HySaX Project — FWF P11160-TEC
Progress Report for the Year 1997

Franz Winkler and Quoc-Nam Tran
Institut für Mathematik and
Research Institute for Symbolic Computation (RISC)
Johannes Kepler Universität Linz, Austria

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In 1997, we continue our research in both theoretical and practical aspects as proposed in the proposal of the project. In the following we describe shortly the main results obtained in this first year of the project. Further information on the project HySaX can be obtained from the web page
`http://www.risc.uni-linz.ac.at/projects/basic/hysax`

<u>Finding a Real Solution of an Arbitrary System of Nonlinear Algebraic Equations</u>

We are interested in methods for finding a real solution of an arbitrary system of nonlinear algebraic equations

$$F : R^h \to R^k,$$

where the system can be overdetermined or underdetermined and its Jacobian matrix can be of any (positive) rank. The system F may have infinitely many solutions, i.e., the manifold or algebraic set $\{x : F(x) = 0\}$ is of positive dimension.

The problem arises in several contexts. For example, in computer aided geometric design (CAGD), visualization and solid modeling, approximation and representation of the intersection of surfaces is a fundamental and difficult problem. In order to trace the intersection, one needs to find a real point on the intersection as a starting point. Moreover, the intersection is usually a composition of several separated components. Therefore, one needs to find a real point on each component of the intersection. The problem of finding the starting points is difficult if not open (see [BHLH88, BK90, PP90, KP91b, KPPW92]). The problem also arises in approximation of solutions of nonlinear boundary-value problems for nonlinear ordinary or partial differential equations by finite differences (see [Rab70]).

Remember that when $F(x)$ and x are vectors of the same dimension (i.e.,

$h = k$), and the Jacobian matrix of the system is nonsingular everywhere, Newton's method (see for example [IK66, Atk78]) or inexact Newton's method (see for example [DES82, EW96]) are well-known methods for finding a real solution of the system.

Unfortunately, when $F(x)$ and x have different dimensions, Newton's methods cannot be applied because the Jacobian matrix is singular or even nonsquare. Moreover, with Newton's methods or any other traditional methods for finding a real solution of a system of nonlinear algebraic equations, we cannot find a solution for each irreducible component of the solution set corresponding to the irreducible decomposition of the solution set.

In [Tran98b], we propose a hybrid symbolic-numeric method which can be applied to these cases as well. The key ideas of our method are: (1) We use symbolic techniques to regularize an arbitrary system of nonlinear algebraic equations obtaining a set of regular systems of algebraic equations whose solution set is corresponding to an irreducible component of the solution set of the original system. (2) We then use Newton-like techniques to find a real solution for the regular systems obtained from the symbolic part of the method.

The main contributions of the new method are: first, it can be used to find a real solution of an arbitrary system of nonlinear algebraic equations, where the system can be overdetermined or underdetermined and its Jacobian matrix can be of any (positive) rank; second, it can be used to find a real solution for each irreducible component of the solution set of the original system F corresponding to the irreducible decomposition of the solution set, i.e., the radical decomposition of the ideal generated by F; and finally, the method treats overdetermined systems or underdetermined systems in the same manner – we do not need to separate them.

We consider some mild conditions for the asymptotic convergence of the numerical part of our method. The asymptotic convergence of the numerical part of our method is still quadratic whereas that of, for example the traditional method of steepest descent where minimization of $\|F\|^2$ is used, is linear. Actually, in practice it has been found that the convergence of the method of steepest descent is often intolerably slow (see [Bro70]). Besides of that, in the new method we do not need to augment the system of equations or its Jacobian matrix. In order to enhance the robustness of the numerical part of the method, techniques like damping can be used as in the regular case (see [EW94]).

