PLANE PARTITIONS VI: STEMBRIDGE'S TSPP THEOREM

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Dedicated to the memory of David Robbins

Abstract. We provide a new proof of Stembridge's theorem which validated the Totally Symmetric Plane Partitions (TSPP) Conjecture. The overall strategy of our proof follows the same general pattern of determinant evaluation as discussed by the first named author in a series of papers. The resulting hypergeometric multiple sum identities turn out to be quite complicated. Their correctness is proved by applying new algorithmic methods from symbolic summation.

1. Introduction

The object of this paper is to provide a new proof of John Stembridge's theorem [Ste95] which validated the Totally Symmetric Plane Partitions (TSPP) Conjecture. We begin with a description of the objects in question.

A plane partition π is an array of nonnegative integers $(\pi_{ij})_{i\geq 1, j\geq 1}$ wherein $\pi_{ij} \geq \pi_{i'j'}$ if $i \geq i'$ and $j \geq j'$. We say this is a plane partition of n if

$$n = |\pi| := \sum_{i,j\geq 1} \pi_{ij}.$$

Obviously this means that all but finitely many of the π_{ij} are zero.

Often a plane partition is displayed in the fourth quadrant (rather than the expected first quadrant). So if $\pi_{1,1} = \pi_{1,2} = 4$, $\pi_{1,3} = 3$, $\pi_{1,4} = 1$, $\pi_{2,1} = 3$, $\pi_{2,2} = \pi_{2,3} = 2$, $\pi_{3,1} = \pi_{3,2} = 1$ and all other $\pi_{ij} = 0$, then π is represented by the diagram

$$\begin{matrix} 4 & 4 & 3 & 1 \\ 3 & 2 & 2 & \\ 1 & 1 & & \end{matrix}$$

and π is a plane partition of 21 with 3 rows and 4 columns.

A plane partition is called *symmetric* if $\pi_{ij} = \pi_{ji}$. A plane partition is called *cyclically symmetric* if the i^{th} row π (considered as an ordinary partition) is the conjugate of the i^{th} column of π.

A plane partition is called *totally symmetric* if it is both symmetric and cyclically symmetric.

In [Sta86a], R. Stanley provides an account of many conjectures on plane partitions (many of which have been settled); of these, we are interested in

Conjecture 1 (cf. [Sta86b, Case 4])**.** *Let T_n equal the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$*

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}. \tag{1}$$

The first author was partially supported by National Science Foundation Grant DMS.

The second author was partially supported by the SFB-grant F1305 of the Austrian FWF.

The third author was supported by the Austrian Academy of Sciences and by grant P16613-N12 of the Austrian FWF.

This assertion is now known as Stembridge's theorem: Stembridge's proof [Ste95] combines a variety of masterful steps involving the combinatorics of Pfaffians and reduction of such to known determinant representations from which (1) follows.

Prior to Stembridge's work, S. Okada had obtained an evaluation of T_n which is easily seen to be equivalent to the following:

Okada's Theorem ([Oka89, Sec. 4])**.** For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases} \tag{2}$$

where

$$M(n) = \big(\mu_1(i,j)\big)_{0\leq i,j\leq n-1}, \tag{3}$$

$$\mu(i,j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0,\ i < j, \\ (-1)^{j-i-1} + \displaystyle\sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases} \tag{4}$$

and

$$\mu_1(i,j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0,\ j > 0, \\ (-1)^i & \text{if } j = 0,\ i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1,j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1,i-1) & \text{if } 1 \leq j < i. \end{cases} \tag{5}$$

In this paper, we shall deduce Stembridge's theorem by exhibiting an upper triangular matrix $W(n)$ with 1's on the main diagonal for which we prove that

$$M(n)W(n) = \begin{pmatrix} A_{00} & 0 & 0 & \dots & 0 \\ A_{10} & A_{11} & 0 & \dots & 0 \\ A_{20} & A_{21} & A_{22} & \dots & 0 \\ \vdots & & & \ddots & \\ A_{n-1,0} & A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n-1} \end{pmatrix} \tag{6}$$

is a lower triangular matrix with the A_{ij} explicitly determined. From the form of the A_{ii} and the immediate fact that

$$\det\big(M(n)\big) = A_{00}A_{11}\dots A_{n-1,n-1}, \tag{7}$$

we obtain Stembridge's theorem.

Remark: We remark that Okada's Theorem 5 [Oka89] is a much more simply stated version of the above result. However, to our surprise our techniques were unsuccessful in studying this apparently simpler formulation of the problem.

The entries of the matrix $W(n)$ will be presented in the next section. They are complicated to say the least. This then naturally suggests the question: Why provide a new proof of Stembridge's theorem under these circumstances?

We believe there are two compelling reasons. First, the five previous entries [And79a, And77, And79b, And87, And94] in this series have been devoted to determinant evaluations precisely along the lines of (6), and it is of interest to see that Stembridge's theorem fits this general pattern. Second, the proof of (6) requires the development of really new results concerning Zeilberger's telescoping and its extensions. The full treatment of these discoveries will be presented in [Sch04a] and [PS04]. Consequently the effort to prove (6) has yielded substantial improvement in the methods of symbolic summation.

The remainder of this paper is structured as follows. Section 2 introduces notation and basic definitions together with Theorem 1 which implies Stembridge's theorem as a corollary. Section 3 presents all the identities we need to verify in order to prove Theorem 1. In Section 4 we briefly describe the general proof method which we used to prove all the identities in question. The major steps of these proofs are made explicit in Section 5 in the form of proof certificates. In Section 6 concluding remarks are made.

2. Notation and Definitions

We begin with the standard rising factorial

$$(x)_n = \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)\dots(x+n-1) & \text{if } n > 0, \end{cases}$$

and the binomial coefficient

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}, \qquad \text{for } n \geq 0.$$

The remaining symbols are all specific to this paper and are not to be confused with other meanings in other settings:

$$\begin{aligned} \left\{ \begin{matrix} x \\ n \end{matrix} \right\} &= \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n} \right); \\ t_1(n) &= \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases} \\ t(n) &= \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases} \end{aligned}$$

We remark in passing that if the right-hand side of (1) is denoted by τ_n, then for $n \geq 1$

$$\frac{\tau_n}{\tau_{n-1}} = \prod_{1 \leq i \leq j \leq n} \frac{i+j+n-1}{i+j+n-2} = \prod_{j=1}^{n} \frac{2j+n-1}{j+n-1} = t_1(n).$$

Therefore to prove (1) we need only prove equivalently

$$T_n^2 = \prod_{j=0}^{n} t_1(j)^2 = \tau_n^2$$

in light of the fact that each of the T_n and τ_n is positive.

Now we come to the more intricate components of $W(n)$:

$$
\begin{aligned}
r_3(s,j) &= 4^{-s}\sum_{k=0}^{s}\frac{(j-k)(j)_k(-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},\\
f_1(c,j) &= (-1)^c\sum_{s=0}^{\lfloor\frac{c}{2}\rfloor}\frac{(-1)^s\binom{j-1-s}{c-2s}(j)_s(-3j+1)_s(3j-3s-1)}{4^s s!(-2j+\frac{3}{2})_s(3j-1)},\\
f_2(c,j) &= (-1)^c\sum_{s=0}^{\lfloor\frac{c}{2}\rfloor}(-1)^s\begin{Bmatrix} j-s\\ c-2s\end{Bmatrix} r_3(s,j),\\
r_2(j) &= \begin{cases}\frac{t_1(j-1)}{2} & \text{if } j \text{ even},\\ \frac{t_1(j-1)}{2}+\frac{f_2(j-2,\frac{j-1}{2})}{2} & \text{if } j \text{ odd},\end{cases}\\
r_1(j) &= \begin{cases} t_1(j-1) & \text{if } j \text{ even},\\ 0 & \text{if } j \text{ odd},\end{cases}\\
e_1(i,j) &= \begin{cases} 0 & \text{if } i>j,\\ 1 & \text{if } i=j,\\ r_1(j) & \text{if } i=0,\ i<j,\\ r_2(j) & \text{if } i=1,\ i<j,\\ f_1(j-i,\frac{j}{2}) & \text{if } 2\le i<j,\ j \text{ even},\\ f_2(j-i,\frac{j-1}{2}) & \text{if } 2\le i<j,\ j \text{ odd},\end{cases}\\
e(i,j) &= \begin{cases} 0 & \text{if } i>j,\\ 1 & \text{if } i=j,\\ e_1(i,j)-t(j-1)x^{(-1)^j}e_1(i,j-1) & \text{if } i<j,\end{cases}\\
W(n) &= \big(e(i,j)\big)_{0\le i,j\le n-1}.
\end{aligned}
$$

Given all of the above, our object in the remainder of the paper will be to prove the following

Theorem 1. *For each $n \ge 1$,*

$$
M(n)W(n)=\begin{pmatrix}
x & 0 & 0 & 0 & \dots & 0\\
A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0\\
A_{20} & A_{21} & t_1(1)^2x & 0 & \dots & 0\\
A_{30} & A_{31} & A_{32} & \frac{t_1(2)^2}{x} & \dots & 0\\
\vdots & & & & \ddots & \vdots\\
A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2x^{\pm 1}
\end{pmatrix}
\tag{8}
$$

where the entry $x^{\pm1}$ in the lower triangular matrix has to be interpreted as x, if n is odd, and as x^{-1}, if n is even.

