2. Sets, Functions, Domains

Sets

Collections of elements.

- Enumeration of elements

  $\{1, \{1, 4, 7\}, 4\}, \{\textit{red}, \textit{yellow}\}, \{\}$

- Defining property

  $\{x | P(x)\}$
  $\{x |\ x \textit{ is an even integer}\}$

Examples

- Natural numbers **N**

  $\{0, 1, 2, \ldots\}$

- Truth values (Booleans) **B**

  $\{\textit{true}, \textit{false}\}$

- Rational numbers **Q**

  $\{x | x = p/q \textit{ for some } p, q \in \mathbf{N}, q \neq 0\}$

Set Predicates

Based on concept of *membership*

- Membership $x \in S$

  Only basic predicate
  (sets are black boxes otherwise)

- Equivalence $R = S$

  $x \in R \Leftrightarrow x \in S$ *(for all* x*)*
  (extensionality principle)

- Subset $R \subseteq S$

  $x \in R \Rightarrow x \in S$ *(for all* x*)*
  $(\{\} \subseteq S,\ S \subseteq S)$

Set Constructions

Composition of sets

- Union $R \cup S$

  $\{x | x \in R \text{ or } x \in S\}$
  $\cup_i S_i = S_{j_1} \cup S_{j_2} \cup \ldots \cup S_{j_n}$

- Intersection $R \cap S$

  $\{x | x \in R \text{ and } x \in S\}$
  $\cap_i S_i = S_{j_1} \cap S_{j_2} \cap \ldots \cap S_{j_n}$

- Powerset $\mathbf{P}(R)$

  $\{x | x \subseteq R\}$
  $(\{\} \in \mathbf{P}(R),\ R \in \mathbf{P}(R))$

Product

Concept of ordered pair

- Constructor (x, y)
- Selectors

  $fst(x,y) = x$
  $snd(x,y) = y$

- Product $R \times S$

  $\{(x, y) | x \in R \text{ and } x \in S\}$

Sum

Concept of disjoint union

- Sum $R + S$

  $\{(\mathit{zero}, x) | x \in R\} \cup \{(\mathit{one}, y) | y \in S\}$

  "tags" to preserve origin of element

- Constructors

  $\mathit{inR}(x) = (\mathit{zero}, x)$ *(for* $x \in R$*)*
  $\mathit{inS}(x) = (\mathit{one}, y)$ *(for* $y \in S$*)*

- Selector

  cases m *of*
  $\mathit{isR}(x) \rightarrow \boxed{\ldots x \ldots}$
  $\mathit{isS}(y) \rightarrow \boxed{\ldots y \ldots}$
  end
  $m = (\mathit{zero}, x) \Rightarrow \boxed{\ldots x \ldots}$
  $m = (\mathit{zero}, y) \Rightarrow \boxed{\ldots y \ldots}$

Functions

Black box that accepts object as input and produces another object as output

Definition in terms of sets

- $f : R \to S$

  f *is function from* R *to* S
  R *domain of* f, S *codomain of* f
  $R \to S$ *arity (functionality) of* f

- Application $f(a)$

  $a \in R$, $f(a) \in S$

- Equality $f = g$

  $f, g : R \to S$ $f(x) = g(x)$ *(for all* x*)*
  (extensionality principle)

- Composition $f \circ g$

  $f : R \to S, g : S \to T$
  $f \circ g : R \to T$
  $(f \circ g)(x) = g(f(x))$

Classification of Mappings

- Injective (one-one, 1-1)

  $f(x) = f(y) \Rightarrow x = y$ *(for all* $x, y \in R$*)*

- Surjective (onto)

  for every $y \in S$ *there is some* $x \in R$ *such that* $f(x) \in y$

- Identity function

  $f : R \to R$
  $f(x) = x$ *(for all* $x \in R$*)*

- Inverse function

  $f : R \to S$ *injective and surjective*
  $g : S \to R, g(y) = x \Leftrightarrow f(x) = y$
  $g = f^{-1}$

Isomorphism

Relationship between sets defined by functions

R and S are isomorphic if there is a pair of functions

$$f : R \to S$$
$$g : S \to R$$
$f \circ g$ is identity on R
$g \circ f$ is identity on S

f and g are then called *isomorphisms*.

Examples

- $R = \{1, 4, 7\}$ is isomorphic to $S = \{2, 4, 6\}$
- $A \times B$ is isomorphic to $B \times A$
- **N** is isomorphic to **Z**

Functions as Sets

Every function $f : R \to S$ can be represented by its *graph*:

$$\begin{aligned} \mathit{graph}(f) &= \{(x, f(x)) | x \in R\} \\ &\subseteq R \times S \end{aligned}$$

Successor function on **Z**
$\{\ldots, (-2,-1), (-1,0), (0,1), (1,2), \ldots\}$

- Function application

  $f(a) = b \Leftrightarrow (a,b) \in \mathit{graph}(f)$
  $f(a) := \mathit{apply}(\mathit{graph}(f), a)$

- Function composition

  $\mathit{graph}(f \circ g) =$
  $\{(x,z) |\ x \in R$ *and, for some* $y \in S$,
  $(x,y) \in \mathit{graph}(f)$ *and*
  $(y,z) \in \mathit{graph}(g)\}$

Examples

- *add* : $(\mathbf{N} \times \mathbf{N}) \to \mathbf{N}$

  $\{((0,0),0),((1,0),1),((0,1),1),$
  $((1,1),2),((2,0),2),((2,1),3),$
  $((2,2),4),\ldots\}$

