GENERALIZED RECIPROCITY LAWS FOR SUMS OF HARMONIC NUMBERS

MARKUS KUBA, HELMUT PRODINGER, AND CARSTEN SCHNEIDER

Abstract. We present summation identities for generalized harmonic numbers, which generalize *reciprocity laws* discovered when studying the algorithm *quickselect*. Furthermore, we demonstrate how the computer algebra system Sigmacan be used in order to find/prove such identities. We also discuss alternating harmonic sums, as well as limiting relations.

1. Introduction

In the study of Hoare's Find Algorithm (quickselect) [3], the sums

$$\sum_{k=1}^{j}\frac{H_{n-k}}{k}$$

occurred, where $H_n=\sum_{1\le k\le n}\frac{1}{k}$ denotes a *harmonic number*; later we will also need harmonic numbers of higher order: $H_n^{(a)}=\sum_{1\le k\le n}\frac{1}{k^a}$.

The best thing would of course be to find a closed form evaluation, which—for a variety of reasons—is not possible for general j and n. However, since there is an intrinsic symmetry in the algorithm between j and $n+1-j$, it was *a priori* clear that some relations must hold. Indeed, the following two *reciprocity laws* were obtained in [3]:

- Reciprocity for Type 1 sums:

$$\sum_{k=1}^{j}\frac{H_{n-k}}{k}+\sum_{k=1}^{n+1-j}\frac{H_{n-k}}{k}=H_jH_{n+1-j}+H_n^2-H_n^{(2)}-\frac{1}{j(n+1-j)}. \tag{1}$$

- Reciprocity for Type 2 sums:

$$\begin{aligned}\sum_{k=1}^{j}\frac{H_{n+k-j}}{k}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}}{k}&=\frac{1}{2}\Big(H_j^2+H_j^{(2)}\Big)+\frac{1}{2}\Big(H_{n+1-j}^2+H_{n+1-j}^{(2)}\Big)\\&\quad+H_jH_{n+1-j}+\frac{1}{j(n+1-j)}+\frac{n+1}{j(n+1-j)}\Big(H_n-H_j-H_{n+1-j}\Big).\end{aligned}$$

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In the present paper, we want to generalize these to Type 1 sums

$$\sum_{k=1}^{j} \frac{H_{n-k}^{(a)}}{k^b}$$

and Type 2 sums

$$\sum_{k=1}^{j} \frac{H_{n+k-j}^{(a)}}{k^b}.$$

Notice that the two reciprocity laws contain two Type 1 resp. Type 2 sums (for j and $n+1-j$) and otherwise only *known* quantities.

Now, in the general case of natural numbers a and b, we are still able to get such a result in the Type 1 instance. In the Type 2 instance the situation is more complicated than in the $a = b = 1$ case. We can either express the sum of two Type 2 sums by known quantities *and* some Type 1 sums, or we can express the sum of *more than two* Type 2 sums by known quantities. This is an intrinsic phenomenon, and simplification *only occurs for* $a = b = 1$.

We also discuss *alternating* harmonic numbers/sums, as well as limiting relations, when $n = 2m+1$ and $j = m+1$ (in this case the sums are invariant under the change $j \leftrightarrow n+1-j$).

One section is devoted to the software package Sigma, created by one of us, which is particulary well suited to handle sums of harmonic numbers. We describe how it can be used to guess and prove the identities in question.

2. Reciprocity for Type 1 sums

The following theorem treats the reciprocity of Type 1 sums.

Theorem 1.

$$\begin{aligned}\sum_{k=1}^{j} \frac{H_{n-k}^{(a)}}{k^b} + \sum_{k=1}^{n+1-j} \frac{H_{n-k}^{(b)}}{k^a} &= -\frac{1}{j^b(n+1-j)^a} + H_j^{(b)} H_{n+1-j}^{(a)} + \sum_{k=1}^{n} \frac{H_{n-k}^{(b)}}{k^a} \\ &= -\frac{1}{j^b(n+1-j)^a} + H_j^{(b)} H_{n+1-j}^{(a)} + R_n^{(a,b)},\end{aligned}$$

with

$$R_n^{(a,b)} = \sum_{i=1}^{a} \binom{i+b-2}{b-1} \zeta_n(i+b-1, a+1-i) + \sum_{i=1}^{b} \binom{i+a-2}{a-1} \zeta_n(i+a-1, b+1-i), \tag{2}$$

where the multiple zeta function and its finite counterpart are defined as follows:

$$\zeta(a_1, \ldots, a_l) := \sum_{n_1 > n_2 > \cdots > n_l \ge 1} \frac{1}{n_1^{a_1} n_2^{a_2} \ldots n_l^{a_l}},$$

$$\zeta_N(a_1, \ldots, a_l) := \sum_{N \ge n_1 > n_2 > \cdots > n_l \ge 1} \frac{1}{n_1^{a_1} n_2^{a_2} \ldots n_l^{a_l}}.$$

Proof.

$$\begin{aligned}\sum_{k=1}^{j}\frac{H_{n-k}^{(a)}}{k^b} &= H_{n-j}^{(a)}H_j^{(b)}+\sum_{k=1}^{j}\frac{1}{k^b}\sum_{l=n+1-j}^{n-k}\frac{1}{l^a} = H_{n-j}^{(a)}H_j^{(b)}+\sum_{l=n+1-j}^{n-1}\frac{1}{l^a}\sum_{k=1}^{n-l}\frac{1}{k^b}\\ &= H_{n-j}^{(a)}H_j^{(b)}+\sum_{l=n+1-j}^{n}\frac{H_{n-l}^{(b)}}{l^a} = H_{n+1-j}^{(a)}H_j^{(b)}-\frac{1}{j^b(n+1-j)^a}+\sum_{l=n+2-j}^{n}\frac{H_{n-l}^{(b)}}{l^a},\end{aligned}$$

which proves the first part. For $R_n^{(a,b)}$ we use the partial fraction decomposition (which appears already in [4].

$$\frac{1}{k^a(n-k)^b}=\sum_{i=1}^{a}\frac{\binom{i+b-2}{b-1}}{n^{i+b-1}k^{a+1-i}}+\sum_{i=1}^{b}\frac{\binom{i+a-2}{a-1}}{n^{i+a-1}(n-k)^{b+1-i}},$$

which leads to

$$\begin{aligned}R_n^{(a,b)}-R_{n-1}^{(a,b)} &= \sum_{k=1}^{n-1}\frac{1}{k^a(n-k)^b} = \sum_{i=1}^{a}\frac{\binom{i+b-2}{b-1}}{n^{i+b-1}}\sum_{k=1}^{n-1}\frac{1}{k^{a+1-i}}+\sum_{i=1}^{b}\frac{\binom{i+a-2}{a-1}}{n^{i+a-1}}\sum_{k=1}^{n-1}\frac{1}{(n-k)^{b+1-i}}\\ &= \sum_{i=1}^{a}\frac{\binom{i+b-2}{b-1}}{n^{i+b-1}}H_{n-1}^{(a+1-i)}+\sum_{i=1}^{b}\frac{\binom{i+a-2}{a-1}}{n^{i+a-1}}H_{n-1}^{(b+1-i)}.\end{aligned}$$

By iterating we get

$$\begin{aligned}R_n^{(a,b)} &= \sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{n}\frac{H_{k-1}^{(a+1-i)}}{k^{i+b-1}}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=1}^{n}\frac{H_{k-1}^{(b+1-i)}}{k^{i+a-1}}\\ &= \sum_{i=1}^{a}\binom{i+b-2}{b-1}\zeta_n(i+b-1,a+1-i)+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta_n(i+a-1,b+1-i).\end{aligned}$$

□

Setting $a=b=1$ in Theorem 1 we get the corresponding result of [3], since

$$R_n^{(1,1)}=H_n^2-H_n^{(2)}. \tag{3}$$

Note that $R_n^{(a,b)}$ equals $R_n^{(b,a)}$, which can easily be seen to be true using the *polylogarithms* $\operatorname{Li}_a(z) := \sum_{n\geq 1} z^n/n^a$.

$$R_n^{(a,b)}=\sum_{k=1}^{n}\frac{H_{n-k}^{(a)}}{k^b}=\sum_{k=1}^{n-1}[z^k]\operatorname{Li}_b(z)[z^{n-k}]\frac{\operatorname{Li}_a(z)}{1-z}=[z^n]\frac{\operatorname{Li}_a(z)\operatorname{Li}_b(z)}{1-z}=\sum_{k=1}^{n}\frac{H_{n-k}^{(b)}}{k^a}=R_n^{(b,a)}.$$

In any case, we treat $R_n^{(a,b)}$ as *known quantities* and notice that they can be expressed by (finite) multiple zeta values.

