
1 - T

5. degree bound(s) for Y = {0}
------------------------

6. equation to solve in Y:
--------------------------
T - T Y[0] == 0

7. solution polynomial Y(k) =
---------------------------
1

8. Gosper solution function G(k) =
   (s.t. F(k) = G(k) - G(k-1) )
--------------------------------
      1
-------------
qfac[q, q, k]

9. G(n) - G(-1) =
---------------
      1
-------------
qfac[q, q, n]
```

The entries `P-factor P_fac(T)`, `Q-numerator Q_num(T)` and `R-denominator R_den(T)`, where T is used as an abbreviation for q^k, correspond to polynomials $\bar{P}, \bar{Q}$ and $\bar{R}$, respectively, of the slightly modified q-GP representation

$$\frac{f(k)}{f(k-1)} = \frac{\bar{P}}{\epsilon^{-1}\bar{P}} \; \frac{\bar{Q}}{\bar{R}},$$

which is easier to handle algorithmically than the original one.

The simplified certificate of the last computation, i.e., the rational function $R(n,k)$ such that $G(n,k) = R(n,k)F(n,k)$, can be obtained by calling the function `Cert` without any parameter.

Finally, by setting the global variable `Simp` to `False` one can suppress the automatic simplification of the solution function and the correction terms. By default, the program applies the rules listed in the file `qSimplify.m` to those expressions. Since the size of the result may grow enormously, this should be done only in case of emergency.

Some Remarks on the Run-Time and Its History

The run-times listed in `qZeilExamples.txt` for certain examples refer to tests on a PC-486/33 with 16 MB memory using Mathematica 2.2 for Windows. All other results were obtained in less than 180 seconds.

Concerning the run-time, the main part of Gosper's algorithm consists in solving a system of homogeneous linear equations with polynomial coefficients. It turned out that the Mathematica functions `NullSpace` and `LinearSolve` are absolutely impracticable even for rather simple applications. To overcome this problem E. Aichinger wrote a Mathematica function `ENullSpace` based on Gaussian elimination, which does the job excellently for most of the examples. The interface `LinSolve` was written by M. Schorn. These two routines are contained in the file `LinSolve.m`.

At the very beginning of the implementation up to 95 percent of the run-time were spent for solving the system of equations. In the meanwhile this amount has decreased to about 30–40 percent average, mainly due to a preprocessing of the system in which all constant factors with respect to the summation variable are extracted. The size of the system essentially depends on the order of the recurrence and the degree of the solution polynomial. For the computer mentioned above it turned out that order 1 and degree 10, order 2 and degree 6, and order 3 and degree 3 seem to be roughly estimated limits, for which we do not run out of memory.

Furthermore, a lot of considerations had to be put into finding a powerful *and* efficient simplification procedure. As a compromise, the strategy is now based on the application of several rewrite rule blocks one by one as following. First, a q-hypergeometric expression involving `qBrackets`', `qFactorial`'s and `qBinomial`'s is transformed into one containing only `qfac`'s. Then we apply rules for manipulating those q-shifted factorials and finally reconstruct the remaining parts of the original expression.

2.3 A Comparison with Koornwinder's Implementation

As already mentioned, Koornwinder implemented Zeilberger's algorithm and its q-analogue in Maple. Furthermore, in [11] he gives a rigorous description of the ordinary algorithm and some remarks how to carry it over to the q-case. His program implements the q-Gosper algorithm for

$$\sum_{k=0}^{n} \frac{(\alpha_1;q)_k\,(\alpha_2;q)_k \cdots (\alpha_r;q)_k}{(q;q)_k\,(\beta_1;q)_k \cdots (\beta_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \zeta^k$$

and the q-Zeilberger algorithm for

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k\,(q^{i_2 n}\alpha_2;q)_k \cdots (q^{i_r n}\alpha_r;q)_k}{(q;q)_k\,(q^{j_1 n}\beta_1;q)_k \cdots (q^{j_s n}\beta_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} (q^{n\nu}\zeta)^k\,,$$

where $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s$ and ζ are rational functions in a fixed number of indeterminates including q (but not q^k) and $i_2,\ldots,i_r,j_1,\ldots,j_s,\nu \in \mathbf{Z}$, such that

$$\begin{array}{ll} {}^q\log\beta_t \notin \mathbf{Z} \text{ if } j_t = -1,-2,\ldots; & {}^q\log\beta_t \neq 0,-1,-2,\ldots \text{ if } j_t = 0; \\ {}^q\log\alpha_t \notin \mathbf{Z} \text{ if } i_t = 0; & \zeta \neq 0. \end{array}$$

Besides the fact that the program is very fast — the Maple functions `solve` and `factor` are a great deal faster than the corresponding Mathematica functions — the input specification described above is quite restrictive. Some shortcomings, which I tried to overcome with my implementation, are:

1. The summation range cannot be changed to an interval different from $[0,n]$ like $[-n,n]$ or $[0,2n]$ etc. Since $(q^{-n};q)_k$ and $(q;q)_k^{-1}$ have to be factors of the summand, the program always assumes finite support. Therefore no inhomogeneous recurrences can be dealt with.

2. Concerning the bases of the q-shifted factorials, no powers of q are accepted. Since the domain of computation is the field of the rational numbers and not of the complex numbers, it is in general impossible to split q-shifted factorials of the form $(\alpha;q^c)_k$ for $c>1$ into $(\alpha^{(1)},\ldots,\alpha^{(c)};q)_k$, where the $\alpha^{(i)}$ are the complex roots of α. Furthermore, no rational powers of indeterminates are allowed.

3. The α's and β's must be free of k, and the index in q-shifted factorial expressions is restricted to be k. One often has to apply expensive transformations to achieve this form.

4. No polynomial part can be specified.

3 Applications

In this section we shall describe additional features of the algorithm and illustrate the wide range of its applicability by giving non-trivial examples. We start with general proof strategies for closed form identities as well as for transformation formulas satisfying higher order recurrences, then comment on the amazing "magic-factor-trick" for decreasing the order of certain recurrences and finally deal with companion and dual identities.

3.1 Proof Strategies

We shall discuss two different types of algorithmic proofs for q-hypergeometric identities. First we introduce the notion of qWZ-pairs for proving identities of the form $\sum_k F(n,k) = rhs(n)$. Then we show how to prove the equality of two sums, i.e., $\sum_k F_1(n,k) = \sum_k F_2(n,k)$, via recurrences.

Proving Closed Form Identities

Suppose we want to prove the identity $\sum_k F(n,k) = rhs(n)$, where $rhs(n)$ is a closed form expression not depending on k. Then, if $rhs(n) \neq 0$ for all n, this problem is equivalent to checking that

$$\sum_k \frac{F(n,k)}{rhs(n)} = 1.$$

Hence, we might think of $F(n,k)/rhs(n)$ as having been the original summand and we want to prove that $\sum_k F(n,k) = 1$. Now, if the algorithm returns a recurrence which is satisfied by 1 and additionally the initial values also equal 1, the proof is complete.

In most instances this works with order one and $\sigma_0(n) = \sigma_1(n) = 1$, implying that we have found a function $G(n,k)$ such that

$$F(n,k) - F(n-1,k) = G(n,k) - G(n,k-1). \tag{3.1.1}$$

In this case we say that F and G form a *qWZ-pair*. If we sum over both sides of (3.1.1), say for k from a to b, we obtain

$$SUM(n) - SUM(n-1) = G(n,b) - G(n,a-1) + CT(n).$$

Suppose that $G(n,b) - G(n,a-1) + CT(n) = 0$, as it happens for instance in case of naturally induced bounds. Then, from

$$SUM(n) - SUM(n-1) = 0,$$

it follows that $SUM(n)$ is constant, i.e., not depending on n, and therefore $SUM(n) = SUM(0)$ holds for all n. The program will recognize this fact returning `SUM[n] == c` with $c := SUM(0)$ evaluated, provided that the bounds were specified explicitly. Since in Turbo-mode the program is not able to compute $SUM(0)$, we obtain `SUM[n] == SUM[n-1]`.

Checking if a recurrence is satisfied by 1 can be done algorithmically by calling the boolean function `Check1[REC]`, where `REC` is a recurrence returned by the algorithm. The only thing that remains to the user is to show that, if d is the order of the recurrence, we have for the initial values $SUM(0) = \ldots = SUM(d-1) = 1$.

The following examples shall illustrate what happens. Let us again consider Jackson's q-analogue of the Pfaff-Saalschütz sum (1.8.4),

$${}_3\phi_2\left[\begin{matrix} q^{-n}, a, b \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.$$

Proceeding as described above, we divide the left hand side by the right hand side to get the following result.