We note before proceeding that in light of the fact that

$$\det\big(W(n)\big)=1,$$

and (2) together with (8) we have the following

Corollary 1 (Stembridge's theorem [Ste95, Th. 0.2])**.** *For each $n \ge 2$,*

$$T_{n-2}^2=\det\big(M(n)_{x=1}\big)=\prod_{j=0}^{n-2}t_1(j)^2.$$

3. The Identities To Prove

3.1. **The odd case $j = 2m + 1$.** The following definitions will be convenient for proving Theorem 1.

Definition 1. For $i, j \geq 0$ denote the entry of the ith row and the jth column of the matrix $M(n)W(n)$ by $G(i,j)$; i.e.,

$$G(i,j) = \sum_{k=0}^{j} \mu_1(i,k)e(k,j).$$

Definition 2. Extend the definition of μ by

$$\mu(-1,k) := \begin{cases} 0, & \text{if } k = 0, \\ (-1)^k, & \text{if } k \geq 1. \end{cases}$$

For this subsection we fix $j = 2m + 1$ with $m \geq 1$.

Definition 3. For $i \geq 0$ and $m \geq 1$ define

$$a_1(i,m) := \sum_{k=1}^{i-2} \mu(k, i-1) f_2(2m-k, m), \qquad a_2(i,m) := \sum_{k=i}^{2m} \mu(i-1, k) f_2(2m-k, m),$$

$$b_1(i,m) := \sum_{k=1}^{i-2} \mu(k, i-1) f_1(2m-k-1, m), \quad b_2(i,m) := \sum_{k=i}^{2m} \mu(i-1, k) f_1(2m-k-1, m).$$

The next fact is immediate from the definitions.

Lemma 1. *We have $f_1(0,j) = 1$ for all $j \geq 0$ and $f_2(0,j) = 1$ for all $j \geq 1$.*

From Lemma 1 and the definitions the following representation is straightforward.

Proposition 1. *For $m \geq 1$ and $0 \leq i \leq 2m+1$,*

$$G(i, 2m+1) = \mu_1(i,1)\frac{t_1(2m) + f_2(2m-1,m)}{2} - a_1(i,m) + a_2(i,m)$$
$$- \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(\mu_1(i,0)t_1(2m-1) + \mu_1(i,1)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m)\Big).$$

Since Proposition 1 involves $\mu_1(i,0)$ and $\mu_1(i,1)$ we have to distinguish three cases: $i = 0$ (case A), $i = 1$ (case B), and $i \geq 2$ (case C).

3.1.1. *Case A: $i = 0$.* In this case, Proposition 1 turns into the following form.

Proposition 2. *For $m \geq 1$,*

$$G(0, 2m+1) = \frac{-t_1(2m) + f_2(2m-1,m)}{2} + a_2(0,m) - \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(\frac{t_1(2m-1)}{2} + b_2(0,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 2, it is immediate that showing $G(0, 2m+1) = 0$ for all $m \geq 1$ is equivalent to proving the following identities for $m \geq 1$; namely,

$$\sum_{k=1}^{2m} (-1)^k f_2(2m-k, m) = \frac{t_1(2m) - f_2(2m-1,m)}{2} \tag{9}$$

and

$$\sum_{k=1}^{2m-1} (-1)^k f_1(2m-k-1, m) = -\frac{t_1(2m-1)}{2}. \tag{10}$$

Proposition 3. *For $m \geq 1$,*

$$\sum_{k=0}^{2m}(-1)^k f_2(2m-k,m) = \frac{1}{2}t_1(2m) \tag{11}$$

and

$$\sum_{k=0}^{2m-1}(-1)^k f_1(2m-k-1,m) = -\frac{1}{2}t_1(2m-1). \tag{12}$$

Proof: See Subsection 5.5. □

Lemma 2. *For $m \geq 1$,*

$$f_1(2m-1,m) = 0 \qquad \textit{and} \qquad f_2(2m-1,m) = 2f_2(2m,m).$$

Proof: The first assertion is immediate. The proof of the second statement is elementary and is left to the reader. □

From Lemma 2 it is easily seen that (11) and (12) imply (9) and (10), respectively. These steps conclude the proof of

$$G(0,2m+1) = 0 \text{ for } m \geq 1. \tag{13}$$

3.1.2. *Case B: $i = 1$.* In this case, Proposition 1 turns into the following form.

Proposition 4. *For $m \geq 1$,*

$$G(1,2m+1) = a_2(1,m) - \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(-t_1(2m-1) + b_2(1,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 4, it is immediate that showing $G(1,2m+1) = 0$ $(m \geq 1)$ is equivalent to proving the following identities for $m \geq 1$; namely,

$$\sum_{k=1}^{2m}(-1)^k f_2(2m-k,m) = \frac{1}{2}\sum_{k=1}^{2m} 2^k f_2(2m-k,m) \tag{14}$$

and

$$\sum_{k=1}^{2m-1}(2^{k-1} - (-1)^k) f_1(2m-k-1,m) = t_1(2m-1). \tag{15}$$

Proposition 5. *For $m \geq 1$,*

$$\sum_{k=0}^{2m} 2^k f_2(2m-k,m) = t_1(2m) - f_2(2m,m) \tag{16}$$

and

$$\sum_{k=0}^{2m-1} 2^k f_1(2m-k-1,m) = t_1(2m-1). \tag{17}$$

Proof: See Subsection 5.5. □

It is easily seen that (16) and (17) together with (9), (10) and Lemma 2 imply (14) and (15), respectively. These steps conclude the proof of

$$G(1,2m+1) = 0 \text{ for } m \geq 1. \tag{18}$$

3.1.3. *Case C: $i \geq 2$.* In this case, Proposition 1 turns into the following form.

Proposition 6. *For $m \geq 1$ and $2 \leq i \leq 2m+1$,*

$$G(i,2m+1) = -\big(2^{i-2}+(-1)^i\big)\frac{t_1(2m)+f_2(2m-1,m)}{2} - a_1(i,m) + a_2(i,m)$$
$$-\frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(-(2^{i-2}-(-1)^i)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 6, it is immediate that showing

$$\forall m \geq 1: \qquad G(i,2m+1) = \begin{cases} 0, & \text{if } 2 \leq i \leq 2m, \\ \frac{1}{x}t_1(2m)^2, & \text{if } i = 2m+1 \end{cases} \tag{19}$$

is equivalent to proving the following identities for $m \geq 1$; namely[1],

$$a_2(i,m) - a_1(i,m) = \big(2^{i-2}+(-1)^i\big)\frac{t_1(2m)+f_2(2m-1,m)}{2} \quad \text{for all } i \text{ s.t. } 2 \leq i \leq 2m+1, \tag{20}$$

and

$$b_2(i,m) - b_1(i,m) = \big(2^{i-2}-(-1)^i\big)\frac{t_1(2m-1)}{2} \quad \text{for all } i \text{ s.t. } 2 \leq i \leq 2m, \tag{21}$$

and

$$b_2(2m+1,m) - b_1(2m+1,m) = -t_1(2m)t_1(2m-1) + \big(2^{2m-1}+1\big)\frac{t_1(2m-1)}{2}. \tag{22}$$

Ideally one would prove (20) to (22) directly with the Sigma package [Sch01, PS03, Sch04b, Sch04a]. However, it turns out that the computational complexity in this formulation of the problem is still too high. More precisely, the left hand sides of (20) and (21) are differences of two 4-fold sums, whereas the left hand side of (22) is a difference of two 3-fold sums. So, in order to be able to use Sigma, we transform the problem by an induction argument which is presented in the next subsection.