We can state some corollories of this reciprocity theorem for Type 1 sums.

Corollary 1.

$$\sum_{k=1}^{j-1}\frac{H_k^{(b)}}{(n-k)^a}+\sum_{k=1}^{n-j}\frac{H_k^{(a)}}{(n-k)^b}=\frac{H_{n-j}^{(a)}}{j^b}+\frac{H_{j-1}^{(b)}}{(n+1-j)^a}+\frac{1}{j^b(n+1-j)^a}-H_j^{(b)}H_{n+1-j}^{(a)}+\sum_{k=1}^{n}\frac{H_{n-k}^{(a)}}{k^b}.$$

Proof. Observing that

$$\sum_{k=1}^{j}\frac{H_k^{(b)}}{(n-k)^a}=\sum_{k=n-j}^{n-1}\frac{H_{n-k}^{(b)}}{k^a}=\frac{H_j^{(b)}}{(n-j)^a}+\frac{H_{j-1}^{(b)}}{(n+1-j)^a}+\sum_{k=n+2-j}^{n-1}\frac{H_{n-k}^{(b)}}{k^a},$$

$$\sum_{k=1}^{n+1-j}\frac{H_k^{(a)}}{(n-k)^b}=\sum_{k=j-1}^{n-1}\frac{H_{n-k}^{(a)}}{k^b}=\frac{H_{n+1-j}^{(a)}}{(j-1)^b}+\frac{H_{n-j}^{(a)}}{j^b}+\sum_{k=j+1}^{n-1}\frac{H_{n-k}^{(a)}}{k^b},$$

and adding these sums to the left hand side of the Type 1 reciprocity identity gives the desired result. □

Corollary 2.

$$\sum_{k=1}^{j}\frac{H_{n-k}^{(a)}}{k^b}-\sum_{k=1}^{j}\frac{H_k^{(b)}}{(n-k)^a}=H_{n-1-j}^{(a)}H_j^{(b)}.$$

Proof.

$$\sum_{k=1}^{j}\frac{H_{n-k}^{(a)}}{k^b}=H_{n-1-j}^{(a)}H_j^{(b)}+\sum_{k=1}^{j}\frac{1}{k^b}\sum_{l=n-j}^{n-k}\frac{1}{l^a}=H_{n-1-j}^{(a)}H_j^{(b)}+\sum_{k=1}^{j}\frac{1}{k^b}\sum_{l=k}^{j}\frac{1}{(n-l)^a}$$
$$=H_{n-1-j}^{(a)}H_j^{(b)}+\sum_{l=1}^{j}\frac{1}{(n-l)^a}\sum_{k=1}^{l}\frac{1}{k^b}.$$

□

3. Reciprocity for Type 2 sums

For the evaluation of Type 2 sums we observe the following:

$$\sum_{k=1}^{j}\frac{H_{n+k-j}^{(b)}}{k^a}=H_j^{(a)}H_n^{(b)}-\sum_{k=1}^{j-1}\frac{1}{k^a}\sum_{l=n+1-j+k}^{n}\frac{1}{l^b}=H_j^{(a)}H_n^{(b)}-S.$$

Further we get

$$S:=\sum_{k=1}^{j-1}\frac{1}{k^a}\sum_{l=n+1-j+k}^{n}\frac{1}{l^b}=\sum_{k=1}^{j-1}\frac{1}{k^a}\sum_{l=k}^{j-1}\frac{1}{(n+1-j+l)^b}=\sum_{k=1}^{j-1}\frac{1}{k^a}\sum_{l=1}^{j-k}\frac{1}{(n+k-j+l)^b},$$

and by using the partial fraction decomposition

$$\frac{1}{k^a(k-r)^b}=\sum_{i=1}^{a}(-1)^{i+1}\frac{\binom{i+b-2}{b-1}}{(-r)^{i+b-1}k^{a+1-i}}+\sum_{i=1}^{b}\frac{(-1)^a\binom{i+a-2}{a-1}}{(-r)^{i+a-1}(k-r)^{b+1-i}}\tag{4}$$

with $-r = n+l-j$

$$S = \sum_{k=1}^{j-1}\sum_{l=1}^{j-k}\Bigg(\sum_{i=1}^{a}(-1)^{i+1}\frac{\binom{i+b-2}{b-1}}{(n+l-j)^{i+b-1}k^{a+1-i}} + \sum_{i=1}^{b}\frac{(-1)^a\binom{i+a-2}{a-1}}{(n+l-j)^{i+a-1}(k+n+l-j)^{b+1-i}}\Bigg).$$

We set $S = S_1 + S_2$. The first sum can easily be simplified into Type 1 sums:

$$\begin{aligned} S_1 &= \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{j-1}\sum_{l=1}^{j-k}\frac{1}{(n+l-j)^{i+b-1}k^{a+1-i}} \\ &= \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{j-1}\frac{H_{n-k}^{(i+b-1)} - H_{n-j}^{(i+b-1)}}{k^{a+1-i}} \\ &= \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{j}\frac{H_{n-k}^{(i+b-1)}}{k^{a+1-i}} - \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}H_{n-j}^{(i+b-1)}H_j^{(a+1-i)}. \end{aligned} \tag{5}$$

Now we use Theorem 1 to translate the sums $\sum_{k=1}^{j}\frac{H_{n-k}^{(i+b-1)}}{k^{a+1-i}}$ into $\sum_{k=1}^{n+1-j}\frac{H_{n-k}^{(a+1-i)}}{k^{i+b-1}}$:

$$\begin{aligned} S_1 &= \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\Bigg(-\frac{1}{j^{a+1-i}(n+1-j)^{i+b-1}} + H_j^{(a+1-i)}H_{n+1-j}^{(i+b-1)} + \sum_{k=1}^{n}\frac{H_{n-k}^{(a+1-i)}}{k^{i+b-1}}\Bigg) \\ &- \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{n+1-j}\frac{H_{n-k}^{(a+1-i)}}{k^{i+b-1}} - \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}H_{n-j}^{(i+b-1)}H_j^{(a+1-i)} \\ &= -\sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{n+1-j}\frac{H_{n-k}^{(a+1-i)}}{k^{i+b-1}} + \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{n}\frac{H_{n-k}^{(a+1-i)}}{k^{i+b-1}} \\ &+ \sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\Bigg(\frac{H_j^{(a+1-i)}}{(n+1-j)^{i+b-1}} - \frac{1}{j^{a+1-i}(n+1-j)^{i+b-1}}\Bigg) = S_{11} + S_{12}, \end{aligned} \tag{6}$$

where S_{11} contains all the Type 1 sums. For S_2 we get

$$\begin{aligned}
S_2 &= (-1)^a \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{k=1}^{j-1} \sum_{l=1}^{j-k} \frac{1}{(n+l-j)^{i+a-1}(k+n+l-j)^{b+1-i}} \\
&= (-1)^a \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{l=1}^{j-1} \sum_{k=1}^{j-l} \frac{1}{(n+l-j)^{i+a-1}(k+n+l-j)^{b+1-i}} \\
&= (-1)^a \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{l=1}^{j} \frac{H_n^{(b+1-i)} - H_{n-j+l}^{(b+1-i)}}{(n+l-j)^{i+a-1}} \\
&= (-1)^a \sum_{i=1}^{b} \binom{i+a-2}{a-1} H_n^{(b+1-i)} (H_n^{(a+i-1)} - H_{n-j}^{(a+i-1)}) - (-1)^a \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{l=1}^{n} \frac{H_k^{(b+1-i)}}{k^{a+i-1}} \\
&+ (-1)^a \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{l=1}^{n-j} \frac{H_k^{(b+1-i)}}{k^{a+i-1}}.
\end{aligned} \tag{7}$$

Now we turn to the next sum $\sum_{k=1}^{n+1-j} \frac{H_{j+k-1}^{(a)}}{k^b}$. We use the following lemma.