```
In[3]:= qZeil[qfac[q^(-n),q,k] qfac[a,q,k] qfac[b,q,k] *
              qfac[c,q,n] qfac[c/(a b),q,n] q^k /
             (qfac[q,q,k] qfac[c,q,k] qfac[a b/c q^(1-n),q,k] *
              qfac[c/a,q,n] qfac[c/b,q,n]),
              {k, 0, n}, n, 1]

Out[3]= SUM[n] == 1
```

The following finite version of Gauss' summation formula,

$$\sum_{k=0}^{2n} (-1)^k q^{(k-n)(k-n-1)} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q = q^{n(n+1)}$$

leads to an inhomogeneous recurrence of order one which cannot be evaluated immediately, but which is easily seen to be satisfied by 1.

```
In[4]:= qZeil[(-1)^k q^((k-n)(k-n-1)) qBinomial[2n+1,k,q] / q^(n(n+1)),
              {k, 0, 2n}, n, 1]

                                  2 n
                   -2 n    (1 - q    ) SUM[-1 + n]
Out[4]= SUM[n] == q      - ---------------------
                                      2 n
                                     q

In[5]:= Check1[%]

Out[5]= True
```

Since $SUM(0) = 1$, we may conclude that $SUM(n) = 1$ is true for all n.

Finally, the following example taken from Paule [13, Eq. (21)] stating that

$$\sum_{k=-n}^{n} (-1)^k q^{k(2k-1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} = (-q^2; q^2)_n \, (q; q^2)_n$$

has minimal recursion order 2 instead of 1, as one would expect.

```
In[6]:= qZeil[(-1)^k qBinomial[2n,n+k,q^2] q^(k(2k-1)) /
             (qfac[-q^2,q^2,n] qfac[q,q^2,n]), {k, -n, n}, n, 2]

                         n         n         -2 + 2 n
                  (-q + q ) (q + q ) (q + q         ) SUM[-2 + n]
Out[6]= SUM[n] == --------------------------------------------- +
                                    2 n        2 n
                               (q - q   ) (1 + q   )

               2         -2 + 4 n
         (1 + q ) (q - q         ) SUM[-1 + n]
         ------------------------------------
                      2 n        2 n
                 (q - q   ) (1 + q   )

In[7]:= Check1[%]

Out[7]= True
```

Again we have $SUM(0) = SUM(1) = 1$, which completes the proof. We want to note that the order can be decreased to the "natural" order 1 by applying the "magic-factor-trick", see Subsection 3.2.

Proving Transformation Formulas

Another large field of applications is based on the fact that Zeilberger's algorithm can be used not only for proving *summation* formulas, but also for handling *transformations*, i.e., identities of the form

$$SUM_1(n) = SUM_2(n) = \ldots = SUM_j(n),$$

where, for $1 \le i \le j$,

$$SUM_i(n) := \sum_k F_i(n,k).$$

For proving that $SUM_{i_1}(n) = SUM_{i_2}(n)$ holds for all n it is sufficient to show that $SUM_{i_1}(n)$ and $SUM_{i_2}(n)$ satisfy the same recurrence, having both the same initial values.

Consider the following transformation formulas taken from Gasper and Rahman [6, (III.11), (III.12)],

$$\begin{aligned} {}_3\phi_2\left[\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q\right] &= \frac{(de/bc;q)_n}{(e;q)_n}\left(\frac{bc}{d}\right)^n {}_3\phi_2\left[\begin{matrix} q^{-n}, d/b, d/c \\ d, de/bc \end{matrix}; q, q\right] \\ &= \frac{(e/c;q)_n}{(e;q)_n}\, c^n\, {}_3\phi_2\left[\begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; q, \frac{bq}{e}\right]. \end{aligned}$$

```
In[8]:= qZeil[qfac[q^(-n),q,k] qfac[b,q,k] qfac[c,q,k] q^k /
           (qfac[d,q,k] qfac[e,q,k] qfac[q,q,k]), {k, 0, n}, n, 2]

                      -1 + n        2        n
                 (-1 + q      ) (b c q  - d e q ) SUM[-2 + n]
Out[8]= SUM[n] == ------------------------------------------- +
                                   n          n
                            (-q + d q ) (-q + e q )

              3      3        2 n        1 + n        2 + n
         ((b q  + c q  + d e q    - d e q      - b c q      -

                2 + n      2 + n        1 + 2 n
            d q      - e q      + d e q        ) SUM[-1 + n]) /

                   n          n
          (q (-q + d q ) (-q + e q ))

In[9]:= qZeil[qfac[q^(-n),q,k] qfac[d/b,q,k] qfac[d/c,q,k] *
           qfac[d e/(b c),q,n] (b c/d)^n q^k /
           (qfac[d,q,k] qfac[d e/(b c),q,k] qfac[q,q,k] qfac[e,q,n]),
           {k, 0, n}, n, 2]
```

```
                  -1 + n          2         n
              (1 - q      ) (-(b c q ) + d e q ) SUM[-2 + n]
Out[9]= SUM[n] == ---------------------------------------------- +
                                n           n
                         (-q + d q ) (-q + e q )

        3      3        2 n        1 + n        2 + n
   ((b q  + c q  + d e q    - d e q      - b c q      -

          2 + n      2 + n        1 + 2 n
       d q      - e q      + d e q        ) SUM[-1 + n]) /

                 n           n
    (q (-q + d q ) (-q + e q ))

