DEAM

diffop *)

Workshop:

Differential Equations by Algebraic Methods

(DEAM)

RISC 2009

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From the 6th of February to the 8th of February 2009, the research project DIFFOP of RISC hosted a workshop about *differential equations by algebraic methods (DEAM)*. The workshop took place in Hagenberg, Austria, in the RISC castle. The main focus was on differential operators and differential polynomials.

The team of DIFFOP consists of:

Dipl.-Math. oec. Univ. Christian Dönch (Ph. D. student, on leave)

Dipl.-Math. Johannes Middeke (Ph. D. student)

Dr. Ekaterina Shemyakova (postdoc)

Prof. Dr. Franz Winkler (project leader)

The following people particpated in the workshop:

DI Chistian Aisleitner (RISC, Austria)

Research Scientist Dr. Evelyne Hubert (INRIA Sophia Antipolis, France)

Prof. Dr. George Labahn (University of Waterloo, Canada)

Dipl. Phys. Arne Lorenz (RWTH Aachen, Germany)

Prof. Dr. Elizabeth Mansfield (University of Kent, UK)

M. Sc. Lâm Xuân Châu Ngô (RISC, Austria)

Ao. Univ.-Prof. Dr. Franz Pauer (Universität Innsbruck, Austria)

Prof. Dr. Wilhelm Plesken (RWTH Aachen, Germany)

Dr. Georg Regensburger (RICAM, Austria)

Dr. Markus Rosenkranz (RICAM, Austria)

Dr. Fritz Schwarz (Fraunhofer SCAI, Germany)

M. Sc. Loredana Tec (RISC, Austria)

This report includes the schedule, the slides that where presented during the workshop, and the minutes of the business meeting.

Johannes Middeke
Ekaterina Shemyakova
Franz Winkler

RISC | RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION

JOHANNES KEPLER UNIVERSITY LINZ | JKU

Workshop "*Differential Equations by Algebraic Methods*"

Preliminary Schedule

Friday, 6th of February in RISC, Hagenberg		
9:00 – 9:20	Morning Coffee	
9:20 – 9:30	Franz Winkler	welcome
9:30 – 10:15	Kate Shemyakova	ta
10:30 – 11:15	Arne Lorenz	Laplace Invariants via Vessiot Equivalence Method
11:30 – 12:15	Johannes Middeke	The Jacobson Normal Form of a Matrix of Differential Operators
Lunch		
14:00 – 14:45	Evelyne Hubert	Differential Invariants of Lie groups: Generating Sets and Syzygies
15:00 – 15:45	Fritz Schwarz	Ideal Intersections in Rings of Partial Differential Operators
Coffee breack		
16:15 – 17:00	Markus Rosenkranz Georg Regensburger	A Skew Polynomial Approach to Integro-Differential Operators (joint work with Johannes Middeke)
17:15 – 17:30	Loredana Tec	Implementation of Integro-Differential Operators

Saturday, 7th of February in RISC, Hagenberg		
9:30 – 10:15	George Labahn	The Popov Normal Form of a Matrix of Differential Polynomials
10:30 – 11:15	Elizabeth Mansfield	ta
11:30 – 12:15	Wilhelm Plesken	Counting Solutions of differential and polynomial systems
Lunch		
14:00 – 14:45	Christian Aistleitner	Differential reduction"s" for differential characteristic set computations
15:00 – 15:45	Lam Xuan Chau Ngo	Rational gen. solutions of first-order non-autonomous parametric ODEs
Conference Dinner ∼ 19:00 in Linz		

Sunday, 8th of February in the hotel	
10:00 – 12:00	Discussions, planning cooperations, etc.
Lunch in Linz	
optional: Linz 2009 European Capital of Culture (Lentos museum: "Best of Austria. An Art Collection" and "Images of the City in Art. Linz view. 1909-2009")	

Other participants: Franz Pauer, Günter Landsmann, Ralf Hemmecke, ...