- *duplicate* : $R \to R \times R, = \{1,4,7\}$

  $\{(1,(1,1)),(4,(4,4)),(7,(7,7))\}$

- *which* : $(\mathbf{B} + \mathbf{N}) \to \{\mathsf{isbool}, \mathsf{isnum}\}$

  $\{((\mathit{zero}, \mathit{true}), \mathit{isbool}),$
  $((\mathit{zero}, \mathit{false}), \mathit{isbool}),$
  $((\mathit{one}, 0), \mathit{isnum}), ((\mathit{one}, 1), \mathit{isnum}),$
  $((\mathit{one}, 2), \mathit{isnum}), \ldots\}$

- *singleton* : $\mathbf{N} \to \mathbf{P}(\mathbf{N})$

  $\{(0,\{0\}),(1,\{1\}),\ldots,(n,\{n\}),\ldots\}$

- nothing : $\mathsf{B} \cap \mathsf{N} \to \mathsf{B}$

  $\{\}$

Examples

- *split-add* : $\mathbf{N} \to (\mathbf{N} \to \mathbf{N})$

$$\{ (0, \underline{\{(0,0),(1,1),(2,2),\ldots\}}),$$
$$(1, \underline{\{(0,1),(1,2),(2,3),\ldots\}}),$$
$$(2, \underline{\{(0,2),(1,3),(2,4),\ldots\}}),$$
$$\ldots,$$
$$(n, \underline{\{(0,n),(1,n+1),(2,n+2),\ldots\}}),$$
$$\ldots\}$$

- *make-succ* : $(\mathbf{N} \to \mathbf{N}) \to (\mathbf{N} \to \mathbf{N})$

$$\{\ldots,$$
$$(\underline{\{(0,1),(1,1),(2,1),(3,6),\ldots\}},$$
$$\underline{\{(0,2),(1,2),(2,2),(3,7),\ldots\}}),\ldots\}$$

- *apply* : $(((\mathbf{N} \to \mathbf{N})) \times \mathbf{N}) \to \mathbf{N}$

$$\{\ldots,$$
$$((\underline{\{(0,1),(1,1),(2,1),(3,6),\ldots\}},0),1),$$
$$((\underline{\{(0,1),(1,1),(2,1),(3,6),\ldots\}},1),1),$$
$$((\underline{\{(0,1),(1,1),(2,1),(3,6),\ldots\}},2),1),$$
$$((\underline{\{(0,1),(1,1),(2,1),(3,6),\ldots\}},3),6),$$
$$\ldots\}$$

Functions as Equations

More convenient form of specification

- *add* : $(\mathbf{N} \times \mathbf{N}) \rightarrow \mathbf{N}$

  $add(m, n) = m + n$

- *duplicate* : $R \rightarrow R \times R, = \{1, 4, 7\}$

  $duplicate(r) = (r, r)$

- *which* : $(\mathbf{B} + \mathbf{N}) \rightarrow \{\text{isbool}, \text{isnum}\}$

  $which(m) =$ *cases* m *of*
  $isB(b) \rightarrow$ *isbool*
  $isN(n) \rightarrow$ *isnum*
  end

- *singleton* : $\mathbf{N} \rightarrow \mathbf{P}(\mathbf{N})$

  $singleton(n) = \{n\}$

- nothing : $B \cap N \rightarrow B$

  no equational definition (domain empty)!

Equations are just function representations!

Evaluation of Equations

- Definition $f : A \to B, f(x) = \alpha$
- Application $f(a_0)$
  1. *Substitution* $[a_0/x]\alpha$
  2. *Simplification to underlying value*

Lambda Notation $f = \lambda x.\alpha$

split-add$(x) = \lambda y.x + y$
slit-add $= \lambda x.\lambda y.x + y$

Updating Functions $[a_0 \mapsto b_0]f$

$([a_0 \mapsto b_0]f)(a_0) = b_0$
$([a_0 \mapsto b_0]f)(a) = f(a)$, *for all* $a \neq a_0$

Semantic Domains

Those sets that are used as value spaces in denotational semantics.

Primitive domains

N, **Z**, **B**, ...

Compound domains

- *Product domains* $A \times B$
- *Sum domains* $A + B$
- *Function domains* $A \to B$
- *Lifted domains* $A_\perp = A \cup \{\perp\}$
  - $\perp$ = *"bottom"*
  - *Non-termination, no value at all*
  - *Strict functions* $f : A_\perp =\!\!\to B_\perp, f = \underline{\lambda} x.\alpha$

    $f(\perp) = \perp$

    $f(a) = [a/x]\alpha$, *for* $a \in A$

Semantic Algebras

Format for presenting semantic domains

Rational Numbers

Domain $Rat = (\mathbf{Z} \times \mathbf{Z})_\perp$

Operations

$makerat: \mathbf{Z} \to (\mathbf{Z} \to Rat)$
$makerat = \lambda p.\lambda q.(q = 0) \to \perp \; [] \; (p, q)$

$addrat: Rat \to (Rat \to Rat)$
$addrat = \underline{\lambda}(p_1, q_1).\underline{\lambda}(p_2, q_2).((p_1 * q_2) + (p_2 * q_1), q1 * q2)$

$multrat: Rat \to (Rat \to Rat)$
$multrat = \underline{\lambda}(p_1, q_1).\underline{\lambda}(p_2, q_2).(p_1 * p_2, q_1 * q_2)$

Choice function $e_1 \to e_2 \; [] \; e_3$
$\Rightarrow e_2$, *if* $e_1 =$ *true*
$\Rightarrow e_3$, *if* $e_1 =$ *false*