3.2. **Transforming the problem by induction.** The following transformation is based on an induction argument. It reduces the 4-fold sums on the left hand sides of (20) and (21) to 3-fold sums, and the 3-fold sums on the left hand side of (22) to 2-fold sums.

3.2.1. *Recurrences for the induction step.*

Definition 4. For $i, j \geq 1$ define

$$\alpha(i,j) := (-1)^{j-i-1} \qquad \text{and} \qquad \beta(i,j) := \sum_{s=i}^{j-1}\binom{i+j-2}{s}. \tag{23}$$

It is convenient to introduce the following lemma. Its proof is elementary.

Lemma 3. *For integers $i, j \geq 1$:*

$$\alpha(i+1,j) = -\alpha(i,j) \qquad \text{and} \qquad \alpha(i,j+1) = -\alpha(i,j); \tag{24}$$
$$\beta(i+1,j) = 2\beta(i,j) - \binom{i+j-1}{i} \quad (1 \leq i \leq j-1); \tag{25}$$
$$\beta(i,j+1) = 2\beta(i,j) - \binom{i+j-1}{j} \quad (1 \leq i \leq j). \tag{26}$$

Lemma 3 implies the following recurrences for μ.

[1]Note: These bounds for i are sharp!

Proposition 7. *For integers*[2] i, j:

$$\mu(i+1,j) = 2\mu(i,j) + 3(-1)^{j-i} - \binom{i+j-1}{i} \qquad (1 \le i \le j-2); \tag{27}$$

$$\mu(i,j+1) = 2\mu(i,j) + 3(-1)^{j-i} + \binom{i+j-1}{j} \qquad (0 \le i \le j-1). \tag{28}$$

Proof: The definition of μ implies

$$\mu(i,j) = \alpha(i,j) + \beta(i,j) \qquad (1 \le i < j).$$

Hence Lemma 3 for $1 \le i \le j-2$ gives

$$\begin{aligned}\mu(i+1,j) = \alpha(i+1,j) + \beta(i+1,j) &= -\alpha(i,j) + 2\beta(i,j) - \binom{i+j-1}{i}\\ &= 2(\alpha(i,j)+\beta(i,j)) - 3\alpha(i,j) - \binom{i+j-1}{i},\end{aligned}$$

which proves (27). Recurrence (28) is proved analogously. □

With Proposition 7 in hand one finds the relations of Proposition 8 and Proposition 9 which will be used in the induction step and which involve the following sums.

Definition 5. For $2 \le i \le 2m+1$ define

$$A_1(i,m) := \sum_{k=1}^{i-3}(-1)^k f_2(2m-k,m), \qquad B_1(i,m) := \sum_{k=1}^{i-3}(-1)^k f_1(2m-k-1,m),$$

$$A_2(i,m) := \sum_{k=i}^{2m}(-1)^k f_2(2m-k,m), \qquad B_2(i,m) := \sum_{k=i}^{2m-1}(-1)^k f_1(2m-k-1,m),$$

and

$$A_0(i,m) := \sum_{k=0}^{2m}\binom{i+k-3}{i-2} f_2(2m-k,m), \quad B_0(i,m) := \sum_{k=0}^{2m-1}\binom{i+k-3}{i-2} f_1(2m-k-1,m).$$

Proposition 8. *For* $3 \le i \le 2m+1$,

$$\begin{aligned}&a_2(i,m) - a_1(i,m) = 2(a_2(i-1,m) - a_1(i-1,m)\\ &\quad - \Big(f_2(2m-i+2,m) + 2f_2(2m-i+1,m)\Big) + 3(-1)^i\Big(A_2(i,m) - A_1(i,m)\Big) - A_0(i,m).\end{aligned} \tag{29}$$

Proof: Applying (27) and (28), respectively, yields

$$\begin{aligned}a_2(i,m) - a_1(i,m) &= \sum_{k=i}^{2m}\mu(i-1,k)f_2(2m-k,m)\\ &\qquad - \Big(\sum_{k=1}^{i-3}\mu(k,i-1)f_2(2m-k,m) + \mu(i-2,i-1)f_2(2m-i+2,m)\Big)\\ &= 2\Big(a_2(i-1,m) - \mu(i-2,i-1)f_2(2m-i+1,m)\Big)\\ &\qquad + 3(-1)^i A_2(i,m) - \sum_{k=i}^{2m}\binom{i+k-3}{i-2}f_2(2m-k,m) - \Big[2a_1(i-1,m) + 3(-1)^i A_1(i,m)\\ &\qquad + \sum_{k=1}^{i-3}\binom{i+k-3}{i-2}f_2(2m-k,m) + \mu(i-2,i-1)f_2(2m-i+2,m)\Big]\\ &= 2\big(a_2(i-1,m) - a_1(i-1,m)\big) + 3(-1)^i\big(A_2(i,m) - A_1(i,m)\big) - A_0(i,m)\end{aligned}$$

[2] Note that the ranges for i and j in both recurrences are sharp.

$$+\left(\binom{2i-5}{i-2}-\mu(i-2,i-1)\right)f_2(2m-i+2,m)$$
$$+\left(\binom{2i-4}{i-2}-2\mu(i-2,i-1)\right)f_2(2m-i+1,m)$$

which gives the right side of (29) by invoking the fact that

$$\mu(i-2,i-1)=1+\binom{2i-5}{i-2}\qquad (i\geq 3). \tag{30}$$

□

Proposition 9. *For* $3\leq i\leq 2m+1$,

$$b_2(i,m)-b_1(i,m)=2\big(b_2(i-1,m)-b_1(i-1,m)\big)\\ -\big(f_1(2m-i+1,m)+2f_1(2m-i,m)\big)+3(-1)^i\big(B_2(i,m)-B_1(i,m)\big)-B_0(i,m). \tag{31}$$

Proof: The proof is completely analogous to that of Proposition 8. □

3.2.2. *Identities for the induction step.* Suppose we have proved the case $i=2$ of (20) and (21). Then proving (20) and (21) in their full generality can be done by induction on i for fixed $m\geq 1$. Namely, due to Proposition 8, identity (20) is equivalent to showing the case $i=2$ and that for any i with[3] $3\leq i\leq 2m+1$ the equality

$$\boxed{\begin{aligned}&\frac{3}{2}(-1)^i\big(t_1(2m)+f_2(2m-1,m)\big)\\ &=-\big(f_2(2m-i+2,m)+2f_2(2m-i+1,m)\big)+3(-1)^i\big(A_2(i,m)-A_1(i,m)\big)-A_0(i,m)\end{aligned}} \tag{32}$$

holds. Analogously, because of Proposition 9, identity (21) is equivalent to showing the case $i=2$ and that for any i with $3\leq i\leq 2m$ the equality

$$\boxed{\begin{aligned}&-\frac{3}{2}(-1)^i t_1(2m-1)\\ &=-\big(f_1(2m-i+1,m)+2f_1(2m-i,m)\big)+3(-1)^i\big(B_2(i,m)-B_1(i,m)\big)-B_0(i,m)\end{aligned}} \tag{33}$$

holds.

The base case $i=2$ of (20) and (21), respectively, simplifies as explained in Subsection 3.2.3 below. Identities (32) and (33) are proved in Subsections 5.4 and 5.3, respectively.

Finally, the proof of (19) is completed by showing (22). Since (21) will be proved independently from it, we can proceed as follows. Applying (31) and then (21) gives

$$b_2(2m+1,m)-b_1(2m+1,m)=\big(2^{2m-2}-1\big)t_1(2m-1)-1+3B_1(2m+1,m)-B_0(2m+1,m).$$

By (12) and Lemma 1 one has

$$B_1(2m+1,m)=-\sum_{k=1}^{2m-2}(-1)^k f_1(k,m)=1-\frac{t_1(2m-1)}{2}. \tag{34}$$

Consequently the proof of (22) is equivalent to showing that for $m\geq 1$,

$$\boxed{B_0(2m+1,m)=t_1(2m)t_1(2m-1)-3t_1(2m-1)+2.} \tag{35}$$

Proof: See Subsection 5.6. □

[3] Note that these bounds for identity (32) are sharp; similarly for identity (33).

3.2.3. *The base case for $i = 2$.* The case $i = 2$ and $m \geq 1$ of identity (20) reads as

$$\sum_{k=2}^{2m} \mu(1,k) f_2(2m-k,m) = t_1(2m) + f_2(2m-1,m). \tag{36}$$

Because of

$$\mu(1,k) = 2^{k-1} + (-1)^k - 1 \qquad (k \geq 2), \tag{37}$$

and then (11) and (16), the left side of (36) turns into

$$-\sum_{k=0}^{2m-2} f_2(k,m) + t_1(2m) - 2f_2(2m,m).$$

Consequently, identity (36) is implied by the following lemma.

