Basic Logic and Primitive Recursive Realizability

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Intuitionistic semantics (Brouwer, Kolmogorov, Heyting)

A sentence A is true iff it has a verification ("a proof").

$A \to B$ is true iff there exists an effective operation which allows to get a justification of A from the justification of B.

$\forall x\, A(x)$ is true iff there exists an effective operation which allows to get a justification of the statement $A(n)$ for every n.

Recursive realizability (Kleene)
Constructive logic (Markov)

Verifications are coded by natural numbers:

a justification = a natural number;

an effective operation = a partial recursive function.

e realizes $A \to B$ iff e is the Gödel numder of a partial recursive function which maps every realization of A to a realization of B.

Thus verifications of $A \to B$ are (the codes of) programs.

e realizes $\forall x\, A(x)$ iff e is the Gödel number of a recursive function mapping every natural number n to a realization of $A(n)$.

The soundness theorem (D. Nelson, 1947): *every formula provable in Intuitionistic Arithmetic is realizable*.

The incompleteness phenomenon (G. Rose, 1953): *there exist realizable propositional formulas which are not deducible in Intuitionistic Propositional Calculus (IPC)*.

Primitive recursive realizability

Effective operation = primitive recursive function.

An obstacle: there is no universal primitive recursive function. Thus application of the code of a primitive recursive function to an argument is not a primitive recursive operation.

Intuitionistic logic is not sound with a primitive recursive analogue of Kleene's realizability (Varpakhovskii, private communication).

Negationless arithmetic is sound with primitive recursive realizability (E. G. K. Lopez-Escobar, 1974).

Indexing of the primitive recursive functions.

A version of the Grzegorczyk's hierarchy of the primitive recursive functions by Axt (1963): classes $\mathcal{E}^n$.

$$\bigcup_{n=0}^{\infty} \mathcal{E}^n = \text{the primitive recursive functions}$$

$\mathrm{e}_{n+1} \in \mathcal{E}^{n+1}$ — a universal function for the class $\mathcal{E}^n$: if $f(x_1, \ldots, x_n)$ is in $\mathcal{E}^n$, then there exists b (an n-index of f) such that

$$\mathrm{e}_{n+1}(b, \langle x_1, \ldots, x_n \rangle) = f(x_1, \ldots, x_n).$$

$In(n, b)$ means that b is an index of a function in $\mathcal{E}^n$.

Strictly primitive recursive realizability by Z. Damnjanovic (1994)

The relation $t \vdash_n A$.

If A is atomic, then $t \vdash_n A$ iff $t = 0$ and A is true.

$t \vdash_n (B \rightarrow C)$ iff $In(n,t)$ and
$(\forall j \geq n) In(j, \mathsf{e}_{j+1}(t,j))$ and
$(\forall j \geq n) \forall b [b \vdash_j B \Rightarrow \mathsf{e}_{j+1}(\mathsf{e}_{j+1}(t,j), b) \vdash_j C]$.

$t \vdash_n \forall x B(x)$ iff $In(n,t)$ and
$\forall m\, \mathsf{e}_{n+1}(t,m) \vdash_n B(m)$.

A closed arithmetic formula A is *strictly primitive recursively realizable* iff $t \vdash_n A$ for some t and n.

Strictly primitive recursive realizability and Intuitionistic Logic

Theorem 1 (Plisko, 2006) *There exists a closed arithmetic formula which is deducible in Intuitionistic Predicate Calculus with identity but is not strictly primitive recursively realizable.*

Proof. The ternary predicate

$$e_{x+1}([y]_0, [y]_1) = z$$

is expressed by a Σ^0_1-formula $\exists u A(x, y, z, u)$, the formula $A(x, y, z, u)$ being atomic. Let $B(x, y, z)$ be $\neg\neg\exists u A(x, y, z, u)$. We define a formula Φ as

$$\forall x(\forall y(0 = 0 \to \exists z B(x, y, z)) \to \forall y \exists z B(x, y, z)).$$

Φ is deducible in Intuitionistic Predicate Calculus with identity but is not strictly primitive recursively realizable.

Basic Propositional Logic (BPL) was introduced by Visser in 1981 as a solution of the equation

$$\frac{X}{K4} = \frac{IPC}{S4}$$

relative to the Göedel relation between propositional and modal logics:

$$IPC \vdash A \Leftrightarrow S4 \vdash A^{\Box}.$$

Ruitenburg introduced Basic Predicate Calculus (BQC) in 1991 and Basic Arithmetic in 1998.

BQC is a subsystem of Intuitionistic Predicate Calculus (IQC) characterized by the Kripke frames with transitive (but not necesserily reflexive) accessibility relations.

Modus ponens does not hold for the relation $\Gamma \vdash_{BQC} A$ with non-empty Γ.

The formula $(\top \to A) \to A$ is not deducible in BPL.

Primitive recursive realizability by Salehi (2000)

The relation $t\,\mathbf{r}^{PR}A$, where t is a natural number, A is a closed formula of the language of Basic Arithmetic.

If A is atomic, then $t\,\mathbf{r}^{PR}A$ iff A is true.

$t\,\mathbf{r}^{PR}\forall\mathbf{x}(B(\mathbf{x}) \to C(\mathbf{x}))$ iff t is the index of a unary primitive recursive function ψ_t and

$$\forall b, \mathbf{x}(y\,\mathbf{r}^{PR}B(\mathbf{x}) \Rightarrow \psi_t(\langle y, \mathbf{x}\rangle)\,\mathbf{r}^{PR}C(\mathbf{x})).$$

A closed arithmetic formula A is primitive recursively realizable iff $t\,\mathbf{r}^{PR}A$ for some t.

Every formula deducible in Basic Arithmetic is primitive recursively realizable (Salehi, 2000).

Theorem 2 (Plisko, 2006) *There exists a closed arithmetic formula which is strictly primitive recursively realizable but is not primitive recursively realizable by Salehi.*

Proof. The ternary predicate

$$\mathrm{e}_{x+1}([y]_0, [y]_1) = z.$$

is expressed by a Σ^0_1-formula $\exists u A(x, y, z, u)$ with atomic $A(x, y, z, u)$. Let $B(x, y, z)$ be

$$\neg\neg\exists u A(x, y, z, u).$$

The binary predicate $\mathrm{e}_x(x, x) = y$ is expressed by a Σ^0_1-formula $\exists v C(x, y, v)$ with atomic $C(x, y, v)$. Let $D(x, y)$ be $\neg\neg\exists v C(x, y, v)$.

The formula

$$\forall x(\forall y(0 = 0 \to \exists z B(x, y, z)) \to \exists y D(x, y))$$

is strictly primitive recursively realizable but is not primitive recursively realizable by Salehi.

Strictly primitive recursive realizability and Basic Logic

Theorem 3 (Plisko, 2007) *Every closed arithmetic formula deducible in the Basic Predicate Calculus is strictly primitive recursively realizable.*