In[10]:= qZeil[qfac[q^(-n),q,k] qfac[c,q,k] qfac[d/b,q,k] *
              qfac[e/c,q,n] c^n (b q/e)^k /
             (qfac[d,q,k] qfac[c/e q^(1-n),q,k] qfac[q,q,k] *
              qfac[e,q,n]), {k, 0, n}, n, 2]

                    -1 + n        2        n
               (-1 + q      ) (b c q  - d e q ) SUM[-2 + n]
Out[10]= SUM[n] == ------------------------------------------- +
                                 n           n
                          (-q + d q ) (-q + e q )

        3      3        2 n        1 + n        2 + n
   ((b q  + c q  + d e q    - d e q      - b c q      -

          2 + n      2 + n        1 + 2 n
       d q      - e q      + d e q        ) SUM[-1 + n]) /

                 n           n
    (q (-q + d q ) (-q + e q ))
```

Since

$$SUM_1(0) = SUM_2(0) = SUM_3(0) = 1$$

and

$$SUM_1(1) = SUM_2(1) = SUM_3(1) = \frac{b + c - bc - d - e + de}{1 - d - e + de},$$

we are done.

3.2 The Magic Factor

As we saw in the last subsection, Zeilberger's algorithm does not always return the recurrence with minimal order. For example, take the following finite version of the so-called first Rogers-Ramanujan identity,

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k\,(q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}}. \tag{3.2.1}$$

Then the left hand side is the solution of a second order recurrence equation, whereas the right hand side is annihilated by a recurrence operator of order five. In other words, after running the algorithm we would still have to carry out some steps by hand for proving the equality of these two sums, cf. Ekhad and Tre [5].

Recently, Paule [14] found an amazing trick how to boil down q-certificates of a large number of applications. This trick is based on the summand's symmetry in k and $-k$. More precisely, assume that the summand $F(n,k)$ satisfies the symmetry condition

$$\sum_{k=a}^{b} F(n,k) = \sum_{k=a}^{b} F(n,-k).$$

Then looking at the decomposition of the summand into its even and its odd part, given by

$$F(n,k) = F_e(n,k) + F_o(n,k) = \frac{F(n,k)+F(n,-k)}{2} + \frac{F(n,k)-F(n,-k)}{2},$$

we find that if we sum over k running from a to b, the sum of the odd parts vanishes and we obtain

$$\begin{aligned}\sum_{k=a}^{b} F(n,k) &= \sum_{k=a}^{b} F_e(n,k) = \sum_{k=a}^{b} \frac{F(n,k)+F(n,-k)}{2} \\ &= \sum_{k=a}^{b} F(n,k)\left(\frac{1+F(n,-k)/F(n,k)}{2}\right).\end{aligned}$$

For $quot(n,k) = F(n,-k)/F(n,k)$, Paule observed the remarkable fact that in many instances — provided that the magic factor $mf(n,k) := (1+quot(n,k))/2$ fulfills the input restrictions of the algorithm — the summand containing the rational function $mf(n,k)$ satisfies a recurrence of smaller order than the original summand.

Paule's trick can be easily generalized by a slight modification of the symmetry condition as following. Suppose that the summand satisfies

$$\text{(SYM)} \qquad \sum_{k=a}^{b} F(n,k) = \sum_{k=a}^{b} F(n,-k-c)$$

for some integer c. Then, replacing $F(n,-k)$ by $F(n,-k-c)$ in the summand's decomposition gives that

$$\sum_{k=a}^{b} F(n,k) = \sum_{k=a}^{b} F(n,k)\left(\frac{1+F(n,-k-c)/F(n,k)}{2}\right).$$

For the right hand side of (3.2.1), the symmetry condition (SYM) is clearly satisfied for $c=0$, and we have $quot(n,k)=q^k$. Therefore, after introducing the magic factor $(1+q^k)/2$, identity (3.2.1) is equivalent to

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k\,(q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{1+q^k}{2}\,\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}}.$$

If we now apply the algorithm, we obtain a recurrence of order two for *both* sides of the identity.

```
In[3]:= qZeil[q^(k^2) / (qfac[q,q,k] qfac[q,q,n-k]), {k, 0, n}, n, 2]

                                          2    2 n    1 + n
                  q SUM[-2 + n]    (-q - q  - q     + q     ) SUM[-1 + n]
Out[3]= SUM[n] == ------------- + -------------------------------------
                          n                             n
                     -1 + q                        q (-1 + q )

In[4]:= qZeil[(1+q^k)/2 (-1)^k q^((5k^2-k)/2) /
             (qfac[q,q,n-k] qfac[q,q,n+k]), {k, -n, n}, n, 2]

                                          2    2 n    1 + n
                  q SUM[-2 + n]    (-q - q  - q     + q     ) SUM[-1 + n]
Out[4]= SUM[n] == ------------- + -------------------------------------
                          n                             n
                     -1 + q                        q (-1 + q )
```

Since

$$SUM_1(0) = SUM_2(0) = 1 \quad \text{and} \quad SUM_1(1) = SUM_2(1) = \frac{1+q}{1-q},$$

the proof is complete.

Introducing the magic factor can also be of fundamental importance when dealing with summation formulas. Consider the following special version of the q-Dixon identity,

$$\sum_{k=-n}^{n} (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^3 q^{k(3k+1)/2} = \frac{(q;q)_{3n}}{(q;q)_n^3}.$$

In this case the program does not find a recurrence of order one and order two. For order three we obtain — after several hours of run-time — an out-of-memory message.

Since (SYM) is satisfied for $c = 0$ with $quot(n,k) = q^{-k}$, we multiply the summand by $(1+q^{-k})/2$ to obtain the following closed form evaluation.

```
In[5]:= qZeil[(1+q^(-k))/2 (-1)^k qBinomial[2n,n+k,q]^3 q^(k(3k+1)/2) *
             qfac[q,q,n]^3 / qfac[q,q,3n], {k, -n, n}, n, 1]

Out[5]= SUM[n] == 1
```

Finally, to illustrate the significance of the shift parameter c in (SYM), consider the non-symmetric analogue of the q-Dixon identity,

$$\sum_{k=-n-1}^{n} (-1)^k \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q^3 q^{k(3k+1)/2} = \frac{(q;q)_{3n+1}}{(q;q)_n^3}.$$

Proceeding in a straightforward manner, we could increase the upper summation bound to $n+1$. Then (SYM) is satisfied for $c = 0$ leading to the rather complex quotient

$$quot(n,k) = \frac{(1-q^{n+k+1})^3}{q^k(1-q^{n-k+1})^3},$$

which is furthermore not defined for $k = n+1$, since $F(n, n+1) = 0$. But by observing that

$$\sum_{k=-n-1}^{n} F(n,k) = \sum_{k=-n-1}^{n} F(n,-k-1),$$

the corresponding expressions for $c = 1$ reduce to $quot(n,k) = -q^{2k+1}$ and therefore $mf(n,k) = (1-q^{2k+1})/2$. Now we again obtain a recurrence of order one which is found to be constant.