Extending Dixon's Resultant and Its Applications

Solving problems from geometric reasoning, CAGD, solid modeling, etc., one often has to eliminate several variables from a system of equations (see [Cho88], [Far88], [Hof89], [HWW95] and [PW95]). The Gröbner bases method [Buc85], Ritt-Wu's characteristic sets method [Rit50, Wu94] and successive Sylvester resultant computations [vdW40] can be used to achieve the demands. However, they are not efficient enough and require a lot of memory space for some large problems.

Meanwhile, Dixon's resultant method is an efficient way to simultaneously eliminate several variables from a system of nonlinear polynomial equations at a time. In contrast with successive resultant computation techniques which eliminate variables one by one and compute numerous intermediate resultants before finally computing the resultant of the whole set, Dixon's resultant method creates a much smaller matrix and avoids such intermediate results, and hence is much faster and also saves memory space. By Dixon's resultant we mean a nonzero multiple of the standard resultant because it provides a solvability criterion for the existence of solutions other than certain known solutions. (A standard resultant furnishes a solvability criterion for the existence of any solutions.)

Unfortunately, Dixon's resultant method does not work for most algebraic and geometric problems. Originally, Dixon's resultant method was for obtaining the resultant of a set of three generic bidegree polynomials (see [Dix08]). This is a generalization of Cayley's formulation [Cay65] of Bezout's efficient method for computing the resultant of two univariate polynomials. Dixon had generalized his method to any $n+1$ generic n-degree polynomials in n variables – a condition which cannot be satisfied by most problems arising in practice. For most practical problems, the matrix set up in Dixon's method – the Dixon matrix – becomes singular or even nonsquare. As a consequence, Dixon's resultant vanishes identically, without providing any information about the common solutions of equations. There are some efforts to obtain nonzero conditions by perturbing the polynomial system (see [Chi90]). However, these efforts suffer from the facts that the method is not automatic, requires human expertise and enlarges the size of Dixon matrix with larger entries. Another approach of Kapur et al. (see [KSY94], [KS95]) is based on the idea of checking if there exists a submatrix of the Dixon matrix with one column less and strictly smaller rank than the Dixon matrix when the Dixon matrix is square and singular. However, this approach requires a precondition on the system of equations.

In [Tran97a], by using techniques from pseudoinverse theory and linear transformations, the author overcomes the above obstacles and extends Dixon's resultant method to an arbitrary system of $n+1$ nonlinear polynomials in n variables where the Dixon matrix can be singular or even nonsquare. Our method does not require any precondition – this is the main contribution of the paper. Several examples show that the new method works well also in situations where other methods may fail to give a correct answer. Note that this approach can also be used to extend other multivariate resultant methods such as Macaulay resultant or sparse resultant. However, extracting the extraneous factors from Dixon's resultants is a difficult task and still needs further investigations.

NURBS Interpolation of Surface-to-Surface Intersections

In [TrVa97b], we describe a hybrid symbolic-numerical algorithm for piecewise interpolation and representation of surface-to-surface intersections – a fundamental and difficult problem in Computer Aided Geometric Design (CAGD) and solid modeling. Given two trivariate polynomials with rational number coefficients, the algorithm produces a space graph which is a non-uniform rational B-spline (NURBS) piecewise interpolation and homeomorphic to the real algebraic set of the intersection (in fact, a space curve) in Euclidean space. Due to its importance, there have been persistent efforts in devising algorithms for this problem (see [BFJP87], [BHLH88], [ACM88], [Hof89], [KPP90], [W$^+$91], [KP91a], [Pat92, Pat93], [Tau94] and [Tra95]). The main issue in the intersection problem is the efficient discovery and description of all features of the solution with high precision commensurate with the tasks required by the underlying geometric modeler. Reliability and efficiency of intersection algorithms are two basic prerequisites for their effective use in any geometric modeling system. It is closely associated with the way the algorithm handles such features as near singular cases, small loops, etc.

In contrast to symbolic methods our method avoids the use of expensive computations such as algebraic set decomposition, computation of birational mappings, projections, reconstructions, etc. while still ensuring the reliability of the method especially near the singularities. The method requires only rational arithmetic such as resultants, greatest common divisors, etc.