Lemma 1.

$$\sum_{k=1}^{n+1-j} \frac{H_{j+k-1}^{(a)}}{k^b} + \sum_{k=j+1}^{n} \frac{H_{k-j}^{(b)}}{k^a} = H_n^{(a)} H_{n+1-j}^{(b)}.$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{n+1-j} \frac{H_{j+k-1}^{(a)}}{k^b} &= H_n^{(a)} H_{n+1-j}^{(b)} - \sum_{k=1}^{n-j} \sum_{l=j+k}^{n} \frac{1}{l^a k^b} = H_n^{(a)} H_{n+1-j}^{(b)} - \sum_{l=j+1}^{n} \sum_{k=1}^{l-j} \frac{1}{l^a k^b} \\
&= H_n^{(a)} H_{n+1-j}^{(b)} - \sum_{k=j+1}^{n} \frac{H_{k-j}^{(b)}}{k^a}.
\end{aligned}$$

□

Thus we have reduced the problem to the computation of $T := \sum_{k=j+1}^{n} \frac{H_{k-j}^{(b)}}{k^a}$:

$$\begin{aligned}
T &= \sum_{k=j+1}^{n} \frac{H_{k-j}^{(b)}}{k^a} = \sum_{k=j+1}^{n} \sum_{l=1}^{k-j} \frac{1}{k^a l^b} = \sum_{k=j+1}^{n} \sum_{l=1}^{k-j} \frac{1}{k^a (k+1-j-l)^b} = \sum_{k=j+1}^{n} \sum_{l=1}^{k-j} \frac{1}{k^a (k-(l+j-1))^b} \\
&= \sum_{k=j+1}^{n} \sum_{l=1}^{k-j} \left(\sum_{i=1}^{a} \frac{(-1)^b \binom{i+b-2}{b-1}}{(l+j-1)^{i+b-1} k^{a+1-i}} + \sum_{i=1}^{b} \frac{(-1)^{i+1} \binom{i+a-2}{a-1}}{(l+j-1)^{i+a-1} (k-l-j+1)^{b+1-i}} \right).
\end{aligned}$$

By splitting $T = T_1 + T_2$ we further get

$$
\begin{aligned}
T_1 &= \sum_{k=j+1}^{n}\sum_{l=1}^{k-j}\sum_{i=1}^{a}\frac{(-1)^b\binom{i+b-2}{b-1}}{(l+j-1)^{i+b-1}k^{a+1-i}} = (-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=j+1}^{n}\frac{H_{k-1}^{(i+b-1)}-H_{j-1}^{(i+b-1)}}{k^{a+1-i}}\\
&= (-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=j+1}^{n}\frac{H_k^{(i+b-1)}}{k^{a+1-i}} - (H_n^{(a+b)}-H_j^{(a+b)})(-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\\
&- (-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}H_{j-1}^{(i+b-1)}(H_n^{(a+1-i)}-H_j^{(a+1-i)})\\
&= (-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{n}\frac{H_k^{(i+b-1)}}{k^{a+1-i}} - (-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{j}\frac{H_k^{(i+b-1)}}{k^{a+1-i}}\\
&- (-1)^b\binom{a+b-1}{b}(H_n^{(a+b)}-H_j^{(a+b)}) - (-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}H_{j-1}^{(i+b-1)}(H_n^{(a+1-i)}-H_j^{(a+1-i)}).
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
T_2 &= \sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\sum_{k=j+1}^{n}\sum_{l=1}^{k-j}\frac{1}{(l+j-1)^{i+a-1}(k-l-j+1)^{b+1-i}}\\
&= \sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\sum_{l=1}^{n-j}\sum_{k=l+j}^{n}\frac{1}{(l+j-1)^{i+a-1}(k-l-j+1)^{b+1-i}}\\
&= \sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\sum_{l=1}^{n-j}\frac{H_{n-(l+j-1)}^{(b+1-i)}}{(l+j-1)^{i+a-1}} = \sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\sum_{l=j}^{n-1}\frac{H_{n-l}^{(b+1-i)}}{l^{i+a-1}}\\
&= \sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\left(\frac{H_{n-j}^{(b+1-i)}}{j^{i+a-1}}+\sum_{l=1}^{n-1}\frac{H_{n-l}^{(b+1-i)}}{l^{i+a-1}}\right) - \sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\sum_{l=1}^{j}\frac{H_{n-l}^{(b+1-i)}}{l^{i+a-1}}\\
&= T_{21}+T_{22},
\end{aligned}
\tag{9}
$$

where T_{22} contains all the $\mathsf{Type\ 1}$ sums. Combination of all the results and using the *basic identity* for harmonic numbers

$$
\sum_{k=1}^{n}\frac{H_k^{(a)}}{k^b}+\sum_{k=1}^{n}\frac{H_k^{(b)}}{k^a} = H_n^{(a)}H_n^{(b)}+H_n^{(a+b)} \tag{10}
$$

leads to the following theorem, which relates $\mathsf{Type\ 2}$ sums to $\mathsf{Type\ 1}$ sums.

Theorem 2.

$$\sum_{k=1}^{j}\frac{H_{n+k-j}^{(b)}}{k^a}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(a)}}{k^b}=$$
$$=\sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}\sum_{k=1}^{j-1}\frac{H_{n-k}^{(b+1-i)}}{k^{i+a-1}}+\sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}\sum_{k=1}^{n-j}\frac{H_{n-k}^{(a+1-i)}}{k^{i+b-1}}$$
$$-\sum_{i=1}^{a}(-1)^{i+1}\binom{i+b-2}{b-1}R_n^{(a+1-i,i+b-1)}-\sum_{i=1}^{b}(-1)^{i+1}\binom{i+a-2}{a-1}R_n^{(b+1-i,i+a-1)}$$
$$+H_n^{(b)}H_j^{(a)}+H_n^{(a)}H_{n+1-j}^{(b)}-(-1)^b\binom{a+b-1}{b}H_{j-1}^{(a+b)}-(-1)^a\binom{a+b-1}{a}H_{n-j}^{(a+b)}$$
$$-(-1)^a\sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta_{n-j}(a+i-1,b+1-i)-(-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\zeta_{j-1}(b+i-1,a+1-i)$$
$$-(-1)^a\sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta_n(b+1-i,a+i-1)-(-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}\zeta_n(a+1-i,b+i-1)$$
$$+(-1)^b\sum_{i=1}^{a}\binom{i+b-2}{b-1}H_{j-1}^{(i+b-1)}H_n^{(a+1-i)}+(-1)^a\sum_{i=1}^{b}\binom{i+a-2}{a-1}H_{n-j}^{(i+a-1)}H_n^{(b+1-i)},$$

where $R_n^{(a,b)}$ is given in (2).

Since this formula is quite involved, it is worthwhile to state the instance $a=b$ explicitly, which is more attractive.