```
In[6]:= qZeil[(1-q^(2k+1))/2 (-1)^k qBinomial[2n+1,n+k+1,q]^3 *
             q^(k(3k+1)/2) qfac[q,q,n]^3 / qfac[q,q,3n+1],
             {k, -n-1, n}, n, 1]

Out[6]= SUM[n] == 1
```

3.3 The Companion Identity

We already saw that qWZ-pairs play an important rôle in q-certification. Moreover, we can use qWZ-pairs to get new identities "for free", i.e., without too much additional effort (cf. Wilf and Zeilberger [21]). One of them is called the companion identity and is based on the symmetry of F and G in the qWZ-equation (3.1.1).

Theorem 3.3.1 (companion identity). Let F and G form a qWZ-pair satisfying the following conditions:

(F) For each integer k, the limit $f_k := \lim_{n\to\infty} F(n,k)$ exists and is finite.

(G) We have $\lim_{k\to-\infty} \sum_{n\geq 0} G(n+1,k) = 0$.

Then the companion identity is given by

$$\sum_{n\geq 0} G(n+1,k) = \sum_{j\leq k} \left(f_j - F(0,j)\right).$$

Proof. Since F and G form a qWZ-pair we have

$$F(n+1,k) - F(n,k) = G(n+1,k) - G(n+1,k-1).$$

Summing both sides for n from 0 to N gives

$$F(N+1,k) - F(0,k) = (Id - K)\Big(\sum_{n=0}^{N} G(n+1,k)\Big).$$

Now we let $N \to \infty$ and use (F) to get

$$f_k - F(0,k) = (Id - K)\Big(\sum_{n\geq 0} G(n+1,k)\Big).$$

If we replace k by j and sum over both sides for j from $-l$ to k, we obtain

$$\sum_{j=-l}^{k} \left(f_j - F(0,j)\right) = \sum_{n\geq 0} G(n+1,k) - \sum_{n\geq 0} G(n+1,-l-1).$$

Letting $l \to \infty$ and using (G) gives the companion identity

$$\sum_{j\leq k} \left(f_j - F(0,j)\right) = \sum_{n\geq 0} G(n+1,k). \qquad \blacksquare$$

The program computes the companion identity, if the global variable `Companion` is set to `True` and F and G in fact form a qWZ-pair. To compute f_k we might have to make the assumption $|q| < 1$, or, to take the limit w.r.t. sequences of formal power (or Laurent) series. The condition (G) has to be checked by the user. Note that (G) is satisfied if F has naturally induced bounds. Default value for `Companion` is `False`, the result is assigned to the variable `CompId`.

Let us consider the q-Chu-Vandermonde identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} = \begin{bmatrix} b+n \\ n \end{bmatrix}_q .$$

Then we have

$$\lim_{n\to\infty} F(n,k) = \lim_{n\to\infty} \frac{(q;q)_b^2\,(q;q)_n^2\,q^{k^2}}{(q;q)_{b+n}\,(q;q)_{b-k}\,(q;q)_k^2\,(q;q)_{n-k}} = \frac{(q;q)_b^2\,q^{k^2}}{(q;q)_{b-k}\,(q;q)_k^2}$$

and

$$F(0,k) = \frac{\begin{bmatrix} 0 \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q}{\begin{bmatrix} b \\ 0 \end{bmatrix}_q}\, q^{k^2} = \delta_{0,k}.$$

Hence we obtain the following result.

```
In[3] := Companion = True;
         qZeil[qBinomial[n,k,q] qBinomial[b,k,q] q^(k^2) /
               qBinomial[n+b,n,q], {k, 0, n}, n, 1, {b}]

Out[4]= SUM[n] == 1

In[5]:= CompId


                       2
                1 + k + k  + n                                      2
Out[5]= Sum[-((q                qBinomial[n, k, q] qfac[q, q, b]

               qfac[q, q, n]) / (qfac[q, q, -1 + b - k] qfac[q, q, k]

                 qfac[q, q, 1 + b + n])), {n, 0, Infinity}] ==

                          2
                        jj
                       q    qBinomial[b, b - jj, q] qfac[q, q, b]
         -(k >= 0) + Sum[-----------------------------------------,
                                        qfac[q, q, jj]

           {jj, -Infinity, k}]
```

Following the notation of Graham, Knuth and Patashnik [10], for any true-or-false predicate *pred* we say that $(pred) := 1$ if *pred* is true and $(pred) := 0$ if *pred* is false.

Hence, for $b \geq 0$ and $k \geq 0$ the companion identity reads as

$$\frac{q^{1+k+k^2}(1-q^{k+1})}{(1-q^{b+1})} \begin{bmatrix} b \\ k+1 \end{bmatrix}_q \cdot \sum_{n \geq k} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n+b+1 \\ n \end{bmatrix}_q} q^n = 1 - \sum_{j=0}^{k} \frac{\begin{bmatrix} b \\ j \end{bmatrix}_q (q;q)_b}{(q;q)_j} q^{j^2}.$$

Since the left hand side vanishes for $b \leq k$, we simultaneously proved the terminating identity

$$\sum_{j=0}^{b} \frac{\begin{bmatrix} b \\ j \end{bmatrix}_q}{(q;q)_j} q^{j^2} = \frac{1}{(q;q)_b}.$$

For the special case $b = n$ of the q-Chu-Vandermonde identity we get the following result.

```
In[6]:= qZeil[qBinomial[n,k,q]^2 q^(k^2) / qBinomial[2n,n,q],
              {k, 0, n}, n, 1]

Out[6]= SUM[n] == 1

In[7]:= CompId


                     2
                1 + k  + n     k    n    1 + 2 n                  2
Out[7]= Sum[-((q          (2 q  - q  - q        ) qBinomial[n, k, q]

                          2            1 + n
               qfac[q, q, n] ) / ((1 + q     ) qfac[q, q, 1 + 2 n])),