On the other hand, our algorithm overcomes the main difficulties of numerical tracing, namely finding a sample point for every real branch and tracing through singularities by using techniques for real root isolation of polynomials with real algebraic number coefficients.

A numerical tracing scheme technically requires formulating and solving a linear system of equations. Such a system is formulated both for determining an approximation to the curve at a given point, as well as when refining an estimate of the location of a point on the curve with Newton iterations. At a curve singularity, the linear system is singular, and nearby the condition number of the system is so large that roundoff errors destroy all accuracy of the estimate. To overcome this difficulty, we use a new technique of the first author's, namely the "Extended Newton Method" for determining the roots of an arbitrary system of equations iteratively, where the equations can be nonlinear algebraic or transcendental, the number of equations of the system can be greater than, less than or equal to the number of variables (see [Tra94, Tra97]).

Furthermore, in order to improve the efficiency of the algorithms and the smoothness of the plots, we utilize NURBS to piecewise interpolate the intersection. We use NURBS because it provides a concise formulation for curves that are composed of many polynomial pieces including conic sections and rational Bézier curves, thereby automatically controlling the overall curve smoothness. In addition, the algorithms are fast, numerically stable and invariant under common affine and projective transformations.

The symbolic tools are implemented in the software system CASA, while the NURBS tools are implemented in the software system CGAL and will become a CGAL application module. The software system CASA has been developed at RISC-Linz over the last seven years. Meanwhile, RISC-Linz is one of the seven academic developing sites of the CGAL library. Our method has been implemented as an interface between those two systems. It calls symbolic or numerical procedures when needed.

Experiments from examples found in the literature show a considerable speed up of our method in comparison with symbolic methods. Meanwhile, the method still works well near the singularities which is much better in comparison with numerical methods.

<u>Converting Bases with the Gröbner Walk</u>

Basis conversion arises in many parts of computational mathematics and computer science such as solving algebraic equations, implicitization of algebraic sets, elimination theory, etc. In [Tran98c], we experimented with parallelizing the Gröbner walk method of Collart et al. to convert a given Gröbnerr basis of a multivariate polynomial ideal of arbitrary dimension into a Gröbnerr basis of the ideal with respect to another term order. This efficient method is a very useful method in CAGD for solving algebraic equations, implicitization of varieties, etc. Our experiments show that our parallel implementation achieves a speed-up of up to six over a sequential implementation, which in turn achieves a speed-up of up to ten over the direct application of Buchberger's algorithm.

<u>Development of the program system CASA</u>

A new version – version 2.3 – of the program system CASA was released in September 1997. Several of the theoretical achievements mentioned above were incorporated in the new version. Among them are:

1. Parametrization over an optimal field of extension. 2. Hybid symbolic-numerical methods for finding (real) solutions of an arbitrary system of non-linear equations. 3. Hybid symbolic-numerical methods for real root isolation. 4. Classical multivariate resultant method. 5. Dixon's multivariate resultant method. 6. Gröbner Walk method. 7. Scientific visualization tools for a. Plotting plane algebraic curves; b. Plotting the intersection of two surfaces. 8. Puiseux series expansion. 9. Computation of rational points on conics. 10. Offset curves.

Additionally, the help feature of CASA has been totally redesigned in order to match with features of the new release of the language Maple. This new version of CASA provides online help pages with hyperlinks. Just click on a highlighted item to pull up its subsequent help page.

A paper [TrWi97] for giving an overview of the system CASA has been published on the Symbolic and Algebraic Computation (SAC) Newsletter in June 1997.

The CASA system is used by many colleagues around the world. More than 100 people have downloaded CASA and probably installed it at their sites. In particular, we know that the system is used at ETH Zürich, universities in Madrid and Santander, TU München, and several other universities in Europe, USA and Australia.

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