Corollary 3.

$$\sum_{k=1}^{j}\frac{H_{n+k-j}^{(a)}}{k^a}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(a)}}{k^a}=$$
$$=\sum_{i=1}^{a}(-1)^{i+1}\binom{i+a-2}{a-1}\Bigg[\sum_{k=1}^{j-1}\frac{H_{n-k}^{(a+1-i)}}{k^{i+a-1}}+\sum_{k=1}^{n-j}\frac{H_{n-k}^{(a+1-i)}}{k^{i+a-1}}\Bigg]$$
$$-2\sum_{i=1}^{a}(-1)^{i+1}\binom{i+a-2}{a-1}R_n^{(a+1-i,i+a-1)}-(-1)^a2\sum_{i=1}^{a}\binom{i+a-2}{a-1}\zeta_n(a+1-i,a+i-1)$$
$$+H_n^{(a)}H_j^{(a)}+H_n^{(a)}H_{n+1-j}^{(a)}-(-1)^a\binom{2a-1}{a}\big(H_{j-1}^{(2a)}+H_{n-j}^{(2a)}\big)$$
$$-(-1)^a\sum_{i=1}^{a}\binom{i+a-2}{a-1}\Big[\zeta_{j-1}(a+i-1,a+1-i)+\zeta_{n-j}(a+i-1,a+1-i)\Big]$$
$$+(-1)^a\sum_{i=1}^{a}\binom{i+a-2}{a-1}\Big[H_{j-1}^{(i+a-1)}H_n^{(a+1-i)}+H_{n-j}^{(i+a-1)}H_n^{(a+1-i)}\Big].$$

We note that the instance $a=1$ is somewhat special, as then the Type 1 sums (the first line of the righthand side) can be explicitly summed using the reciprocity law for Type 1 sums. Then using the special case

$$\zeta_n(a,a) = \frac{1}{2}\big(H_n^{(a)}{}^2 - H_n^{(2a)}\big) \tag{11}$$

of (10) we get the second reciprocity law from page 1. For $a \geq 2$, this pleasant feature unfortunately fails to hold. Nevertheless, we can use the identity of Theorem 1 in the form

$$\sum_{k=1}^{n-j} \frac{H_{n-k}^{(b)}}{k^a} = -\sum_{k=1}^{j} \frac{H_{n-k}^{(a)}}{k^b} - \frac{H_{j-1}^{(b)}}{(n+1-j)^a} - \frac{1}{j^b(n+1-j)^a} + H_j^{(b)} H_{n+1-j}^{(a)} + R_n^{(a,b)}$$

in order to eliminate the sums of type $\sum_{k=1}^{n-j} \frac{H_{n-k}^{(b)}}{k^a}$ in terms of the sums of type $\sum_{k=1}^{j} \frac{H_{n-k}^{(a)}}{k^b}$. Using in addition (11) we obtain for instance for $a = 2$ the identity

$$\begin{aligned}
&\sum_{k=1}^{j} \frac{H_{n+k-j}^{(2)}}{k^2} + \sum_{k=1}^{n+1-j} \frac{H_{j+k-1}^{(2)}}{k^2} \\
&\quad = (H_n^{(2)}(H_j^{(2)} + H_{j-1}^{(2)} + H_{n-j}^{(2)} + H_{n+1-j}^{(2)} - H_n^{(2)}) + 2H_n(H_{j-1}^{(3)} + H_{n-j}^{(3)}) + \frac{2}{j^4} - \frac{5H_j^{(4)}}{2} \\
&\qquad + H_j^{(2)}\Big(\frac{1}{j^2} - \frac{1}{2}H_j^{(2)}\Big) + H_{n-j}^{(2)}(H_{j-1}^{(2)} - \frac{1}{2}H_{n-j}^{(2)}) - 2H_j H_{n-j}^{(3)} + H_n^{(4)} - \frac{5}{2}H_{n-j}^{(4)} \\
&\qquad - 2\sum_{k=1}^{j-1} \frac{H_{n-k}}{k^3} + 2\sum_{k=1}^{j} \frac{H_{n-k}^{(3)}}{k} - 2\zeta_{j-1}(3,1) - 2\zeta_n(1,3) - 2\zeta_{n-j}(3,1).
\end{aligned} \tag{12}$$

Although such an expression is "shorter," it doesn't display the symmetry between j and $n+1-j$ anymore.

As mentioned in the Introduction, we henceforth try to find a *linear combination* of Type 2 sums, which *can* be explicitly evaluated. To do so, we start with Type 1 sums

$$U_{n,j}^{(a,b)} = \sum_{k=1}^{j} \frac{H_{n-k}^{(b)}}{k^a}.$$

Then by similar arguments as in previous computations, we get

$$U_{n,j}^{(a,b)} - U_{n-1,j}^{(a,b)} = \sum_{k=1}^{j} \frac{1}{k^a(n-k)^b} = \sum_{i=1}^{a} \frac{\binom{i+b-2}{b-1}}{n^{i+b-1}} H_j^{(a+1-i)} + \sum_{i=1}^{b} \frac{\binom{i+a-2}{a-1}}{n^{i+a-1}} (H_{n-1}^{(b+1-i)} - H_{n-1-j}^{(b+1-i)}). \tag{13}$$

Hence by iterating,

$$\begin{aligned}
U_{n,j}^{(a,b)} = &U_{j,j}^{(a,b)} + \sum_{i=1}^{a} \binom{i+b-2}{b-1} \sum_{k=j+1}^{n} \frac{H_j^{(a+1-j)}}{k^{i+b-1}} \\
&+ \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{k=j+1}^{n} \frac{H_{k-1}^{(b+1-i)}}{k^{i+a-1}} - \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{k=j+1}^{n} \frac{H_{k-1-j}^{(b+1-i)}}{k^{i+a-1}} \\
&= R_j^{(a,b)} + \sum_{i=1}^{a} \binom{i+b-2}{b-1} H_j^{(a+1-i)} (H_n^{(i+b-1)} - H_j^{(i+b-1)}) + \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{k=1}^{n} \frac{H_{k-1}^{(b+1-i)}}{k^{i+a-1}}
\end{aligned}$$

$$-\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=1}^{j}\frac{H_{k-1}^{(b+1-i)}}{k^{i+a-1}}-\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=j+1}^{n}\frac{H_{k-j}^{(b+1-i)}}{k^{i+a-1}}$$
$$+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=j+1}^{n}\frac{1}{(k-j)^{b+1-i}k^{i+a-1}},$$

where $R_j^{(a,b)}$ is accessible by (2). By using Lemma 1 we get further

$$-\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=j+1}^{n}\frac{H_{k-j}^{(b+1-i)}}{k^{i+a-1}}=\sum_{i=1}^{b}\binom{i+a-2}{a-1}\Big(-H_n^{(i+a-1)}H_{n+1-j}^{(b+1-i)}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(i+a-1)}}{k^{b+1-i}}\Big),$$

and thus

$$U_{n,j}^{(a,b)}=R_j^{(a,b)}+\sum_{i=1}^{a}\binom{i+b-2}{b-1}H_j^{(a+1-i)}(H_n^{(i+b-1)}-H_j^{(i+b-1)})+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=1}^{n}\frac{H_{k-1}^{(b+1-i)}}{k^{i+a-1}}$$
$$-\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=1}^{j}\frac{H_{k-1}^{(b+1-i)}}{k^{i+a-1}}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=j+1}^{n}\frac{1}{(k-j)^{b+1-i}k^{i+a-1}}$$
$$-\sum_{i=1}^{b}\binom{i+a-2}{a-1}H_n^{(i+a-1)}H_{n+1-j}^{(b+1-i)}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(i+a-1)}}{k^{b+1-i}}.$$