          {n, 0, Infinity}] == -(k >= 0) +

                2
               jj
              q    qfac[q, q, Infinity]
          Sum[-------------------------, {jj, -Infinity, k}]
                                 2
                   qfac[q, q, jj]
```

For $k = 0$ the companion identity becomes

$$1 + \sum_{n \geq 0} \frac{q^{n+1}\left(q^{2n+1} + q^n - 2\right)(q;q)_n^2}{(1+q^{n+1})\,(q;q)_{2n+1}} = (q;q)_\infty.$$

These companion identities — and many others — appear to be new identities in the q-hypergeometric database.

3.4 The Dual Identity

In this subsection we shall present another method for discovering new identities based on the fact that to any qWZ-pair we can associate a dual pair that may produce new identities. This mapping will turn out to be an involution up to constant factors.

Once we have found a qWZ-pair, we can easily construct other ones as listed in the following theorem, which is taken from Gessel [7], carried over to the q-case.

Theorem 3.4.1. Let (F, G) be a qWZ-pair.

(i) For integers a and b, $(\widetilde{F}(n,k), \widetilde{G}(n,k)) := (F(n+a, k+b), G(n+a, k+b))$ is a qWZ-pair.

(ii) For any complex number c, $(\widetilde{F}(n,k), \widetilde{G}(n,k)) := (cF(n,k), c\,G(n,k))$ is a qWZ-pair.

(iii) $(\widetilde{F}(n,k), \widetilde{G}(n,k)) := (F(-n,k), -G(-n+1,k))$ is a qWZ-pair.

(iv) $(\widetilde{F}(n,k), \widetilde{G}(n,k)) := (F(n,-k), -G(n,-k-1))$ is a qWZ-pair.

(v) $(\widetilde{F}(n,k), \widetilde{G}(n,k)) := (G(k,n), F(k,n))$ is a qWZ-pair.

(vi) If
$$\frac{F(n,k)}{F(n-1,k)} = \frac{\widetilde{F}(n,k)}{\widetilde{F}(n-1,k)}, \quad \frac{F(n,k)}{F(n,k-1)} = \frac{\widetilde{F}(n,k)}{\widetilde{F}(n,k-1)}$$
and
$$\frac{G(n,k)}{F(n,k)} = \frac{\widetilde{G}(n,k)}{\widetilde{F}(n,k)},$$
then $(\widetilde{F}(n,k), \widetilde{G}(n,k))$ is a qWZ-pair.

Proof. (i) - (v): Straightforward by plugging in $\widetilde{F}$ and $\widetilde{G}$ into the qWZ-equation (3.1.1).

(vi): Dividing the qWZ-equation (3.1.1) by $F(n,k)$ we get
$$\begin{aligned} 1 - \frac{F(n-1,k)}{F(n,k)} &= \frac{G(n,k)}{F(n,k)} - \frac{G(n,k-1)}{F(n,k)} \\ &= \frac{G(n,k)}{F(n,k)} - \frac{G(n,k-1)}{F(n,k-1)}\,\frac{F(n,k-1)}{F(n,k)}. \end{aligned}$$

By our assumptions we may replace F and G by $\widetilde{F}$ and $\widetilde{G}$, respectively. Multiplying through by $\widetilde{F}(n,k)$ proves that $(\widetilde{F}, \widetilde{G})$ form a qWZ-pair. ∎

As in the $q = 1$ case (cf. Gessel [7], Wilf [20] or Wilf and Zeilberger [21], [22]), we introduce the operation of *shadowing*. Let, for instance, $a(n) = (q;q)_n$ for $n \geq 0$. Then the defining property of $a(n)$ is that it satisfies the recurrence equation $a(n) = (1 - q^n)\, a(n-1)$ together with the initial condition $a(0) = 1$. But why should we restrict ourselves to non-negative integers? We could ask for a function $\bar{a}(n)$ such that $\bar{a}(n) = (1 - q^n)\, \bar{a}(n-1)$ holds for negative integers. A function that satisfies this condition is

$$\bar{a}(n) = \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_{-n-1}} \quad \text{for } n \leq -1.$$

We call $\bar{a}(n)$ the *shadow* of $a(n)$. More generally, for $a(n,k) = (A;q^d)_{an+bk+c}$, the shadow is given by

$$\bar{a}(n,k) = \frac{(-1)^{an+bk+c} A^{an+bk+c+1} q^{d\left[\binom{an+bk+c}{2}-1\right]}}{(q^{2d}/A;q^d)_{-an-bk-c-1}},$$

with the property that

$$\frac{a(n,k)}{a(n-1,k)} = \frac{\bar{a}(n,k)}{\bar{a}(n-1,k)} \quad \text{and} \quad \frac{a(n,k)}{a(n,k-1)} = \frac{\bar{a}(n,k)}{\bar{a}(n,k-1)}.$$

The shadow of $F(n,k)$ is defined to be the result of formally replacing every factor of the form $(A;q^d)_{an+bk+c}$ in F according to the shadowing rule described above. Since $F(n,k)/F(n-1,k) = \bar{F}(n,k)/\bar{F}(n-1,k)$ and $F(n,k)/F(n,k-1) = \bar{F}(n,k)/\bar{F}(n,k-1)$, we apply Theorem 3.4.1 (vi) — the third assumption is trivially satisfied, because the certificate is not influenced by taking the shadow — to obtain the following result.

Corollary 3.4.2. If F and G form a qWZ-pair, then so do $\bar{F}$ and $\bar{G}$.

The shadow of $F(n,k)$ satisfies the same linear recurrence equations as $F(n,k)$. The only difference of these two functions is in their domain of definition, and when they vanish. Clearly, one can apply the shadow mapping to only some of the factors of $F(n,k)$ and $G(n,k)$ and not to others getting different shadow pairs. Following Wilf [20], a choice that seems to give fruitful results, is to shadow only those factors $(A;q^d)_{an+bk+c}$ for which $a + b \neq 0$. Hence, we will apply this kind of default shadowing in the algorithm. In this case, to avoid trivial qWZ-pairs like $(0,0)$ etc., we have to cancel all q-shifted factorial expressions in $F(n,k)$ and $G(n,k)$ being free of n and k, which again gives us, by Theorem 3.4.1 (ii), a qWZ-pair.

The final step in the dualization is to pass from the shadow pair $(\bar{F}, \bar{G})$ to the dual pair (F', G') by a flip of variables and functions transforming the domain of n back to the non-negative integers. The dual pair is defined as

$$(F'(n,k), G'(n,k)) := \left(\bar{G}(-k,-n-1), \bar{F}(-k-1,-n)\right),$$

which, by Theorem 3.4.1 (iii), (iv), (v) and (i), does not influence the fact that the functions form a qWZ-pair, but which does alter the certificate via the same change of variables.

Note that in general dualization does not commute with specialization, i.e., the dual identity of some special case of an identity is not the same as the specialization of the dual identity. However, dualization is an involution up to constant factors.

The program computes the dual qWZ-pair, if the global variable `Dual` is set to `True`, and F and G actually form a qWZ-pair. The result is assigned to the variable `DualPair`. Default value for `Dual` is `False`.

For the q-Chu-Vandermonde identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} = \begin{bmatrix} b+n \\ n \end{bmatrix}_q$$

we get the following result.

```
In[3]:= Dual = True;
        qZeil[qBinomial[n,k,q] qBinomial[b,k,q] q^(k^2) /
              qBinomial[n+b,n,q], {k, 0, n}, n, 1, {b}]

Out[4]= SUM[n] == 1

In[5]:= DualPair

                                 2                   2
            k + n  k/2 + b k + k /2 - n/2 - b n - n /2
Out[5]= {((-1)      q

          qBinomial[n, k, q] qfac[q, q, -1 - b + k]

          qfac[q, q, -1 - b - n] qfac[q, q, n]) / qfac[q, q, k],

                                   2                   2
            k + n  b + k/2 + b k + k /2 + n/2 - b n - n /2
         ((-1)      q

          qBinomial[-1 + n, k, q] qfac[q, q, -b + k]

          qfac[q, q, -1 - b - n] qfac[q, q, -1 + n]) / qfac[q, q, k]}
```

Hence, after replacing b by $-b-1$ the dual identity becomes

$$\sum_{k=0}^{n} (-1)^{n+k} q^{(n-k)(2b-k-n+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ k \end{bmatrix}_q = \begin{bmatrix} b \\ n \end{bmatrix}_q,$$

which is the same as the original identity aside from a renaming of the parameters. An identity satisfying this property is called *self-dual*.

As mentioned above, for the special case $b = n$ we do not obtain just the dual identity with b replaced by n, but the following result.