The workshop is supported by FWF project DIFFOP (F. Winkler and E. Shemyakova) and EU project SCIEnce: Symbolic Computation Infrastructure for Europe (F. Winkler).

My Resent Results on Symbolic Methods for LPDOs

Ekaterina Shemyakova

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Outline

1. Research Interests of Our Group
2. Linear Partial Differential Operators
3. Laplace Transformations Method
4. First Constructive Factorization Algorithm: Grigoriev and Schwarz
5. Invariants of LPDOs
6. Invariants of Bivariate, Hyperbolic, Third-Order LPDOs
7. Moving Frames for Laplace Invariants
8. Existence of Factorizations for Hyperbolic, $\mathrm{ord} = 3$ LPDOs
9. A Maple-Package for LPDOs with Parametric Coefficients

Research Interests of Our Group

Franz Winkler Team leader. Besides the work with the members of the team, cooperation with M.Zhou on Groebner Bases. This work is based on the work of F.Pauer (a participant), "Groebner Bases in Rings of Differential Operators", in: B.Buchberger,F.Winkler, "Groebner Bases and Applications", Cambridge Univ. Press (1998).

Johannes Middeke Smith normal form for matrices with DE entries (a talk). Cooperation with M. Rozenkranz and G. Regensburger on Integro-Differential Operators (a talk).

Chau Ngo Feng and Gao have shown how to use the parametrization of algebraic curves (Sendra/Winkler) for exact solution of a DEs of the form $F(y, y') = 0$, where F is a polynomial; Chau is working on extending of this results (a talk).

Ekaterina Shemyakova Moving Frames for computation of Laplace invariants of different sorts with E. Mansfield (a talk). Transformation Methods for exact solution of PDEs with S. Tsarev (a talk).

Linear Partial Differential Operators

- K: a field, $char(K) = 0$ with commuting ∂_x, ∂_y.
- $K[D] = K[D_x, D_y]$: the ring of LPDOs

$$L = \sum_{i+j=0}^{d} a_{ij} D_x^i D_y^j \quad a_{ij} \in K \ ,$$

the principal symbol is the formal polynomial

$$\mathrm{Sym}_L = \sum_{i+j=d} a_{ij} X^i Y^j \ .$$

- K is differentially closed, i.e. contains solutions of (non-linear in the generic case) differential equations with coefficients from K.

Analogously we work with arbitrary number of independent variables.

Laplace Transformations Method

Laplace (1749 - 1827)

Darboux (1842 - 1917)

it is a method of exact integration of $L(u(x,y)) = 0$ with

$$L = D_x \circ D_y + aD_x + bD_y + c \ , \quad a = a(x,y), b = b(x,y), c = c(x,y) \ .$$

Lemma 1.

$$L = D_x \circ D_y + aD_x + bD_y + c = (D_x + b) \circ (D_y + a) + h$$
$$= (D_y + a) \circ (D_x + b) + k,$$

where $h = c - a_x - ab$, $k = c - b_y - ab$.

h, k form a generating set of differential invariants under the gauge transformations!

Consider $\mathcal{L} = \{L = D_x \circ D_y + aD_x + bD_y + c\}$ and the gauge action on $\mathcal{L}$

$$\exp(g(x,y)) * L = \exp(-g(x,y)) \circ L \circ \exp(g(x,y)) \ .$$

The gauge action can be defined also the coefficients of L's as

$$\begin{aligned} \exp(g) * a &= a + g_y \ , \\ \exp(g) * b &= b + g_x \ , \\ \exp(g) * c &= c + g_{xy} + g_x g_y + a g_x + b g_y \ . \end{aligned}$$

Differential Invariant: an algebraic function of coefficients a, b, c and finite number of their derivatives that is invariant under the given transformations.
$h = h(a,b,c)$ and $k = k(a,b,c)$ are differential invariants for $\mathcal{L}$ under the gauge transformations, i.e.

$$h(a,b,c) = h(\exp(g) * a, \exp(g) * b, \exp(g) * c) \ ,$$

$$k(a,b,c) = k(\exp(g) * a, \exp(g) * b, \exp(g) * c) \ .$$

Lemma 2.