Lemma 4. *For $m \geq 1$,*

$$\sum_{k=0}^{2m} f_2(k,m) = -f_2(2m,m). \tag{38}$$

Proof: See Subsection 5.2. □

The case $i = 2$ and $m \geq 1$ of identity (21) reads as

$$\sum_{k=2}^{2m-1} \mu(1,k) f_1(2m-k-1,m) = 0. \tag{39}$$

The case $m = 1$ is trivial. Because of (37) and then (12) and (16), the left side of (39) turns into $-\sum_{k=2}^{2m-1} f_1(2m-k-1,m)$. Consequently, identity (39) is implied by Lemma 2 and

Lemma 5. *For $m \geq 1$,*

$$\sum_{k=0}^{2m-1} f_1(2m-k-1,m) = f_1(2m-2,m). \tag{40}$$

Proof: See Subsection 5.2. □

3.3. **The even case: $j = 2m$.** Analogously to Proposition 1, from all the definitions we can derive the following representation.

Proposition 10. *For $m \geq 2$ and $0 \leq i \leq 2m$,*

$$\begin{aligned} G(i,2m) = &\big(2\mu_1(i,0) + \mu_1(i,1)\big)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m) \\ &- x\frac{t_1(2m-1)}{t_1(2m-2)}\Big(\frac{\mu_1(i,1)}{2}\big(t_1(2m-2) + f_2(2m-3,m-1)\big) - a_1(i,m-1) + a_2(i,m-1)\Big). \end{aligned} \tag{41}$$

Suppose we have proved $G(i,2m+1) = 0$ for $0 \leq i \leq 2m+1$. From Proposition 1 we know that the "x–free part" of the right side of (41) is 0 for $0 \leq i \leq 2m$, and also that the "x–part" of the right side of (41) is 0 for $0 \leq i \leq 2m-1$. Consequently, to complete the proof of

$$\forall m \geq 2: \qquad G(i,2m) = \begin{cases} 0, & \text{if } 0 \leq i \leq 2m-1, \\ xt_1(2m-1)^2, & \text{if } i = 2m, \end{cases} \tag{42}$$

it remains to show that

$$\begin{aligned} -x\frac{t_1(2m-1)}{t_1(2m-2)}\Big[&\frac{\mu_1(2m,1)}{2}\big(t_1(2m-2) + f_2(2m-3,m-1)\big) \\ &+ a_2(2m,m-1) - a_1(2m,m-1)\Big] = xt_1(2m-1)^2 \end{aligned} \tag{43}$$

for all $m \geq 2$.

If we define

$$a(m) := a_2(2m, m-1) - a_1(2m, m-1) \qquad (m \geq 2) \tag{44}$$

then (43) is equivalent to

$$a(m) = \big(2^{2m-3} + \frac{1}{2}\big)\big(t_1(2m-2) + f_2(2m-3, m-1)\big) - t_1(2m-1)t_1(2m-2) \qquad (m \geq 2). \tag{45}$$

On the other hand, from (29) with $m \to m-1$ and $i = 2m$ we obtain that

$$a(m) = 2\big(a_2(2m-1, m-1) - a_1(2m-1, m-1)\big) \\ - 1 - A_0(2m, m-1) - 3\sum_{k=1}^{2m-3}(-1)^k f_2(2m-k-2, m-1),$$

which by (20) [with $m \to m-1$ and $i = 2m-1$] and by (11) can be simplified further to

$$a(m) = 2 - A_0(2m, m-1) + 3f_2(2m-2, m-1) \\ - \frac{3}{2}t_1(2m-2) + \big(2^{2m-3} - 1\big)\big(t_1(2m-2) + f_2(2m-3, m-1)\big). \tag{46}$$

Comparing this to (45) and using the elementary property of f_2 described in Lemma 2, to prove identity (43) it suffices to show that for all $m \geq 2$,

$$\boxed{t_1(2m-1)t_1(2m-2) - 3t_1(2m-2) + 2 = A_0(2m, m-1).} \tag{47}$$

Proof: See Subsection 5.6. □

4. The Proof Method

In this section we explain the method that we used for proving **all** of the identities stated in this article. The method consists in a combination of various algorithmic steps which are briefly described below. Remarkably most of the proving steps can be represented in the form of *proof certificates* which are identities that can be verified by elementary calculations independently from the way the algorithm has obtained them. This allows us to produce compact descriptions of the proofs of the identities under consideration which can be found in Section 5.

Except for the task of combining recurrences described in Subsection 4.1, all other algorithmic steps were executed by the computer algebra package Sigma [Sch01, PS03, Sch04b] developed by the third author. It should be noted that in order to handle the conjectured TSPP multiple sum identities, the Sigma tool-box was extended significantly. A detailed description of this work can be found in [Sch04a].

4.1. Combining Recurrences.

Combining Recurrences. Hypergeometric sequences are special instances of P-finite sequences where the latter are defined to be sequences that satisfy a linear recurrence with polynomial coefficients. Such recurrences are called P-finite recurrences. P-finite sequences enjoy various closure properties; see [Sta80].

For our purpose we need to combine P-finite recurrences additively and multiplicatively, namely:

Given P-finite sequences $\big(a_n\big)$ and $\big(b_n\big)$ satisfying the recurrences

$$p_\alpha(n)a_{n+\alpha} + \cdots + p_0(n)a_n = 0 \quad (n \geq 0)^4$$

and

$$q_\beta(n)b_{n+\beta} + \cdots + q_0(n)b_n = 0 \quad (n \geq 0)$$

respectively, **compute** a P-finite recurrence

$$r_\gamma(n)c_{n+\gamma} + \cdots + r_0(n)c_n = 0 \quad (n \geq 0) \tag{48}$$

which is satisfied by the sum sequence $\big(c_n\big)$ with $c_n := a_n + b_n$. — The analogous product problem is with $c_n := a_n\, b_n$.

Algorithms which compute the recurrence (48) for the sum (resp. product) sequence $\big(c_n\big)$ are described in [SZ94]. For our computations we have used Mallinger's Mathematica package GeneratingFunctions [Mal96].

4.2. **The General Proof Strategy.** All the identities we need to prove in this article are of the form

$$c_n^{(1)} + \cdots + c_n^{(k)} = 0 \quad (n \geq 0) \tag{49}$$

where k is a fixed positive integer and where each of the $c_n^{(i)}$ is a P-finite sequence. To prove that (49) is valid indeed for all $n \geq 0$ we proceed as follows. First we compute P-finite recurrences for all of the $c_n^{(i)}$, unless such recurrences are already given. Then, as described in Section 4.1, a P-finite recurrence for the sum sequence $s_n := c_n^{(1)} + \cdots + c_n^{(k)}$ is computed. Finally, we show that $s_n = 0$ for all $n \geq 0$ by checking sufficiently many initial values. — Note that the leading polynomial coefficient of the recurrence for s_n must not have any nonnegative integer root.

In our context the $c_n^{(i)}$ are given as hypergeometric sequences, as single, double and triple sums over hypergeometric sequences, or as the product of such sequences. So in view of Section 4.1 and of our general strategy described above, there remains the task to derive P-finite recurrences for such sums.

In this section, for the sake of simplicity we restrict ourselves to (multiple) sums where all summations are taken over finite summand supports. This means, all sums are understood to extend over all integers, positive and negative, but only finitely many terms contribute. For example, in $\sum_s \binom{r}{s}$, r an integer, the summand vanishes if $s < 0$ or $s > n$. With this restriction *homogeneous* sum recurrences are guaranteed.

4.3. **Single Sums.** Here the basic task is as follows.

Given a summand $F(r, s)$ which is hypergeometric[5] in r and s, **compute** a P-finite recurrence

$$p_\gamma(r)f(r+\gamma) + \cdots + p_0(r)f(r) = 0 \quad (n \geq 0) \tag{50}$$

which is satisfied by the sum $f(r) := \sum_s F(r, s)$.

In the case that $F(r, s)$ satisfies some mild side conditions this problem can be solved by applying Zeilberger's algorithm [PWZ96] which computes polynomials $p_i(r)$, free of s, and $G(r, s)$ such that

$$p_\gamma(r)F(r+\gamma, s) + p_{\gamma-1}(r)F(r+\gamma-1, s) + \cdots + p_0(r)F(r, s) = \Delta_s G(r, s). \tag{51}$$

Note that Δ_s denotes the (forward) difference operator defined as usual by $\Delta_s G(s) = G(s+1) - G(s)$.