By changing the roles of a, b and j, $n+1-j$ we immediately get

$$U_{n,n+1-j}^{(b,a)}=R_{n+1-j}^{(b,a)}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}H_{n+1-j}^{(b+1-i)}(H_n^{(i+a-1)}-H_{n+1-j}^{(i+a-1)})$$
$$+\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{n}\frac{H_{k-1}^{(a+1-i)}}{k^{i+b-1}}-\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{n+1-j}\frac{H_{k-1}^{(a+1-i)}}{k^{i+b-1}}$$
$$+\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=n+2-j}^{n}\frac{1}{(k-(n+1-j))^{a+1-i}k^{i+b-1}}$$
$$-\sum_{i=1}^{a}\binom{i+b-2}{b-1}H_n^{(i+b-1)}H_j^{(a+1-i)}+\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{j}\frac{H_{n+k-j}^{(i+b-1)}}{k^{a+1-i}}.$$

For the convience of the reader we introduce the notation $V_{n,j}^{(a,b)}=\sum_{k=j+1}^{n}\frac{1}{k^a(k-j)^b}$, where $V_{n,j}^{(a,b)}$ can be decomposed into harmonic numbers in the following way:

$$V_{n,j}^{(a,b)}=\sum_{k=j+1}^{n}\frac{1}{k^a(k-j)^b}=\sum_{k=j+1}^{n}\Bigg[\sum_{i=1}^{a}(-1)^{i+1}\frac{\binom{i+b-2}{b-1}}{(-j)^{i+b-1}k^{a+1-i}}+\sum_{i=1}^{b}\frac{(-1)^a\binom{i+a-2}{a-1}}{(-j)^{i+a-1}(k-j)^{b+1-i}}\Bigg]$$
$$=\sum_{i=1}^{a}(-1)^b\binom{i+b-2}{b-1}\frac{H_n^{(a+1-i)}-H_j^{(a+1-i)}}{j^{i+b-1}}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}(-1)^{i+1}\frac{H_{n-j}^{(b+1-i)}}{j^{i+a-1}}. \tag{14}$$

Of course, $V_{n,j}^{(a,b)}$ is considered to be a known and explicit quantity.

This finally leads to the following theorem, in which an explicit evaluation of a linear combination of Type 2 sums is stated.

Theorem 3.

$$\begin{aligned}
&\sum_{i=1}^{a}\binom{i+b-2}{b-1}\sum_{k=1}^{j}\frac{H_{n+k-j}^{(i+b-1)}}{k^{a+1-i}}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(i+a-1)}}{k^{b+1-i}}\\
&\quad=-\frac{1}{j^b(n+1-j)^a}+H_j^{(b)}H_{n+1-j}^{(a)}\\
&\quad+\sum_{i=1}^{a}\binom{i+b-2}{b-1}H_j^{(a+1-i)}H_j^{(i+b-1)}+\sum_{i=1}^{b}\binom{i+a-2}{a-1}H_{n+1-j}^{(b+1-i)}H_{n+1-j}^{(i+a-1)}\\
&\quad-\sum_{i=1}^{b}\binom{i+a-2}{a-1}V_{n,j}^{(i+a-1,b+1-i)}-\sum_{i=1}^{a}\binom{i+b-2}{b-1}V_{n,n+1-j}^{(i+b-1,a+1-i)}\\
&\quad-\sum_{i=1}^{a}\binom{i+b-2}{b-1}\zeta_j(i+b-1,a+1-i)-\sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta_{n+1-j}(i+a-1,b+1-i),
\end{aligned}$$

where $V_{n,j}^{(a,b)}$ is given by (14).

Again, we offer the more attractive instance $a=b$ explicitly.

Corollary 4.

$$\begin{aligned}
&\sum_{i=1}^{a}\binom{i+a-2}{a-1}\Bigg[\sum_{k=1}^{j}\frac{H_{n+k-j}^{(i+a-1)}}{k^{a+1-i}}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(i+a-1)}}{k^{a+1-i}}\Bigg]=-\frac{1}{j^a(n+1-j)^a}+H_j^{(a)}H_{n+1-j}^{(a)}\\
&+\sum_{i=1}^{a}\binom{i+a-2}{a-1}\Big[H_j^{(a+1-i)}H_j^{(i+a-1)}+H_{n+1-j}^{(a+1-i)}H_{n+1-j}^{(i+a-1)}\Big]\\
&-\sum_{i=1}^{a}\binom{i+a-2}{a-1}\Big[V_{n,j}^{(i+a-1,a+1-i)}+V_{n,n+1-j}^{(i+a-1,a+1-i)}\Big]\\
&-\sum_{i=1}^{a}\binom{i+a-2}{a-1}\Big[\zeta_j(i+a-1,a+1-i)+\zeta_{n+1-j}(i+a-1,a+1-i)\Big].
\end{aligned}$$

The instance $a=1$ produces the old reciprocity for Type 2 sums by using the simplification (11). Similarly, the case $a=2$ can be written in the form

$$\begin{aligned}
&\Bigg[\sum_{k=1}^{j}\frac{H_{n+k-j}^{(2)}}{k^2}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(2)}}{k^2}\Bigg]+2\Bigg[\sum_{k=1}^{j}\frac{H_{n+k-j}^{(3)}}{k}+\sum_{k=1}^{n+1-j}\frac{H_{j+k-1}^{(3)}}{k}\Bigg]=\\
&-\frac{H_j^{(2)}}{j^2}+\frac{1}{2}{H_j^{(2)}}^2+\Big(\frac{1}{j^2}+\frac{1}{(n+1-j)^2}\Big)H_n^{(2)}+\frac{1}{2}\big(H_{n-j}^{(2)}\big)^2+2H_{j-1}H_j^{(3)}+2\Big(\frac{1}{j}+\frac{1}{n+1-j}\Big)H_n^{(3)}\\
&\qquad+\frac{H_j^{(4)}}{2}+\frac{1}{2}H_{n-j}^{(4)}+H_{n-j}^{(2)}H_{j-1}^{(2)}+2H_{n-j}H_{n+1-j}^{(3)}-2\zeta_j(3,1)-2\zeta_{n+1-j}(3,1). \qquad (15)
\end{aligned}$$

4. Alternating harmonic numbers and reciprocity laws.

Let $\overline{H}_n^{(a)} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^a}$ denote the alternating counterpart of the general harmonic numbers. Since we can set up three different linear Euler sums including alterning harmonic numbers

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k^{(a)}}{k^b}, \qquad \sum_{k=1}^{n}\frac{\overline{H}_k^{(a)}}{k^b}, \qquad \sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_k^{(a)}}{k^b},$$

we get two corollaries as an immediate consequence of the general reciprocity.

Corollary 5.

$$\sum_{k=1}^{j}(-1)^{k-1}\frac{\overline{H}_{n-k}^{(a)}}{k^b} + \sum_{k=1}^{n+1-j}(-1)^{k-1}\frac{\overline{H}_{n-k}^{(b)}}{k^a} = \frac{(-1)^n}{j^b(n+1-j)^a} + \overline{H}_j^{(b)}\overline{H}_{n+1-j}^{(a)} + \sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_{n-k}^{(b)}}{k^a}.$$

Corollary 6.

$$\sum_{k=1}^{j}(-1)^{k-1}\frac{H_{n-k}^{(a)}}{k^b} + \sum_{k=1}^{n+1-j}\frac{\overline{H}_{n-k}^{(b)}}{k^a} = \frac{(-1)^j}{j^b(n+1-j)^a} + \overline{H}_j^{(b)}H_{n+1-j}^{(a)} + \sum_{k=1}^{n}\frac{\overline{H}_{n-k}^{(b)}}{k^a}.$$

There is no third corollary, since we can change the roles of a, b and j, $n+1-j$ in Corollary 6.

There exist analogous results to Theorem 2 and Theorem 3 for alternating harmonic numbers, which can be derived in a similar fashion as the respective theorems themselves. However, since the resulting formulæ are quite involved, we refrain from stating them explicitly.