```
In[6]:= qZeil[qBinomial[n,k,q]^2 q^(k^2) / qBinomial[2n,n,q],
             {k, 0, n}, n, 1]

Out[6]= SUM[n] == 1

In[7]:= DualPair

               2 k    n    k + n
Out[7]= {((-2 q    + q  + q     ) qBinomial[2 k, k, q]

                           2      k       k
            qBinomial[n, k, q] ) / (q  (1 + q )),

            -1 - k + n                          2
           q           qBinomial[-1 + n, k, q]  qfac[q, q, 1 + 2 k]
          -(---------------------------------------------------------)}
                                                2
                                   qfac[q, q, k]
```

Therefore the dual identity reads as

$$\sum_{k=0}^{n} \begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \frac{q^{n-k} + q^n - 2q^k}{1+q^k} = 0.$$

Next, let us consider the q-Saalschütz identity

$$\sum_{k=0}^{n} \begin{bmatrix} r-s+m \\ k \end{bmatrix}_q \begin{bmatrix} s-r+n \\ n-k \end{bmatrix}_q \begin{bmatrix} s+k \\ m+n \end{bmatrix}_q q^{(n-k)(r-s+m-k)} = \begin{bmatrix} r \\ n \end{bmatrix}_q \begin{bmatrix} s \\ m \end{bmatrix}_q.$$

The program computes the following dual qWZ-pair.

```
In[8]:= qZeil[q^((n-k)(r-s+m-k)) qBinomial[r-s+m,k,q] *
              qBinomial[s-r+n,n-k,q] qBinomial[s+k,m+n,q] /
             (qBinomial[r,n,q] qBinomial[s,m,q]),
              {k, 0, n}, n, 1, {m, r, s}]

Out[8]= SUM[n] == 1

In[9]:= DualPair

                 2
           k + k  - n - k n + k r - n r
Out[9]= {(q

            qBinomial[-1 + k - m, -1 + n - s, q] qBinomial[n, k, q]

            qfac[q, q, n + r - s] qfac[q, q, -1 - m - n - r + s]) /

          (qfac[q, q, -1 - k - r] qfac[q, q, k + r - s]),

                         2
             1 + 2 k + k  - n - k n + r + k r - n r
         -((q

              qBinomial[k - m, -1 + n - s, q] qBinomial[-1 + n, k, q]

              qfac[q, q, -1 + n + r - s] qfac[q, q, -1 - m - n - r + s]

              ) / (qfac[q, q, -2 - k - r] qfac[q, q, k + r - s]))}
```

Again we have to replace the free variables m, r and s by $-m-1, -r-1$ and $-s-1$, respectively, to obtain the dual identity

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix}_q \begin{bmatrix} s \\ r-k \end{bmatrix}_q \begin{bmatrix} m-s \\ n-k \end{bmatrix}_q q^{(n-k)(r-k)} = \begin{bmatrix} m+r-s \\ n \end{bmatrix}_q \begin{bmatrix} n+s \\ r \end{bmatrix}_q.$$

Renaming the parameters we find the q-Saalschütz identity also to be self-dual.

For the special case $m = n$ and $r = s$, the process of dualization leads to the following result.

```
In[10]:= qZeil[q^((n-k)^2) qBinomial[n,k,q]^2 qBinomial[s+k,2n,q] /
               qBinomial[s,n,q]^2,
               {k, 0, n}, n, 1, {s}]

Out[10]= SUM[n] == 1
```

```
In[11]:= DualPair

            k    2 k      n      1 + 3 k + s    1 + k + n + s
Out[11]= {((q  + q    - 2 q  - 2 q            + q              +

               1 + 2 k + n + s
              q                ) qBinomial[2 k, k, q]

                              2
            qBinomial[n, k, q]  qfac[q, q, -2 - 2 k + n - s]) /

           2 k      k                         2
          (q    (1 + q ) qfac[q, q, -1 - k - s]  qfac[q, q, -1 + n - s]),

             -k + n + s                        2
          -((q          qBinomial[-1 + n, k, q]  qfac[q, q, 1 + 2 k]

              qfac[q, q, -3 - 2 k + n - s]) /

                          2                       2
            (qfac[q, q, k]  qfac[q, q, -2 - k - s]

              qfac[q, q, -1 + n - s]))}
```

Therefore, after replacing s by $-s-1$, the dual identity turns out to be

$$\sum_{k=0}^{n} \frac{1+q^{-k}-2q^{n-2k}-2q^{k-s}+q^{n-s}+q^{n-s-k}}{1+q^k} \cdot \begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \frac{(q;q)_{n+s-2k-1}}{(q;q)_{s-k}^2} = 0,$$

where $s \geq n+1$.

Finally, for the q-Dixon identity

$$\sum_k (-1)^k \begin{bmatrix} n+b \\ n+k \end{bmatrix}_q \begin{bmatrix} n+c \\ c+k \end{bmatrix}_q \begin{bmatrix} b+c \\ b+k \end{bmatrix}_q q^{k(3k-1)/2} = \begin{bmatrix} n+b+c \\ n,b,c \end{bmatrix}_q$$

we get, by applying the magic-factor-trick, the following result.

```
In[12]:= qZeil[(1+q^k)/2 (-1)^k qBinomial[n+b,n+k,q] *
             qBinomial[n+c,c+k,q] qBinomial[b+c,b+k,q] q^(k(3k-1)/2) *
             qfac[q,q,n] qfac[q,q,b] qfac[q,q,c] / qfac[q,q,n+b+c],
             {k, -Infinity, Infinity}, n, 1, {b, c}]

Out[12]= SUM[n] == SUM[-1 + n]
```

```
In[13]:= DualPair

                        2                2
              k  k/2 + k /2 - n - k n - n
Out[13]= {((-1)  q

            qBinomial[-b + n, -b + k, q] qfac[q, q, -1 - b - c + k]

            qfac[q, q, -1 - b - n] qfac[q, q, -1 - c - n]

            qfac[q, q, -c + n] qfac[q, q, k + n]) /

          (2 qfac[q, q, k] qfac[q, q, -c + k]),

                        2                2
              k  k/2 + k /2 - n - k n - n        n
          ((-1)  q                            (1 + q )

            qBinomial[-1 - b + n, -b + k, q] qfac[q, q, -b - c + k]

            qfac[q, q, -1 - b - n] qfac[q, q, -1 - c - n]