One can show that $G(r, s)$ is a rational function multiple of $F(r, s)$. Hence recurrence (50) is obtained from (51) by summation over all s. — Consequently, all that is needed to prove the correctness of (50) is the knowledge of (51) which is called "certificate recurrence". Note that after dividing (51) by $F(r, s)$, its verification reduces to checking equality of rational functions, a simple check which is independent from the way the algorithm computed (51).

Remark: It can be that for a fixed order γ there exists only the trivial solution, i.e., where all the $p_i(r)$ are 0. In this case one has to increase the order γ incrementally until a non-trivial solution is computed. Its existence is guaranteed by the theory explained in [PWZ96].

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[4] For the sake of simplicity we have chosen $n \geq 0$; in concrete cases the initial values might be different from 0.

[5] $F(r)$ is hypergeometric in r iff $F(r+1)/F(r) = g(r)$ for some fixed rational function $g(r)$.

4.3.1. *A slight but important variation.* Many TSPP identities involve summands in more than one independent variable. For instance, instead of the summand $F(r,s)$ take the summand $F(n,r,s)$, now hypergeometric in r,s and n. For the following it is important to note that completely analogously to (51) one can compute

$$p'_\gamma(n,r)F(n+1,r,s)+p'_{\gamma-1}(n,r)F(n,r+\gamma-1,s)+\cdots+p'_0(n,r)F(n,r,s)=\Delta_s G'(n,r,s) \quad (52)$$

if it exists. Also for such cases one can prove that $G'(m,r,s)$ is a rational function multiple of $F(n,r,s)$. — Recurrences like (52) are related to contiguous relations [Pau04]; see also [Sch04a]. For instance, summing (52) over all s (assuming finite support) yields

$$p'_\gamma(n,r)f(n+1,r)+p'_{\gamma-1}(n,r)f(n,r+\gamma-1)+\cdots+p'_0(n,r)f(n,r)=0 \quad (n\geq 0) \qquad (53)$$

with $f(n,r)=\sum_s F(n,r,s)$ and where the $p'_i(n,r)$ are polynomials in n and r.

Remark: There remains the question whether relations like (53) or (52) do exist. However, in [Pau04] an existence theory is presented which closely relates to the situation of Zeilberger's algorithm; for multiple sums in [PS04] this question is analysed in further details.

4.4. **Double Sums.** Here the basic task is as follows.

Given a summand $F(n,r,s)$ which is hypergeometric in n,r and s, **compute** a P-finite recurrence

$$p_\gamma(n)S(n+\gamma)+\cdots+p_0(n)S(n)=0 \quad (n\geq 0) \qquad (54)$$

which is satisfied by the sum $S(n):=\sum_r\sum_s F(n,r,s)$.

In principle, one could apply the WZ method which is based on ideas of Sister Celine Fasenmyer and which is described in [PWZ96] . However, it turns out that all available implementations of this approach or of variations of it (e.g., Wegschaider's algorithm [Weg97]) meet serious problems of computational complexity when applied to the TSPP identities in question. As a consequence we will follow a different approach which can be viewed as a new, surprisingly simple variant of Chyzak's algorithm [Chy00]. The basic ideas of this method are as follows; a full account of the details and a comparison to [Chy00] is given in [Sch04a].

The overall goal of the method is to compute a certificate recurrence of type (51), i.e.,

$$p_\gamma(n)f(n+\gamma,r)+\cdots+p_0(n)f(n,r)=\Delta_r g(n,r) \qquad (55)$$

where $f(n,r)$ is defined to be the inner sum, i.e.,

$$f(n,r):=\sum_s F(n,r,s),$$

and where $g(n,r)$ is suitably chosen. From (55) the desired recurrence (54) for $S(n)$ is obtained by summing over all r — as in Zeilberger's algorithm for single sums.

In order to find (55) we proceed as follows. First one computes recurrences of the following form,

$$f(n,r+\delta)=\lambda_0(n,r)f(n,r)+\cdots+\lambda_{\delta-1}(n,r)f(n,r+\delta-1), \qquad (56)$$

and

$$f(n+1,r)=\mu_0(n,r)f(n,r)+\cdots+\mu_{\delta-1}(n,r)f(n,r+\delta-1), \qquad (57)$$

where the $\lambda_i(n,r)$ and $\mu_i(n,r)$ are rational functions in n and r. This can be accomplished by following the description to compute (50) and (53) via (51) and (52), respectively.

Second, for $g(n,r)$ one starts with an expression with undetermined coefficients of the following form,

$$g(n,r)=\phi_0(n,r)f(n,r)+\cdots+\phi_{\delta-1}(n,r)f(n,r+\delta-1). \qquad (58)$$

In the third step, the unknown polynomials $p_i(n)$, free of r, and the unknown rational function coefficients $\phi_i(n,r)$ for $g(n,r)$ are computed such that the certificate recurrence (55) holds. In view of (56) and (57), the key observation is that any shift in n and r of $f(n,r)$ and also $g(n,r)$ can be represented as a linear combination of $f(n,r),\ldots,f(n,r+\delta-1)$ over rational functions in n and r. Then rewriting both sides of (55) in terms of these generators, allows — in all our applications — to compute the unknown data by comparing the coefficients of all the $f(n, r+i)$ involved.

The corresponding computational steps are carried out as follows. For the sake of simplicity we restrict to $\gamma = \delta = 2$; the general case works completely analogously and is described in [Sch04a]. From the relations (56) and (57) one can find rational functions $\nu_i(n, r)$ such that

$$f(n+2, r) = \nu_0(n, r)f(n, r) + \nu_1(n, r)f(n, r+1). \tag{59}$$

This together with (57) implies for the left hand side of (55) that

$$\begin{aligned} &p_2(n)f(n+2, r) + p_1(n)f(n+1, r) + p_0(n)f(n, r) \\ &= \big(p_0(n) + p_1(n)\mu_0(n, r) + p_2(n)\nu_0(n, r)\big)f(n, r) + \big(p_1(n)\mu_1(n, r) + p_2(n)\nu_1(n, r)\big)f(n, r+1). \end{aligned} \tag{60}$$

To represent the right hand side of (55) in terms of the generators $f(n, r+i)$ one invokes (56) which gives that

$$\begin{aligned} \Delta_r g(n, r) = \big(-\phi_0(n, r) + \phi_1(n, r+1)\lambda_0(n, r)\big)f(n, r) \\ + \big(\phi_0(n, r+1) - \phi_1(n, r) + \phi_1(n, r+1)\lambda_1(n, r)\big)f(n, r+1). \end{aligned} \tag{61}$$

Finally comparing the coefficients of $f(n, r)$ and $f(n, r+1)$ on the right hand sides of (60) and (61), respectively, after triangulation leads to the problem of solving the system

$$\begin{aligned} &\lambda_0(n, r+1)\phi_1(n, r+2) + \lambda_1(n, r)\phi_1(n, r+1) - \phi_1(n, r) \\ &\qquad = p_0(n) + \big(\mu_0(n, r+1) + \mu_1(n, r)\big)p_1(n) + \big(\nu_0(n, r+1) + \nu_1(n, r)\big)p_2(n) \end{aligned} \tag{62}$$

and

$$\phi_0(n, r) = \phi_1(n, r+1)\lambda_0(n, r) - \big(p_0(n) + p_1(n)\mu_0(n, r) + p_2(n)\nu_0(n, r)\big). \tag{63}$$

Equation (62) is a parameterized difference equation which has to be solved for a rational function $\phi_1(n, r)$ and for the polynomials $p_i(n)$. This is done by the Sigma package by using a refinement of Abramov's algorithm [Abr89]. Finally $\phi_0(n, r)$ is computed from (63). It is important to note that not only for $\gamma = \delta = 2$, but also for general γ and δ the approach works entirely the same. In particular, as pointed out in [Sch04a] triangularization of the system arising from this coefficient comparison can be avoided completely since the uncoupled system can be represented by a generic formula.

SUMMARY: The key identity for deriving a P-finite recurrence for the double sum $S(n) = \sum_r \sum_s F(n, r, s) = \sum_r f(n, r)$ is the certificate identity (55). Knowing (56) and (57) together with the $\phi_i(n, r)$ in (58), the reader can check the correctness of (55) independently from the steps of the method. Note that the correctness of (56) and (57) can be verified by standard creative telescoping. As a consequence, to certify that double sums satisfy certain P-finite recurrences, in Section 5 we restrict ourselves to provide the data contained in (54), (55), (56), and (57).