If L is factorable, then the equation $L(z) = 0$ is integrable in quadratures.
If, say, $h = 0$, the problem of the solution of $L(z) = 0$ is reduced to the problem of the integration of the two first order equations:

$$\begin{cases} (D_x + b)(z_1) = 0, \\ (D_y + a)(z) = z_1. \end{cases}$$

Accordingly one gets the general solution of $z_{xy} + az_x + bz_y + c = 0$:

$$z = \left(A(x) + \int B(y) e^{\int ady - bdx} dy\right) e^{-\int ady}$$

with two arbitrary functions $A(x)$ and $B(y)$.

The Method of Laplace

If L is not factorable, i.e. $h \neq 0$ and $k \neq 0$, consider L_1 and L_{-1}, which are the results of the differential substitutions

$$z_1 = (D_y + a)(z), \quad z_{-1} = (D_x + b)(z) \ ,$$

correspondingly. Straightforward computation yields

$$L_1 = D_{xy} + \Big(a - \ln|h|_y\Big) D_x + bD_y + c + b_y - a_x - b\ln|h|_y \ ,$$

$$L_{-1} = D_{xy} + aD_x + \Big(b - \ln|k|_x\Big) D_y + c - b_y + a_x - a\ln|k|_x \ .$$

Note that the new operators belong to $\mathcal{L}$.

The Laplace invariants of L_1 and L_{-1}:

$$h_1 = 2h - k - \partial_{xy}(ln|h|) \ , \quad k_1 = h \neq 0 \ ,$$
$$h_{-1} = k \neq 0 \ , \quad k_{-1} = 2k - h - \partial_{xy}(ln|k|) \ .$$

$$\begin{array}{ll} \cdots \leftarrow L_{-2} \leftarrow L_{-1} \leftarrow & L, \\ & L \ \rightarrow L_1 \rightarrow L_2 \rightarrow \ldots \end{array}$$

Lemma

- $L = h^{-1}(L_1)_{-1}h$,
- the Laplace invariants do not change under such substitution.

Therefore, we have essentially ONE chain:

$$\cdots \leftrightarrow L_{-2} \leftrightarrow L_{-1} \leftrightarrow L \leftrightarrow L_1 \leftrightarrow L_2 \leftrightarrow \ldots,$$

and the corresponding chain of invariants is

$$\cdots \leftrightarrow \underbrace{h_{-2}}_{=k_{-1}} \leftrightarrow \underbrace{h_{-1}}_{=k} \leftrightarrow h \leftrightarrow \underbrace{h_1}_{=k_2} \leftrightarrow h_2 \leftrightarrow \ldots$$

Theorem [Goursat/Darboux]

If the chain of invariants if finite in both directions, then one can obtain a quadrature free expression of the general solution of the $L(u(x,y)) = 0$.

Generalizations and Variations of the Laplace Method

- non-linear, 2nd-order, scalar PDEs of the form

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0.$$

  [Darboux] (via linearization)
- as above, but non-scalar [Anderson, Juras, Kamran] (via analysis of the higher degree conservation laws).
- 2nd-order, arbitrary many independent variables, an idea [Dini]
- existence proved [Tsarev] for

$$L = \sum_{i+j+k \leq 2} a_{ijk}(x, y, z) D_x D_y D_z$$

- systems whose order coincides with the number of independent variables [Athorne and Yilmaz]
- attempt for arbitrary order operators in two independent variables [Roux]
- arbitrary order hyperbolic operators [Tsarev]

First Constructive Factorization Algorithm: Grigoriev and Schwarz

Theorem[Constructive Proof] Let the symbol of some LPDO $L \in K[D_{x_1}, \dots, D_{x_n}]$ factors as

$$\mathrm{Sym}_L = S_1 \cdot S_2 \dots S_k \ , \qquad (*)$$

where $S_1, \dots, S_k$ are coprime. Then there exists At MOST one factorization

$$L = F_1 \circ \dots \circ F_k \ ,$$

such that

$$\mathrm{Sym}_{F_i} = S_i \ , \ i = 1, \dots k \ .$$

Note: theorem does not require L to be hyperbolic.
Proof. Consider L and F_i's as the sums of their homogeneous components. Substitute into $(*)$ and equate the corresponding homogeneous components. Obtained polynomial equations can be solved ALGEBRAICALLY when solve them in the descending order one after another.