            qfac[q, q, -1 - c + n] qfac[q, q, k + n]) /

          (2 qfac[q, q, k] qfac[q, q, -c + k])}
```

Replacing b by $-b$ and c by $-c-1$ leads to the dual identity

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-n(1+k+n)} \begin{bmatrix} b+c+k \\ b-1 \end{bmatrix}_q \begin{bmatrix} b+n \\ b+k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q = \begin{bmatrix} b+c \\ c+n+1 \end{bmatrix}_q \begin{bmatrix} c \\ n \end{bmatrix}_q,$$

which is a specialization of the q-Saalschütz identity.

A Errata for the q-Hypergeometric Database

We give a list of q-hypergeometric identities stated incorrectly in the literature which could be corrected in interaction with the program.

Gasper and Rahman [6]

$$\sum_{k=0}^{n} \frac{(1-aq^{2k})(a,q^{-n};q)_k}{(1-a)(q,aq^{n+1};q)_k}\, q^{2k^2}(a^2q^n)^k$$
$$= (aq;q)_n \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\,(-aq^{n+1})^k q^{k(k-1)/2}, \qquad (2.7.5)$$

$$ {}_4\phi_3\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b \\ a^{\frac12}, -a^{\frac12}, aq/b \end{matrix}; q, t\right] = \frac{(aq,bt;q)_\infty}{(t,aq/b;q)_\infty}\, {}_2\phi_1(b^{-1},t;bqt;q,aq), \qquad (\text{Ex. } 2.2)$$

$$\sum_{k=0}^{n} \frac{(1-adp^{3k})(1-b/dp^k)}{(1-ad)(1-b/d)}\,\frac{(a,b;p)_k\,(p^{-2n},ad^2p^{2n}/b;p^2)_k}{(dp^2,adp^2/b;p^2)_k\,(adp^{2n+1},bp^{1-2n}/d;p)_k}\,p^{2k}$$
$$= \frac{(1-d)(1-ad/b)(1-adp^{2n})(1-dp^{2n}/b)}{(1-ad)(1-d/b)(1-dp^{2n})(1-adp^{2n}/b)}, \qquad (3.8.1)$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2 : -aq/w, q^{-n} \\ a, -a : w, -aq^{n+1} \end{matrix}; q^2, q; \frac{wq^{n-1}}{a}\right] = \frac{(-aq,aq^2/w,w/aq;q)_n}{(-q,aq/w,w;q)_n}, \qquad (3.10.5^{\dagger})$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, b^2 : -aq^n/b^2, q^{-n} \\ a, -a, a^2q^2/b^2 : b^2q^{1-n}, -aq^{n+1} \end{matrix}; q^2, q; q\right]$$
$$= \frac{(-aq,a/b^2;q)_n\,(1/b^2;q^2)_n}{(-q,1/b^2;q)_n\,(a^2q^2/b^2;q^2)_n}\,q^n, \qquad (3.10.7)$$

$$\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, b^2 : -aq^{n-1}/b^2, q^{-n} \\ a, -a, a^2q^2/b^2 : b^2q^{2-n}, -aq^{n+1} \end{matrix}; q^2, q; q^2\right]$$
$$= \frac{(-aq,a/qb^2;q)_n\,(aq/b^2,1/b^2q^2;q^2)_n}{(-q,1/qb^2;q)_n\,(a/qb^2,a^2q^2/b^2;q^2)_n}\,q^n, \qquad (3.10.8)$$

[†] The right hand side of the equation is a simplification of the original one.

$$
{}_4\phi_3\left[\begin{matrix} a, qa^{\frac{1}{2}}, -w/qa^{\frac{1}{2}}, q^{-n} \\ a^{\frac{1}{2}}, w, -a^{\frac{1}{2}}q^{1-n} \end{matrix}; q, q\right] = \frac{(w/aq, -a^{\frac{1}{2}}, aq^2/w; q)_n}{(-a^{-\frac{1}{2}}, aq/w, w; q)_n}, \tag{3.10.9‡}
$$

$$
\begin{aligned}
&\Phi\left[\begin{matrix} a^2, aq^2, -aq^2, b^2, c^2 : -aq/w, q^{-n} \\ a, -a, a^2q^2/b^2, a^2q^2/c^2 : w, -aq^{n+1} \end{matrix}; q^2, q; \frac{awq^{n+1}}{b^2c^2}\right] \\
&\quad = \frac{(-aq, aq^2/w, w/aq; q)_n}{(-q, aq/w, w; q)_n} \\
&\qquad \cdot {}_5\phi_4\left[\begin{matrix} aq, aq^2, a^2q^2/b^2c^2, a^2q^2/w^2, q^{-2n} \\ a^2q^2/b^2, a^2q^2/c^2, aq^{2-n}/w, aq^{3-n}/w \end{matrix}; q^2, q^2\right],
\end{aligned} \tag{3.10.14‡}
$$

$$
\begin{aligned}
&\Phi\left[\begin{matrix} a^2, -aq^2, b^2, c^2, d^2 : -\lambda q^{n+1}/a, q^{-n} \\ -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^2/d^2 : a^2q^{-n}/\lambda, -aq^{n+1} \end{matrix}; q^2, q; q\right] \\
&\quad = \frac{(-aq, \lambda q/a; q)_n\,(\lambda q^2/a^2; q^2)_n}{(-q, \lambda q/a^2; q)_n\,(\lambda q^2; q^2)_n} \\
&\qquad \cdot {}_{10}\phi_9\left[\begin{matrix} \lambda, q^2\lambda^{\frac{1}{2}}, -q^2\lambda^{\frac{1}{2}}, a, aq, \frac{\lambda b^2}{a^2}, \frac{\lambda c^2}{a^2}, \frac{\lambda d^2}{a^2}, \frac{\lambda^2 q^{2n+2}}{a^2}, q^{-2n} \\ \lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}}, \frac{\lambda q^2}{a}, \frac{\lambda q}{a}, \frac{a^2q^2}{b^2}, \frac{a^2q^2}{c^2}, \frac{a^2q^2}{d^2}, \frac{a^2q^{-2n}}{\lambda}, \lambda q^{2n+2} \end{matrix}; q^2, \frac{qa^2}{\lambda}\right],
\end{aligned} \tag{3.10.15}
$$

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1-n}, a, aq \\ qb^2, d, dq \end{matrix}; q^2, q^2\right] = \frac{(d/a; q)_n}{(d; q)_n}\, a^n\, {}_4\phi_2\left[\begin{matrix} q^{-n}, a, b, -b \\ b^2, aq^{1-n}/d \end{matrix}; q, -q/d\right], \tag{Ex. 3.4}
$$

$$
{}_5\psi_5\left[\begin{matrix} b, c, d, q^{n+1}/bcd, q^{-n} \\ q/b, q/c, q/d, bcdq^{-n}, q^{n+1} \end{matrix}; q, q\right] = \frac{(q, q/bc, q/bd, q/cd; q)_n}{(q/b, q/c, q/d, q/bcd; q)_n}, \tag{Ex. 5.18 (iii)}