5. Limit results from reciprocity laws

We obtain for $n = 2m+1$, $j = m+1$ from Theorem 1

$$\begin{aligned}\sum_{k=1}^{m+1}\frac{H_{2m+1-k}^{(a)}}{k^b} + \sum_{k=1}^{m+1}\frac{H_{2m+1-k}^{(b)}}{k^a} &= -\frac{1}{(m+1)^{a+b}} + H_{m+1}^{(a)}H_{m+1}^{(b)} + \sum_{k=1}^{2m+1}\frac{H_{2m+1-k}^{(a)}}{k^b}\\ &= -\frac{1}{(m+1)^{a+b}} + H_{m+1}^{(a)}H_{m+1}^{(b)} + R_{2m+1}^{(a,b)},\end{aligned}$$

and by taking the limit $m \to \infty$

Theorem 4.

$$\begin{aligned}\lim_{m\to\infty}\left(\sum_{k=1}^{m+1}\frac{H_{2m+1-k}^{(a)}}{k^b} + \sum_{k=1}^{m+1}\frac{H_{2m+1-k}^{(b)}}{k^a}\right) &= \zeta(a)\zeta(b) + \sum_{i=1}^{a}\binom{i+b-2}{b-1}\zeta(i+b-1,a+1-i)\\ &\quad + \sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta(i+a-1,b+1-i).\end{aligned}$$

For $a = b$, this looks even more attractive:

$$\lim_{m\to\infty}\sum_{k=1}^{m+1}\frac{H_{2m+1-k}^{(a)}}{k^a} = \frac{1}{2}\zeta^2(a) + \sum_{i=1}^{a}\binom{i+a-2}{a-1}\zeta(i+a-1,a+1-i).$$

For the alternating harmonic number we introduce the following notation.

$$\overline{\zeta}(a_1) := \sum_{n\geq 1}\frac{(-1)^{n-1}}{n^{a_1}} = (1-2^{1-a_1})\zeta(a_1), \qquad \overline{\zeta}(a_1,a_2) := \sum_{n_1>n_2\geq 1}\frac{(-1)^{n_1-1}}{n_1^{a_1}n_2^{a_2}}, \tag{16}$$

$$\hat{\zeta}(a_1, a_2) := \sum_{n_1 > n_2 \geq 1} \frac{(-1)^{n_1-1}(-1)^{n_2-1}}{n_1^{a_1} n_2^{a_2}}, \tag{17}$$

where $\overline{\zeta}(1) = \log 2$. Thus we obtain for $n = 2m+1$, $j = m+1$ from Corollary 5

$$\sum_{k=1}^{m+1} (-1)^{k-1} \left(\frac{\overline{H}_{2m+1-k}^{(a)}}{k^b} + \frac{\overline{H}_{2m+1-k}^{(b)}}{k^a} \right) = \frac{(-1)^{2m+1}}{(m+1)^{a+b}} + \overline{H}_{m+1}^{(a)} \overline{H}_{m+1}^{(b)} + \sum_{k=1}^{2m+1} (-1)^{k-1} \frac{\overline{H}_{2m+1-k}^{(b)}}{k^a}, \tag{18}$$

and by taking the limit $m \to \infty$

Theorem 5.

$$\lim_{m\to\infty} \sum_{k=1}^{m+1} (-1)^{k-1} \left(\frac{\overline{H}_{2m+1-k}^{(a)}}{k^b} + \frac{\overline{H}_{2m+1-k}^{(b)}}{k^a} \right) = \overline{\zeta}(a)\overline{\zeta}(b)$$
$$- \sum_{i=1}^{a} \binom{i+b-2}{b-1} \overline{\zeta}(i+b-1, a+1-i) \quad - \sum_{i=1}^{b} \binom{i+a-2}{a-1} \overline{\zeta}(i+a-1, b+1-i).$$

For $a = b$, this reads

$$\lim_{m\to\infty} \sum_{k=1}^{m+1} (-1)^{k-1} \frac{\overline{H}_{2m+1-k}^{(a)}}{k^a} = \frac{1}{2}\overline{\zeta}^2(a) - \sum_{i=1}^{a} \binom{i+a-2}{a-1} \overline{\zeta}(i+a-1, a+1-i).$$

For Corollary 6 we proceed as before, getting

$$\sum_{k=1}^{n} \frac{\overline{H}_{n-k}^{(b)}}{k^a} = -\sum_{i=1}^{a} \binom{i+b-2}{b-1} \sum_{k=1}^{n} (-1)^{k-1} \frac{\overline{H}_{k-1}^{(a+1-i)}}{k^{i+b-1}} + \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{k=1}^{n} \frac{\overline{H}_{k-1}^{(b+1-i)}}{k^{i+a-1}}.$$

Now we make use of the basic relation

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{H_k^{(a)}}{k^b} + \sum_{k=1}^{n} \frac{\overline{H}_k^{(b)}}{k^a} = H_n^{(a)} \overline{H}_n^{(b)} + \overline{H}_n^{(a+b)} :$$

This leads to

$$\sum_{k=1}^{n} \frac{\overline{H}_{n-k}^{(b)}}{k^a} = -\sum_{i=1}^{a} \binom{i+b-2}{b-1} \sum_{k=1}^{n} (-1)^{k-1} \frac{\overline{H}_{k-1}^{(a+1-i)}}{k^{i+b-1}} - \sum_{i=1}^{b} \binom{i+a-2}{a-1} \sum_{k=1}^{n} (-1)^{k-1} \frac{H_{k-1}^{(i+a-1)}}{k^{b+1-i}}$$
$$+ \sum_{i=1}^{b} \binom{i+a-2}{a-1} H_n^{(i+a-1)} \overline{H}_n^{(b+1-i)} - \binom{a+b-1}{a} \overline{H}_n^{(a+b)}.$$

For $j = m+1$ and $n = 2m+1$ (6) turns into

$$\sum_{k=1}^{m+1} (-1)^{k-1} \frac{H_{2m+1-k}^{(a)}}{k^b} + \sum_{k=1}^{m+1} \frac{\overline{H}_{2m+1-k}^{(b)}}{k^a} = \frac{(-1)^{m+1}}{(m+1)^{a+b}} + H_{m+1}^{(a)} \overline{H}_{m+1}^{(b)} + \sum_{k=1}^{2m+1} \frac{\overline{H}_{2m+1-k}^{(b)}}{k^a},$$

and by taking the limit $m \to \infty$ we get

Theorem 6.

$$\lim_{m\to\infty}\Big(\sum_{k=1}^{m+1}(-1)^{k-1}\frac{H_{2m+1-k}^{(a)}}{k^b}+\sum_{k=1}^{m+1}\frac{\overline{H}_{2m+1-k}^{(b)}}{k^a}\Big)=\zeta(a)\overline{\zeta}(b)-\binom{a+b-1}{a}\overline{\zeta}(a+b)$$
$$+\sum_{i=1}^{b}\binom{i+a-2}{a-1}\zeta(i+a-1)\overline{\zeta}(b+1-i)-\sum_{i=1}^{a}\binom{i+b-2}{b-1}\hat{\zeta}(i+b-1,a+1-i)$$
$$-\sum_{i=1}^{b}\binom{j+a-2}{a-1}\overline{\zeta}(b+1-i,i+a-1).$$

There are no worthwhile limiting relations coming out of the reciprocity for $\mathsf{Type}\ 2$ sums.

In the remainder of this section, we state three theorems about the decomposition of $\zeta(a,b)$, $\overline{\zeta}(a,b)$ and $\hat{\zeta}(a,b)$ into single valued zeta functions, which could be used for simplifications in the theorems.

Theorem 7 (D. Borwein, J. M. Borwein and Girgensohn, 1995)**.** *For odd weight $w=a+b$ it holds the following.*

$$\zeta(a,b)=\zeta(w)\Big(\frac{1}{2}-\frac{(-1)^b}{2}\binom{w-1}{a}-\frac{(-1)^b}{2}\binom{w-1}{b}\Big)+\frac{1-(-1)^b}{2}\zeta(a)\zeta(b)-\zeta(w)$$
$$+(-1)^b\sum_{k=1}^{\lfloor b/2\rfloor}\binom{w-2k-1}{a-1}\zeta(2k)\zeta(m-2k)+(-1)^b\sum_{k=1}^{\lfloor a/2\rfloor}\binom{w-2k-1}{b-1}\zeta(2k)\zeta(m-2k), \tag{19}$$

where $\zeta(1)$ should be interpreted as zero whenever appearing.