4.5. **Triple Sums.** Based on what we said about single and double sums we are in the position to solve the following problem.

Given a summand $F(m, n, r, s)$ which is hypergeometric in m, n, r and s, **compute** a P-finite recurrence

$$p_\gamma(m)S(m+\gamma) + \cdots + p_0(m)S(m) = 0 \quad (m \geq 0) \tag{64}$$

which is satisfied by the sum $S(m) := \sum_n \sum_r \sum_s F(m, n, r, s)$.

As with double sums the overall goal of the method is to compute a certificate recurrence of the form

$$p_\gamma(m)h(m+\gamma, n) + \cdots + p_0(m)h(m, n) = \Delta_n g(m, n) \tag{65}$$

where we define $h(m, n)$ as

$$h(m, n) := \sum_{r} \sum_{s} F(m, n, r, s), \tag{66}$$

and where $g(m, n)$ is suitably chosen. Then from (65) the desired recurrence (64) for $S(m)$ is obtained by summation over all n.

To find (65) we proceed analogously to the double sum case. Namely, we first derive recurrences of the form

$$h(m, n + \delta) = \lambda_0(m, n)h(m, n) + \cdots + \lambda_{\delta-1}(m, n)h(m, n + \delta - 1), \tag{67}$$

and

$$h(m + 1, n) = \mu_0(m, n)h(m, n) + \cdots + \mu_{\delta-1}(m, n)h(m, n + \delta - 1), \tag{68}$$

and afterwards we apply the same method as in the double sum case in order to compute all the components for the certificate recurrence (65). In particular, due to our starting point (58), $g(m, n)$ will be of the form

$$g(m, n) = \phi_0(m, n)h(m, n) + \cdots + \phi_{\delta-1}(m, n)h(m, n + \delta - 1), \tag{69}$$

where the $\phi_i(m, n)$, $\lambda_i(m, m)$ and $\mu_i(m, n)$ are rational functions in m and n.

Obviously, from (67), (68) and (69) the correctness of (65) can be verified independently from the steps of our algorithm.

In order to apply the above strategy there remains the task to compute the recurrences (67) and (68). In principle, we could apply our techniques from above. Namely, with our description from Subsection 4.4 we can obtain a recurrence of the type (67) for the double sum (66). Similarly we can derive a recurrence of the form (68) by a slight variation of the same strategy which is described in [Sch04a]. Roughly spoken, this way we reduce triple summation first to double and then to single summation by recursion. Summarizing, we have a general method in hand to derive (67) and (68). But, by observing that the summand in the given TSPP triple sums are all of the type

$$h(m, n) = \sum_{r} H(m, n, r) \sum_{s=0}^{r} F(m, n, s)$$

where $H(m, n, r)$ is hypergeometric in m, n and r and where $F(m, n, s)$ is free of r, we can follow a more direct approach. Namely, as Zeilberger's algorithm can compute the required recurrences (56) and (57) for the double sum case, Sigma can produce in an analogous fashion the recurrences (67) and (68) together with recurrence certificates for $h(m, n)$.

5. The Proof Certificates

In this section we give compact descriptions of the proofs of the identities under consideration. The underlying algorithmic ideas of the proving steps are described in Section 4. As already pointed out, most of these steps can be represented in the form of proof certificates which are identities that can be verified by elementary calculations independently from the way the algorithm has derived them. Such verifications are left to the reader (resp. to the computer of the reader).

In order to stay within reasonable page limits, various technical details such as the consideration of exceptional points (e.g., poles of rational function coefficients arising at summation bounds) or the explicit presentation of more elementary proof certificates have been left out from this description. All such issues are discussed in [APS04] where the algorithmic proofs of ALL the identities under consideration are given in full detail.

5.1. **Preparation.** In all our multi-sum identities under consideration either the sum $f_1(2m-k-1,m)$ or $f_2(2m-k,m)$ occurs in the summand expression; so for convenience we will use the abbreviation

$$h_1(k,m) := f_1(2m-k-1,m) \qquad \text{and} \qquad h_2(k,m) := f_2(2m-k,m). \tag{70}$$

In order to invoke our proof methods according to the Sections 4.4 and 4.5, we need recurrences of the forms (56), (57) or (67), (68) for $h_1(k,m)$ and $h_2(k,m)$, respectively. The task of finding these recurrences is left entirely to the package Sigma which delivers:

$$\begin{aligned}
&-2(2+k)^2(1+k-2m)(k+2m)h_1(k,m)\\
&+(29k^3+5k^4+k(46+20m-40m^2)+k^2(58+6m-12m^2)+12(1+m-2m^2))h_1(k+1,m)\\
&-(26k^3+4k^4+k(55+14m-28m^2)+k^2(59+6m-12m^2)+6(3+m-2m^2))h_1(k+2,m)\\
&\qquad+(1+k)^2(3+k-2m)(2+k+2m)h_1(k+3,m)=0,
\end{aligned}\tag{71}$$

$$\begin{aligned}
&2(k+2m)(-6k-13k^2-3k^3+11k^4+9k^5+2k^6+12m-6km-72k^2m-72k^3m-2k^4m\\
&+24k^5m+8k^6m+66m^2+138km^2-20k^2m^2-192k^3m^2-160k^4m^2-48k^5m^2+42m^3+342km^3\\
&+306k^2m^3+186k^3m^3+96k^4m^3-240m^4+108km^4+72k^2m^4-108k^3m^4-312m^5-144km^5+72k^2m^5)\,h_1(k,m)\\
&+(1+4m)(18k^2+39k^3+9k^4-33k^5-27k^6-6k^7-6km+46k^2m+112k^3m+100k^4m\\
&+56k^5m+16k^6m-60m^2-162km^2+14k^2m^2+252k^3m^2+172k^4m^2+24k^5m^2-180m^3-528km^3\\
&-504k^2m^3-192k^3m^3-24k^4m^3-240km^4-432k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5)\,h_1(k+1,m)\\
&+(2+k-2m)(1+4m)(-3k^2-5k^3+k^4+5k^5+2k^6-14k^2m-18k^3m-4k^4m+12m^2\\
&+24km^2-42k^2m^2-54k^3m^2-12k^4m^2+48m^3+120km^3-36k^3m^3+48m^4+144km^4+72k^2m^4)\,h_1(k+2,m)\\
&\qquad+4(1+k)^2(-1+k-2m)(k-2m)m(1+2m)(-1+4m)(1+4m)\,h_1(k,m+1)=0,
\end{aligned}\tag{72}$$

$$\begin{aligned}
&2(2+k)^2(k-2m)(1+k+2m)h_2(k,m)\\
&-(29k^3+5k^4+k(46-20m-40m^2)-12(-1+m+2m^2)-2k^2(-29+3m+6m^2))h_2(k+1,m)\\
&+(26k^3+4k^4+k(55-14m-28m^2)+k^2(59-6m-12m^2)-6(-3+m+2m^2))h_2(k+2,m)\\
&\qquad-(1+k)^2(2+k-2m)(3+k+2m)h_2(k+3,m)=0,
\end{aligned}\tag{73}$$

and

$$\begin{aligned}
&(1+k+2m)(6+141k-18k^2-141k^3-36k^4+18k^5+12k^6-216m+795km\\
&+392k^2m-357k^3m-180k^4m-48k^5m+16k^6m-1242m^2+1266km^2+1274k^2m^2\\
&-150k^3m^2-32k^4m^2-96k^5m^2-2436m^3+396km^3+1260k^2m^3-60k^3m^3\\
&+192k^4m^3-2040m^4-504km^4+504k^2m^4-216k^3m^4-624m^5-288km^5+144k^2m^5)\,h_2(k,m)\\
&+(3+4m)(-30-111k-41k^2+125k^3+99k^4+k^5-19k^6-6k^7-120m\\
&-549km-489k^2m+148k^3m+254k^4m+80k^5m+16k^6m-30m^2-774km^2\\
&-1210k^2m^2-252k^3m^2+136k^4m^2+24k^5m^2+420m^3+72km^3-1008k^2m^3-480k^3m^3\\
&-24k^4m^3+600m^4+840km^4-72k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5)\,h_2(k+1,m)\\
&-(-1-k+2m)(3+4m)(12+30k-16k^2-32k^3-4k^4+5k^5+2k^6+72m\\
&+186km-20k^2m-99k^3m-16k^4m+156m^2+420km^2+66k^2m^2-108k^3m^2\\
&-12k^4m^2+144m^3+408km^3+144k^2m^3-36k^3m^3+48m^4+144km^4+72k^2m^4)\,h_2(k+2,m)\\
&+4(1+k)^2(-2+k-2m)(-1+k-2m)(1+m)(1+2m)(1+4m)(3+4m)\,h_2(k,m+1)=0
\end{aligned}\tag{74}$$

that hold for all $m,k\geq 0$. Rigorous correctness proofs are spelled out in the Remarks 1, 6, 11, and 14 of [APS04].