$$

$$
\begin{aligned}
&{}_5\psi_5\left[\begin{matrix} b, c, d, q^{n+3}/bcd, q^{-n} \\ q^2/b, q^2/c, q^2/d, bcdq^{-n-1}, q^{n+2} \end{matrix}; q, q\right] \\
&\quad = \frac{(1-q)(q^2, q^2/bc, q^2/bd, q^2/cd; q)_n}{(q^2/b, q^2/c, q^2/d, q^2/bcd; q)_n},
\end{aligned} \tag{Ex. 5.18 (iv)}
$$

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}, c, -c \\ e, c^2q/e, -q \end{matrix}; q, q\right] \\
&\quad = \frac{(eq^{-n}, eq^{n+1}, c^2q^{1-n}/e, c^2q^{n+2}/e; q^2)_\infty}{(e, c^2q/e; q)_\infty}\, q^{n(n+1)/2},
\end{aligned} \tag{II.19}
$$

‡The right hand side of the equation is a simplification of the original one.

$$\left(1-\frac{a}{q}\right)\left(1-\frac{b}{q}\right)\sum_{k=0}^{n}\frac{(ap^k,bp^{-k};q)_{n-1}\,(1-ap^{2k}/b)}{(p;p)_k\,(p;p)_{n-k}\,(ap^k/b;p)_{n+1}}\,(-1)^k p^{\binom{k}{2}}=\delta_{n,0}, \tag{II.37}$$

$$ {}_2\phi_1(q^{-n},b;c;q,z)=\frac{(c/b;q)_n}{(c;q)_n}\;b^n\;{}_3\phi_1\left[\begin{matrix}q^{-n},b,q/z\\ bq^{1-n}/c\end{matrix};q,z/c\right]. \tag{III.8}$$

Gessel and Stanton [8]

$$\begin{aligned}\sum_{k=0}^{n}&\frac{(q^{-n},AD^2q^{n+1/2},A/D^2;q)_k\,(A,qA/C,Cq^{-1/2}/A;q^{1/2})_k}{(q,C,A^2q^{3/2}/C;q)_k\,(Aq^{n+1/2},q^{-n}/D^2,D^2q^{1/2};q^{1/2})_k}\\ &\cdot\frac{(1-Aq^{3k/2})}{(1-A)}\;q^{k/2}\\ &=\frac{(q^{1/2}/A;q^{1/2})_{2n}\,(CD^2/A,D^2Aq^{3/2}/C;q)_n}{(D^2q^{1/2};q^{1/2})_{2n}\,(C,A^2q^{3/2}/C;q)_n},\end{aligned} \tag{1.4}$$

$$ {}_6\phi_5\left[\begin{matrix}A,A^{1/2}q,-A^{1/2}q,qA/C,C/B,q^{-n}\\ A^{1/2},-A^{1/2},C,qAB/C,Aq^{n+1}\end{matrix};q,Bq^n\right]=\frac{(B,Aq;q)_n}{(C,qAB/C;q)_n}, \tag{4.7}$$

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(A;q)_n\,(C;q^{1/2})_{2n}}{(q,C^2;q)_n}\;q^{-n^2/2}(-x)^n\\ &=\frac{(Ax;q)_\infty}{(x;q)_\infty}\;\sum_{k=0}^{\infty}\frac{(A;q)_k\,q^{-\binom{k+1}{2}}(-xq^{1/2})^k}{(q^{1/2},-C;q^{1/2})_k\,(1-x/q)\cdots(1-x/q^k)},\end{aligned} \tag{5.13}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{(A,Bq^{-1/2},qA/B;q)_k\,(q^{-n/2};q^{1/2})_k\,(1-Aq^{3k/2})}{(Aq^{1+n/2};q)_k\,(q^{1/2},Bq^{-1/2},qA/B;q^{1/2})_k\,(1-A)}\;q^{[2nk-\binom{k}{2}]/2}\\ &=\begin{cases}0, & n\text{ odd},\\ \dfrac{(Aq,q^{1/2};q)_N}{(B,Aq^{3/2}/B;q)_N}, & n=2N\text{ even},\end{cases}\end{aligned} \tag{6.13}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{(A,qA/B,Bq^{-1/2};q)_k\,(q^{-n/2},qA/F,Fq^{(n-1)/2};q^{1/2})_k}{(Aq^{1+n/2},F,Aq^{(3-n)/2};q)_k\,(q^{1/2},Bq^{-1/2},qA/B;q^{1/2})_k}\\ &\cdot\frac{(1-Aq^{3k/2})}{(1-A)}\;q^{k/2}\end{aligned}$$

$$= \begin{cases} 0, & n \text{ odd}, \\ \dfrac{(Aq, Fq^{1/2}/B, BF/Aq, q^{1/2}; q)_N}{(F, Aq^{3/2}/B, B, Fq^{-1/2}/A; q)_N}, & n = 2N \text{ even}, \end{cases} \tag{6.14}$$

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k\,(Bq^{-1/3}, q^{2/3}/B; q^{1/3})_k}{(q^{1/3};q^{1/3})_{2k}\,(q^{-n};q^{1/3})_k}\, q^{k/3} = \frac{(B, q/B, q; q)_n}{(q^{1/3};q^{1/3})_{3n}}. \tag{6.20}$$

Slater [19]

$${}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-N} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+N} \end{matrix}; q, aq^{1+N}/bc\right] = \frac{(aq, aq/bc; q)_N}{(aq/b, aq/c; q)_N}. \qquad \text{(3.3.1.4), (IV.9)}$$

$${}_5\phi_4\left[\begin{matrix} a, b, c, d, q^{-N} \\ aq/b, aq/c, aq/d, a^2q^{-N}/k^2 \end{matrix}; q, q\right] = \frac{(kq/a, k^2q/a; q)_N}{(kq, k^2q/a^2; q)_N}$$
$$\cdot\ {}_{12}\phi_{11}\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, \sqrt{a}, -\sqrt{a}, \sqrt{aq}, \\ \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k\sqrt{q/a}, -k\sqrt{q/a}, \end{matrix}\right.$$
$$\left.\begin{matrix} -\sqrt{aq}, k^2q^{N+1}/a, q^{-N} \\ kq/\sqrt{a}, -kq/\sqrt{a}, aq^{-N}/k, kq^{N+1} \end{matrix}; q, q\right], \tag{3.4.1.6}$$

$$(a;q)_{-n} = \frac{(a;q)_\infty}{(aq^{-n};q)_\infty}, \qquad \text{(7.1.2), (II.8)}$$

$${}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, a^2q/bcde\right]$$
$$= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, a^2q/bcde; q)_\infty}, \tag{7.1.1.1}$$

$$\sum_{r=-[n/3]}^{[n/3]} \frac{(1-q^{6r+1})(-1)^r (e;q^3)_r\, q^{r(9r+1)/2}}{(q;q)_{n+3r+1}\,(q;q)_{n-3r}\,(q^4/e;q^3)_r\, e^r} = \frac{(q^2/e;q^3)_n}{(q;q)_{2n}\,(q^2/e;q)_n}, \tag{7.3.1.2}$$

$${}_4\phi_3\left[\begin{matrix} a, -q\sqrt{a}, b, q^{-n} \\ -\sqrt{a}, aq/b, aq^{1+n} \end{matrix}; q, q^{1+n}\sqrt{a}/b\right] = \frac{(aq, q\sqrt{a}/b; q)_n}{(q\sqrt{a}, aq/b; q)_n}. \tag{IV.5}$$

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