Theorem 8 (Flajolet and Salvy, 1998)**.** *For odd weight $w=a+b$ it holds the following.*

$$\overline{\zeta}(a,b)=\frac{(1-(-1)^b)\overline{\zeta}(a)\zeta(b)+\overline{\zeta}(w)}{2}-\overline{\zeta}(w)$$
$$+\sum_{l+2k=a}\binom{a+l-1}{l-1}(-1)^{l+1}\overline{\zeta}(a+l)\overline{\zeta}(2k)+(-1)^b\sum_{i+2k=a}\binom{b+i-1}{i-1}\zeta(b+i)\overline{\zeta}(2k),$$

where $\zeta(0)$ should be interpreted as 1 and $\zeta(1)$ as zero whenever appearing.

Theorem 9 (Flajolet and Salvy, 1998)**.** *For odd weight $w=a+b$ it holds the following.*

$$\big((-1)^b-(-1)^a\big)\hat{\zeta}(a,b)=\big((-1)^{b+1}+(-1)^a\big)\zeta(w)+(-1)^{b+1}\zeta(w)+(1-(-1)^b)\overline{\zeta}(a)\overline{\zeta}(b)$$
$$+2\sum_{l+2k=b}\binom{a+l-1}{a-1}\overline{\zeta}(a+l)\zeta(2k)-2(-1)^b\sum_{i+2k=a}\binom{b+i-1}{b-1}(-1)^i\overline{\zeta}(b+i)\zeta(2k),$$

where $\zeta(0)$ should be interpreted as 1 and $\zeta(1)$ as zero whenever appearing.

6. Generalizations

Some of the identities can also be formulated for general sequences. Let (x_i), (y_i) be arbitrary well defined sequences. We use the following notation for the partial sums

$$X_n:=\sum_{k=1}^{n}x_k,\quad Y_n:=\sum_{k=1}^{n}y_k.$$

Then the general reciprocity can be reformulated as follows.

Theorem 10.

$$\sum_{k=1}^{j} x_k Y_{n-k} + \sum_{k=1}^{n+1-j} y_k X_{n-k} = -x_j y_{n+1-j} + X_j Y_{n+1-j} + \sum_{k=1}^{n-1} y_k X_{n-k}.$$

For $x_k = 1/k^b$ and $y_k = 1/k^a$ this is just Theorem 1. Note that using the generating functions $X(z) = \sum_{k\geq 1} x_k z^k$, $Y(z) = \sum_{k\geq 1} y_k z^k$ it holds

$$\sum_{k=1}^{n} y_k X_{n-k} = \sum_{k=1}^{n-1} [z^k] Y(z) [z^{n-k}] \frac{X(z)}{1-z} = [z^n] \frac{Y(z)X(z)}{1-z} = \sum_{k=1}^{n} x_k Y_{n-k}.$$

We get the corollaries

Corollary 7.

$$\sum_{k=1}^{j} X_k y_{n-k} + \sum_{k=1}^{n-j} Y_k x_{n-k} = X_{j-1} y_{n+1-j} + Y_{n-j} x_j +_j y_{n+1-j} - X_j Y_{n+1-j} + \sum_{k=1}^{n-1} y_k X_{n-k}.$$

Corollary 8.

$$\sum_{k=1}^{j} Y_{n-k} x_k - \sum_{k=1}^{j} X_k y_{n-k} = Y_{n-1-j} X_j.$$

Corollary 9.

$$\sum_{k=1}^{n+1-j} Y_{j+k-1} x_k + \sum_{k=j+1}^{n} X_{k-j} y_k = Y_n X_{n+1-j}.$$

Here is the generalization of the basic identity (10).

Lemma 2.

$$\sum_{k=1}^{n} x_k Y_k + \sum_{k=1}^{n} y_k X_k = Y_n X_n + \sum_{i=1}^{n} x_i y_i.$$

The formulæ for Type 2 sums are of a more subtle nature and depend on the explicit form of the summands; partial fraction decomposition plays a vital rôle here, and thus there is no generality.

7. Finding and proving identities with SIGMA

Subsequently, we illustrate how the symbolic summation package Sigma [10], implemented in the computer algebra system Mathematica, can assist us to find identities like the above. First, we show how one can use Sigma for deriving identities like those in Theorem 1 for specific values $a, b \in \mathbb{N}$. Then given this collection of identities we are able to conjecture the general form with arbitrary $a, b \in \mathbb{N}$. Moreover, we can extract information from our computations that helps us to prove those identities.

Identities with concrete values $\mathbf{a}, \mathbf{b}$. We work out how one can discover identity (1) with the summation principles presented in [7]. In a first attempt one might try to simplify the left hand side by indefinite summation: given the summand $f(k) = H_{n-k}/k$, find $g(k) \in \mathbb{Q}(n)(k)(H_{n-k})$ such that the telescoping equation

$$g(k+1) - g(k) = f(k) \tag{20}$$

holds. Then summing this equation over k from 1 to j (resp. from 1 to $n+1-j$) would give a closed form for $\sum_{k=1}^{j}\frac{H_{n-k}}{k}$ (resp. for $\sum_{k=1}^{n+1-j}\frac{H_{n-k}}{k}$).

Since this approach fails, we rewrite the identity to

$$\sum_{k=1}^{j}\frac{H_{n-k}}{k}+\sum_{k=1}^{n+1-j}\frac{H_{n-k}}{k}=\sum_{k=1}^{j}\Big(\frac{H_{n-k}}{k}-\frac{H_{k-2}}{n-k+2}\Big)+\sum_{k=1}^{n}\frac{H_{n-k}}{k}.$$

Now, given $f(k)=\frac{H_{n-k}}{k}-\frac{H_{k-2}}{(n-k+2)}$, we can compute

$$g(k)=-\frac{1}{(k-1)(n-k+2)}+H_{k-1}H_{n-k+2}$$

such that (20) holds for all $2\le k\le j$. This gives the closed form evaluation

$$\sum_{k=1}^{j}f(k)=\sum_{k=2}^{j}f(k)+f(1)=g(j+1)-g(2)+f(1)=\frac{1}{j(n+1-j)}+H_jH_{n+1-j}.$$

To this end, we simplify the sum $R_n^{(1,1)}=\sum_{k=1}^{n}\frac{H_{n-k}}{k}$. Since telescoping fails, we apply the following definite summation strategy:

(1) Given $f(n,k):=\frac{H_{n-k}}{k}$, find for some $d\in\mathbb{N}$ constants $c_0(n),\dots,c_d(n)\in\mathbb{Q}(n)$, free of k, and $g(n,k)\in\mathbb{Q}(n)(k)(H_{n-k})$ such that the creative telescoping equation

$$c_0(n)f(n,k)+c_1(n)f(n+1,k)+\cdots+c_d(n)f(n+d,k)=g(n,k+1)-g(n,k) \qquad (21)$$

holds for $1\le k\le n$. We find for $d=2$ the solution $c_0(n)=n+1$, $c_1(n)=-(2n+3)$, $c_2(n)=n+2$, and $g(n,k)=-\frac{1}{n+2-k}$. Hence summing this equation over k from 1 to n produces the recurrence relation

$$(n+1)^2R_n^{(1,1)}-(n+1)(2n+3)R_{n+1}^{(1,1)}+(n+1)(n+2)R_{n+2}^{(1,1)}=2,\quad n\ge 1.$$

(2) Next, we solve the recurrence by computing the solutions 1 and H_n for the homogeneous version and by finding the particular solution $H_n^2-H_n^{(2)}$ of the inhomogeneous recurrence itself; note that the correctness of these solutions can be checked easily. Hence we make the ansatz $R_n^{(1,1)}=H_n^2-H_n^{(2)}+k_1\cdot 1+k_2H_n$ with $k_1,k_2\in\mathbb{Q}$. By checking the first initial values it turns out that $c_1=c_2=0$, i.e., we obtain (3). Summarizing, we have discovered and proven identity (1).