5.2. **Lemmas 4 and 5.** *Certificate proof of (40):* Recalling (70) the main step of the proof is the certificate recurrence

$$\Delta_k g(k) = h_1(k,m) \tag{75}$$

where

$$\begin{aligned} g(k,m) := \frac{1}{2\,(1+k)\;m\;(-1+2\,m)}\Big(&-2\,\left(3\,k^2+k^3+3\,(1-2\,m)\,m+k\,\left(2+m-2\,m^2\right)\right)\,h_1(k,m)\\ &+\left(9\,k^2+3\,k^3+2\,(1-2\,m)\,m+k\,\left(6+4\,m-8\,m^2\right)\right)\,h_1(k+1,m)\\ &-k\,\left(2+3\,k+k^2+2\,m-4\,m^2\right)\,h_1(k+2,m)\Big). \end{aligned} \tag{76}$$

The recurrence (75) holds for all $0 \le k \le 2m-4$ which can be checked as follows: Express $\Delta_k g(k,m)$ in terms of the generators $h_1(k,m), h_1(k+1,m), h_1(k+2,m)$ by using (71) and verify equality (75) by polynomial arithmetic. After this verification, the summation of (75) gives

$$\sum_{k=0}^{2m-4} h_1(k,m) = g(2m-3,m) - g(0,m). \tag{77}$$

Finally, we prove (40) for $m \ge 2$ by adding $h_1(2m-3,m)+h_1(2m-2,m)+h_1(2m-1,m)$ to both sides of (77) and by using Lemma 2 together with

$$h_1(2m-3,m) = \frac{-6+21m-18m^2+4m^3}{2(-3+4m)}, \quad h_1(2m-2,m) = 1-m, \quad h_1(2m-1,m) = 1.$$

The proof of the special case $m=1$ is trivial. ☐

Certificate proof of (38): uses a certificate similar to (76) such that $\Delta_k g(k) = h_2(k,m)$; details are given in [APS04, Remark 10] or [Sch04a, Example 2]. ☐

5.3. **Identity (33).** Using Proposition 3 and Lemma 2, identity (33) is simplified to

$$6(-1)^i B_2(i,m) - B_0(i,m) + 2h_1(i-2,m) - 5h_1(i-1,m) + 3(-1)^i t_1(2m-1) = 0 \tag{78}$$

for all $2 \le i \le 2m$. Subsequently, we apply our proof strategy of Subsection 4.2 to prove (78).

(1) We derive recurrences for the P-finite sequences in (78) that hold for all $2 \le i \le 2m$; trivially, for $M(i) := 3(-1)^i t_1(2m-1)$ we obtain $M(i+1)+M(i)=0$, and for $h_1(i-1,m)$ and $h_1(i-2,m)$ we can take (71) by replacing k with $i-1$ or $i-2$, respectively. Finally, with our method given in Section 4.4 we derive recurrences for $B_0(i,m)$ and $B_2(i,m)$, namely

$$\begin{aligned} &(-2-i-i^2)(-1+i-2m)(-2+i+2m)B_0(i,m)\\ &\quad+(3+i)(-2+2i-i^2+i^3-2m+4m^2)B_0(i+1,m)\\ &\quad+(-3+i)(2+2i+i^2+i^3+2m-4m^2)B_0(i+2,m)\\ &\quad+(-2+i-i^2)(2+i-2m)(1+i+2m)B_0(i+3,m)=0 \end{aligned} \tag{79}$$

and

$$\begin{aligned} &2(2+i)(3+i)(1+i-2m)(i+2m)B_2(i,m)\\ &\quad+(3+i)(3i^2+i^3+8m(-1+2m)+i(2-2m+4m^2))B_2(i+1,m)\\ &\quad-2(1+i)(24+14i^2+2i^3+5m-10m^2+i(32+m-2m^2))B_2(i+2,m)\\ &\quad-2(2+i)(6+6i^2+i^3+i(11+2m-4m^2))B_2(i+3,m)\\ &\quad+2(1+i)(3+i)(8+6i+i^2+m-2m^2)B_2(i+4,m)\\ &\quad+(1+i)(2+i)(4+i-2m)(3+i+2m)B_2(i+5,m)=0. \end{aligned} \tag{80}$$

• *Certificate proof of (79):* Define $f(k,i,m) := \binom{i+k-3}{i-2} h_1(k,m)$. The main step of the proof is the certificate recurrence

$$\Delta_k g(k,i,m) = c_0(i,m)f(k,i,m) + \cdots + c_3(i,m)f(k,i+3,m) \tag{81}$$

given by

$$\begin{aligned}
c_0(i,m) &= (1-i)i(1+i)(2+i+i^2)(-1+i-2m)(-2+i+2m),\\
c_1(i,m) &= (-1+i)i(1+i)(3+i)(-2+2i-i^2+i^3-2m+4m^2),\\
c_2(i,m) &= (-3+i)(-1+i)i(1+i)(2+2i+i^2+i^3+2m-4m^2),\\
c_3(i,m) &= (1-i)i(1+i)(2-i+i^2)(2+i-2m)(1+i+2m),
\end{aligned}$$

and

$$\begin{aligned}
g(k,i,m) = \Big[p_0(i,m)h_1(k,i,m) + p_1(i,m)h_1(k+1,i,m) + p_2(i,m)h_1(k+2,i,m)\Big]&\\
\times \frac{k-1}{(1+k)^2}\binom{i+k-3}{i-2}&
\end{aligned}$$

where

$$\begin{aligned}
p_0(i,m) =& -(12-6i-12i^2+6i^3+26k-82ik+7i^2k+16i^3k-3i^4k-24k^2\\
&-106ik^2+73i^2k^2-4i^3k^2+i^4k^2-60k^3+16ik^3+59i^2k^3-16i^3k^3\\
&+9i^4k^3-12k^4+60ik^4-i^2k^4+4i^3k^4+5i^4k^4+14k^5+12ik^5-4i^2k^5+6i^3k^5+4k^6\\
&-2ik^6+2i^2k^6+12m-90im+54i^2m+10i^3m-2i^4m-46km-20ikm\\
&+76i^2km-14i^3km+8i^4km-24k^2m+64ik^2m-4i^2k^2m+2i^3k^2m+6i^4k^2m\\
&+14k^3m+12ik^3m-4i^2k^3m+6i^3k^3m+4k^4m-2ik^4m+2i^2k^4m-24m^2+180im^2\\
&-108i^2m^2-20i^3m^2+4i^4m^2+92km^2+40ikm^2-152i^2km^2+28i^3km^2-16i^4km^2\\
&+48k^2m^2-128ik^2m^2+8i^2k^2m^2-4i^3k^2m^2-12i^4k^2m^2-28k^3m^2-24ik^3m^2\\
&+8i^2k^3m^2-12i^3k^3m^2-8k^4m^2+4ik^4m^2-4i^2k^4m^2),\\
p_1(i,m) =&(-2+i+k)(-10+10i^2-32k+39ik+22i^2k-i^3k-11k^2+85ik^2+5i^2k^2+5i^3k^2\\
&+34k^3+53ik^3-7i^2k^3+10i^3k^3+29k^4+4ik^4+3i^2k^4+4i^3k^4+6k^5-3ik^5+3i^2k^5\\
&-10m+18im+10i^2m-2i^3m-2km+40ikm+4i^2km+2i^3km+20k^2m+16ik^2m\\
&-6i^2k^2m+6i^3k^2m+8k^3m-4ik^3m+4i^2k^3m+20m^2-36im^2-20i^2m^2\\
&+4i^3m^2+4km^2-80ikm^2-8i^2km^2-4i^3km^2-40k^2m^2-32ik^2m^2\\
&+12i^2k^2m^2-12i^3k^2m^2-16k^3m^2+8ik^3m^2-8i^2k^3m^2),\\
p_2(i,m) =& -(-2+i+k)(-1+i+k)(2+2i+5k+ik+2k^2\\
&-ik^2+i^2k^2)(2+k-2m)(1+k+2m).
\end{aligned}$$

The recurrence (81) holds for all $2 \le k \le 2m-4$ which can be checked as follows: Express (81) in terms of the generators $h_1(k,m), h_1(k+1,m), h_1(k+2,m)$, and $q(k,i) := \binom{i+k-3}{i-2}$. To this end, one uses (71) and the relations $q(k,i+j) = q_j\, q(k,i)$ where $q_1 = \frac{i+k-2}{i-1}$, $q_2 = \frac{(i+k-2)(i+k-1)}{(i-1)i}$, and $q_3 = \frac{(i+k-2)(i+k-1)(i+k)}{(i-1)i(i+1)}$; afterwards verify equality (81) by polynomial arithmetic. Finally, summing equation (81) over k from 0 to $2m-4$ and compensating missing terms leads to recurrence (79); see also [APS04, Remark 2].

- *Certificate proof of (80):* Analogous to the proof of (79); see [APS04, Remark 3].

(2) To prove (78) we follow the proof method in Section 4.1; i.e., we combine the recurrences for the P-finite components of (78) into one recurrence of order 9 that is satisfied by the left hand side of (78) for all $2 \le i \le 2m$. By showing that its leading coefficient[6] does not vanish and by checking that the first nine initial values[7] are 0, identity (78), and thus (33), is proven. ☐

[6] We used the CAD method [CH91] to check that the leading coefficient does not vanish; see [APS04, Remark 4].

[7] The initial values are given by the left hand side of (78) with $i \in \{2,\ldots,10\}$. Checking the initial values boils down to indefinite summation problems of the type (75); see [APS04, Remark 5].

5.4. **Identity (32).** Using Proposition 3 and Lemma 2 identity (32) is simplified to

$$6(-1)^i A_2(i,m) - A_0(i,m) + 2h_2(i-2,m) - 5h_2(i-1,m) - 3(-1)^i t_1(2m-1) = 0 \tag{82}$$

for all $3 \leq i \leq 2m+1$. We can carry out the proof of identity (82) in a fashion completely analogous to the proof of identity (33). Namely, we first produce and verify with the methods from Section 4.5 the recurrences

$$\begin{aligned}
&(-2-i-i^2)(2-3i+i^2-2m-4m^2)A_0(i,m)\\
&\qquad + (3+i)(-2+2i-i^2+i^3+2m+4m^2)A_0(i+1,m)\\
&\qquad + (-3+i)(2+2i+i^2+i^3-2m-4m^2)A_0(i+2,m)\\
&\qquad\qquad - (2-i+i^2)(2+3i+i^2-2m-4m^2)A_0(i+3,m) = 0
\end{aligned}$$

and

$$\begin{aligned}
&2(6+5i+i^2)(i+i^2-2m(1+2m))A_2(i,m)\\
&\qquad + (3+i)(3i^2+i^3+8m(1+2m)+2i(1+m+2m^2))A_2(i+1,m)\\
&\qquad - 2(1+i)(24+14i^2+2i^3-5m-10m^2-i(-32+m+2m^2))A_2(i+2,m)\\
&\qquad - 2(2+i)(6+6i^2+i^3+i(11-2m-4m^2))A_2(i+3,m)\\
&\qquad + 2(3+4i+i^2)(8+6i+i^2-m-2m^2)A_2(i+4)\\
&\qquad\qquad + (2+3i+i^2)(7i+i^2-2(-6+m+2m^2))A_2(i+5,m) = 0.
\end{aligned}$$

Both recurrences hold for $3 \leq i \leq 2m+1$. In the next step we combine them together with (73) (k substituted with $i-1$ or $i-2$) and with $M(i) + M(i+1) = 0$ into one recurrence of order 9 that is satisfied by the left hand side of (82). A check of initial values completes the proof; all details, including the proof certificates, are given in [APS04, Section 7]. □

5.5. **Propositions 3 and 5.** *Certificate proof of (12):* Define $f(k,m) := (-1)^k h_1(k,m)$. The main step of the proof is the certificate recurrence

$$\Delta_k g(k,m) = c_0(m) f(k,m) + c_1(m) f(k,m+1) \tag{83}$$

given by $c_0(m) = 12m(1+2m)(-1+3m)(1+3m)$, $c_1(m) = -4m(1+2m)(-1+4m)(1+4m)$ and

$$\begin{aligned}
g(k,m) = &-\frac{k(4m+1)(-1)^k}{(k+1)^2(2m-k+1)(2m-k)}\\
&\qquad \times \left[p_0(k,m)h_1(k,m) + p_1(k,m)h_1(k+1,m) + p_2(k,m)h_1(k+2,m)\right]
\end{aligned}$$

where

$$\begin{aligned}
p_0(k,m) =& 2(-2k-5k^2-2k^3+4k^4+4k^5+k^6+8m+6km-11k^2m-19k^3m-15k^4m\\
&-5k^5m+8m^2+32km^2+6k^2m^2-2k^3m^2+4k^4m^2-64m^3-36km^3+52k^2m^3\\
&+36k^3m^3-192km^4-144k^2m^4+192m^5+144km^5),\\
p_1(k,m) =& 6k+15k^2+6k^3-12k^4-12k^5-3k^6-16m-2km+37k^2m+45k^3m+33k^4m\\
&+11k^5m-40m^2-16km^2+54k^2m^2+46k^3m^2+4k^4m^2+8m^3-12km^3\\
&-92k^2m^3-60k^3m^3+96km^4+144k^2m^4-96m^5-144km^5,\\
p_2(k,m) =& (2+k-2m)(1+k+2m)(-k-k^2+k^3+k^4+2m-3km-2k^2m-3k^3m+6m^2\\
&-12km^2+12m^3).
\end{aligned}$$

The recurrence (83) holds for all $3 \leq k \leq 2m-4$ which can be checked as follows: Represent (83) in terms of the generators $h_1(k,m)$, $h_1(k+1,m)$, $h_1(k+2,m)$ and $(-1)^k$ by using the relations (71) and (72), and verify (83) by polynomial arithmetic. Next, summing equation (83) over k from 0 to $2m-4$ and compensating missing terms leads to the recurrence

$$3(3m-1)(3m+1)S(m)-(4m-1)(4m+1)S(m+1)=0 \tag{84}$$

that is satisfied for all $m\geq 1$ by the left hand side of (12); see [APS04, Remark 12]. Since $t_1(2m-1)$ is also a solution of (84), identity (12) follows by checking that both sides are equal at $m=1$. □

Certificate proofs of (11), (16) and (17): In analogous fashion; see the Remarks 12, 15, and 16 in [APS04]. □

5.6. **Identities (35) and (47).** The proofs of the identities (35) and (47) are completely analogous; see Remarks 13 and 15 in [APS04]. We only sketch the

Certificate proof of (35): Define $f(k,m):=\binom{2m+k-2}{2m-1}h_1(k,m)$. The main step of the proof is the certificate recurrence

$$\Delta_k g(k,m)=c_0(m)f(k,m)+\cdots+c_3(m)f(k,m+3) \tag{85}$$

where

$$g(k,m)=\frac{-(k-1)\big[p_0(k,m)h(k,m)+p_1(k,m)h(k+1,m)+p_2(k,m)h(k+2,m)\big]}{(k+1)^2\prod_{i=0}^{5}(2m-k+i)\prod_{i=2}^{4}(2m+k+i)} \times\binom{2m+k-2}{2m-1},$$

and where the polynomials[8] $c_i(m)$ and $p_i(k,m)$ are given in [APS04, Remark 13]. The recurrence (85) holds for all $0\leq k\leq 2m-4$. Similarly to the proof of identity (12) we can produce from (85) a recurrence of order three (together with a correctness proof) that is satisfied for $m\ \geq 1$ by the left hand side of (35). Since we can construct the same recurrence for the right hand side (35) by the method given in Subsection 4.1, identity (35) is proved by checking that both sides are equal for the first three initial values. □

6. Conclusion

There are two essential observations arising from this work. First, as noted in the introduction, the challenge of proving equation (6) has led to significant new discoveries in methods of summation [Sch04a] and [PS04].

Second, and most tantalizing, Okada's theorem [Oka89, Sec. 4] as stated in the introduction was, in fact, originally given for the general q case not just $q=1$ as presented here. So one would very much like to produce the q-analog of (6) which would then complete the proof of all the classical finite plane partition product formulas [Sta86b]. Unfortunately, preliminary study suggests that the q-analog of the matrix $W(n)$ in (6) is substantially more intricate than the already very elaborate $W(n)$ constructed for Theorem 1 in this paper. In addition, it is also reasonable to suppose that the solution of the full Andrews-Robbins conjecture [Sta86b, p. 106] as formulated by Okada for general q along the lines suggested would again lead to further development and refinement of current summations methods.

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