Remark. With the same techniques of indefinite summation (telescoping) and definite summation (computing a recurrence and solving the recurrence) we can attack, in principle, most of the identities in Sections 2, 3, 4 for concrete values a,b. We should mention that already for small a,b, like in (12), (15), the computations get quite involved.

Identities with parameters $\mathbf{a},\mathbf{b}$. In general, finding and proving identities with arbitrary parameters a in $H_k^{(a)}$, like in Theorem 1, is out of scope of our algorithmic machinery. Nevertheless, we can discover the identity in Theorem 1 with $a=b$ if we have a closer look at the presented computer-proof of identity (1). First observe that

$$\sum_{k=1}^{j}\frac{H_{n-k}^{(a)}}{k^b}+\sum_{k=1}^{n+1-j}\frac{H_{n-k}^{(b)}}{k^a}=\sum_{k=1}^{j}\Big(\frac{H_{n-k}^{(a)}}{k^b}-\frac{H_{k-2}^{(b)}}{(n-k+2)^a}\Big)+\sum_{k=1}^{n}\frac{H_{n-k}^{(b)}}{k^a}$$

for arbitrary $a,b\ge 1$. Now we try to solve the telescoping problem (20) with $f(k)=\frac{H_{n-k}^{(a)}}{k^b}-\frac{H_{k-2}^{(b)}}{(n-k+2)^a}$ for various concrete values $a,b\in\mathbb{N}$. As it turns out, we always succeed. Even more, we find the

general pattern
$$g(n,k) = -\frac{1}{(k-1)^b(n-k+2)^a} + H_{k-1}^{(b)} H_{n-k+2}^{(a)};$$
note that the conjectured solution can be easily verified for all $2 \le k \le n$ and for all $a, b \ge 1$. Hence, we have discovered and proven
$$\sum_{k=1}^{j} \Big(\frac{H_{n-k}^{(a)}}{k^b} - \frac{H_{k-2}^{(b)}}{(n-k+2)^a} \Big) = g(j+1) - g(2) + f(1) = H_j^{(b)} H_{n+1-j}^{(a)} - \frac{1}{j^b(n+1-j)^a}$$
which gives the first line in Theorem 1.

Remark. Similarly, we can find and prove the identities in Lemma 1 and in the Corollaries 1, 2, 5, 6.

Next, we turn to the second sum $R_n^{(a,b)} = \sum_{k=1}^{n} \frac{H_{n-k}^{(b)}}{k^a}$. E.g., for $R_n^{(2,2)} = \sum_{k=1}^{n} f(n,k)$ we can compute for $d = 3$ the solution $c_0(n) = -(n+1)^3$, $c_1(n) = 17 + 27n + 15n^2 + 3n^3$, $c_2(n) = -43 - 51n - 21n^2 - 3n^3$, $c_3(n) = (3+n)^3$ and $g(n,k) = \frac{9-4k+9n-2kn+2n^2}{(-3+k-n)^2(-2+k-n)^2}$ for (21); this results in the recurrence relation
$$\begin{aligned} &-(n+1)^4(n+2)^2 R_n^{(2,2)} + (n+1)(n+2)^2(17+27n+15n^2+3n^3) R_{n+1}^{(2,2)} \\ &\quad - (n+1)(n+2)^2(43+51n+21n^2+3n^3) R_{n+2}^{(2,2)} + (n+1)(n+2)^2(3+n)^3 R_{n+3}^{(2,2)} \\ &\qquad = -2(5+2n). \end{aligned}$$
Afterwards, we compute the solutions 1, $H_n^{(3)}$, $H_n^{(2)} - H_n^{(3)}$ of the homogeneous version plus a particular solution $-1 - H_n^{(4)} - 6H_n^{(3)} + 6H_n^{(2)} + H_n^{(2)} + 4\zeta_n(3,1)$ of the recurrence itself. Combining these solutions accordingly gives $R_n(2,2) = \big(H_n^{(2)}\big)^2 - H_n^{(4)} + 4\zeta_n(3,1)$. Similarly, we find by this method the closed forms
$$\begin{aligned} R_n(3,3) &= \big(H_n^{(3)}\big)^2 - H_n^{(6)} + 6\zeta_n(4,2) + 12\zeta_n(5,1), \\ R_n(4,4) &= \big(H_n^{(4)}\big)^2 - H_n^{(8)} + 8\zeta_n(5,3) + 20\zeta_n(6,2) + 40\zeta_n(7,1), \\ R_n(5,5) &= \big(H_n^{(5)}\big)^2 - H_n^{(10)} + 10\zeta_n(6,4) + 30\zeta_n(7,3) + 70\zeta_n(8,2) + 140\zeta_n(9,1). \end{aligned}$$
Together with (11) this leads to the conjecture
$$R_n(a,a) = \sum_{i=1}^{4} s_i(a)\zeta_n(a-1+i, a+1-i)$$
for some $s_i(a) \in \mathbb{N}_0$. Then using, for example, Sloane's On-Line Encyclopedia of Integer Sequences [9] we can guess that $s_i(a) = 2\binom{i+a-2}{a-1}$. Summarizing, we can guess identities like in Theorem 1 ($a = b$).

But can we also prove them? Unfortunately, we failed to extract any information from these computations that could assist us in proving Theorem 1 for general a, b. E.g., analyzing several instances $f(n,k) = H_{n-k}^{(b)}/k^a$ with $a, b \in \mathbb{N}$ we did not find any pattern in the resulting creative telescoping solution $c_i(n) \in \mathbb{Q}(n)$ and $g(n,k) \in \mathbb{Q}(n)(k)(H_{n-k}^{(b)})$.

Note that we are more successful if we extend the domain in which we search $g(n,k)$. For instance, if we take $a = b = 1$, we can compute the creative telescoping solution
$$f(n+1,k) - f(n,k) = g(n,k+1) - g(n,k)$$

with $g(n,k) = \sum_{i=1}^{k} \frac{1}{i(n+1-i)}$. It is important to mention that the sum $\sum_{i=1}^{k} \frac{1}{i(n+1-i)}$ has been computed completely automatically; for further information we refer to [6]. Trying further cases with $a, b \in \mathbb{N}$ we arrive at the general solution $g(n,k) = \sum_{i=1}^{k} \frac{1}{i^b(n+1-i)^a}$; the correctness can be verified immediately. Consequently, we obtain

$$R_{n+1}^{(a,b)} - R_n^{(a,b)} = \sum_{i=1}^{n} \frac{1}{i^b(n+1-i)^a}$$

which is one the crucial observations to prove Theorem 1.

Remark. Similarly, we can compute the recurrence $U_{n,j}^{(a,b)} - U_{n-1,j}^{(a,b)} = \sum_{k=1}^{j} \frac{1}{k^a(n-k)^b}$ in (13).

8. Conclusion

We have demonstrated how a class of nontrivial identities involving sums over harmonic numbers can be discovered and proved, both, by humans and computers.

Since harmonic numbers are defined as a sum, the objects of this paper are *double sums.*

It will be the subject of further research to look out for "similar" identities in the world of *triple* (and multiple) sums.

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Markus Kuba, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr. 8-10/104, 1040 Wien, Austria

E-mail address: `markus.kuba@tuwien.ac.at`

Helmut Prodinger, Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa

E-mail address: `hproding@sun.ac.za`

Carsten Schneider, Research Institute for Symbolic Computation, J. Kepler University Linz, Altenbergstrasse 69, 4040 Linz, Austria

E-mail address: `Carsten.Schneider@risc.uni-linz.ac.at`