Example

Consider NON-hyperbolic $L = D_{xyy} + D_{xx} + D_{xy} + D_{yy} + xD_x + D_y + x$, and factorization of its symbol $Sym_L = (X) \cdot (Y^2)$. Consider

$$\widehat{Sym_L} + \sum_{i=0}^{2} L_3 = (D_x + G_0) \circ (D_{yy} + H_1 + H_0)$$

with $G_0 = r(x, y)$, $H_1 = a(x, y)D_x + b(x, y)D_y$, and $H_0 = c(x, y)$. Equate the components on the both sides of the equality:

$$\begin{cases} L_2 &= (aX + bY)X + rY^2 \ , \\ L_1 &= (c + ra + a_x)X + (b_x + rb)Y \ , \\ L_0 &= rc + c_x \ , \end{cases}$$

$$L_2 = X^2 + XY + Y^2, \quad L_1 = xX + Y, \quad L_0 = x \ .$$

The first equation gives us $a = b = r = 1$.
We plug this to the second equation, and get $c = x - 1$, that makes the last (third) equation of the system identity. Thus,

$$L = (D_x + 1) \circ (D_{yy} + D_x + D_y + x - 1) \ .$$

Concluding Remark on Grigoriev-Schwarz's algorithm

- It reveals a large class of LPDOs which factor uniquely.
- In general, given a factorization of the symbol, the corresponding factorization of LPDOs is not obligatory unique!

Invariants of LPDOs

The Laplace Transformation method of integration is based on invariant description of invariant properties.

1. Laplace transformations are invariant w.r.t. the G.T. Thus, we can consider a chain of invariants instead of a chain of operators.
2. The existence of a factorization is an invariant property, therefore, can be described in terms of invariants.
   In our particular case,
   $$L = D_{xy} + a(x,y)D_x + b(x,y)D_y + c(x,y) =$$
   $$(D_x + b)(D_y + a) + h = (D_y + a)(D_x + b) + k \ ,$$
   and a factorization exists if and only if $h = 0$ or $k = 0$.

Invariants for Hyperbolic Bivariate LPDOs of Third-Order

Such the operators have the normalized form

$$L = (p(x,y)D_x + q(x,y)D_y)D_xD_y + \sum_{i+j=0}^{2} a_{ij}(x,y)D_x^iD_y^j$$

- Symbol with Constant Coefficients: 4 invariants were determined, but they are not sufficient to form a generating set [Kartaschova].
- Arbitrary Symbol: ideas how one can get some invariants [Tsarev], also some [Kartaschova], but again insufficient to form a generating set.
- Arbitrary Symbol: a generating set is found [Shemyakova, Winkler].

Generating Set of Invs for $L = (pD_x + qD_y)D_xD_y + \dots$

Theorem. The following is a generating set of invariants:

$$
\begin{aligned}
I_p &= p \,, \\
I_q &= q \,, \\
I_1 &= 2a_{20}q^2 - a_{11}pq + 2a_{02}p^2 \,, \\
I_2 &= a_{20x}pq^2 - a_{02y}p^2q + a_{02}p^2q_y - a_{20}q^2p_x \,, \\
I_3 &= a_{10}p^2 - a_{11}a_{20}p + a_{20}(2q_yp - 3qp_y) + a_{20}^2q - a_{11,y}p^2 + a_{11}p_yp + a_{20} \\
I_4 &= a_{01}q^2 - a_{11}a_{02}q + a_{02}(2qp_x - 3pq_x) + a_{02}^2p - a_{11,x}q^2 + a_{11}q_xq + a_{02} \\
I_5 &= a_{00}p^3q - p^3a_{02}a_{10} - p^2qa_{20}a_{01} + p^2a_{02}a_{20}a_{11} + pqp_xa_{20}a_{11} + \\
&\quad (pI_1 - pq^2p_y + qp^2q_y)a_{20x} + (qq_xp^2 - q^2p_xp)a_{20y} \\
&\quad +(4q^2p_xp_y - 2qp_xq_yp + qq_{xy}p^2 - q^2p_{xy}p - 2qq_xpp_y)a_{20} \\
&\quad +(\frac{1}{2}p_{xy}p^2q - p_xp_ypq)a_{11} - \frac{1}{2}p^3qa_{11xy} \\
&\quad +\frac{1}{2}a_{11x}p_yp^2q + \frac{1}{2}a_{11y}p_xp^2q - 2p_xq^2a_{20}^2 - 2p^2p_xa_{20}a_{02} \,.
\end{aligned}
$$

The results mentioned above (ours and not) have been obtained using some generalization of the Laplace methods.

Problems on the way of future development:

- output invariants are not independent;
- they are not enough to form a generating set;
- how to treat non-hyperbolic case?
- joint invariants of a pair of operators?

Moving Frames Method

Laplace-like methods were developing independently from, and without reference to the methods of moving frames.

1. The idea of moving frames is associated with Cartan, but in fact was used earlier for studying geometric properties of submanifolds and their invariants under the action of a transformation group.
2. Fels and Olver formulated a new, constructive approach to equivariant moving frame theory for the finite-dimensional group actions. The methods have been applied in various areas of mathematics (in particular, Mansfield and Moroz have been applying them for PDEs).
3. Recently Olver and Pohjanpelto, also Cheh explore infinite-dimensional case, and pave the way for computer algebra applications.

Moving Frames for Laplace Invariants

Consider this method for the simplest possible example: hyperbolic bivariate LPDOs of 2-nd order:

$$L = D_{xy} + a(x,y)D_x + b(x,y)D_y + c(x,y) \ .$$

For such the class a generating set is known:
$\{h = c - a_x - ab \ , k = c - b_y - ab\}$.

The action of the gauge transformations $L \longrightarrow L^f = L^{\exp(g(x,y))}$:

$$\begin{cases} \widetilde{x} = x \ , \\ \widetilde{y} = y \ , \end{cases} \quad \& \quad \begin{cases} \widetilde{a} = a + g_y \ , \\ \widetilde{b} = b + g_x \ , \\ \widetilde{c} = c + ag_x + bg_y + g_{xy} + g_x g_y \ . \end{cases}$$

In a neighborhood of some generic point (x_0, y_0):

$g(x,y) =$
$g(x_0,y_0)+g_x(x_0,y_0)(x-x_0)+g_y(x_0,y_0)(y-y_0)+\frac{1}{2}g_{xx}(x_0,y_0)(x-x_0)^2+\dots$
One can assume $g(x_0, y_0) = 0$. Now $g_J(x_0, y_0)$, $J \in \mathbb{N}_0^2$ are independent parameters of the prolonged action.
Further below we omit the designation of the dependence on x_0 and y_0.

The Cartan normalization procedure: construct a cross-section by choosing some normalization equations.

Set the values of the parameters g_x and g_y:

$$\begin{cases} \widetilde{a} &= a + g_y = 0 \ , \\ \widetilde{b} &= b + g_x = 0 \ . \end{cases}$$

Set the values of the parameters g_{xy}, g_{yy}, g_{xx}:

consider some formulae of the first prolongation:

$$\begin{cases} \widetilde{a}_x &= a_x + g_{xy} = 0 \ , \\ \widetilde{a}_y &= a_y + g_{yy} = 0 \ , \\ \widetilde{b}_x &= b_x + g_{xx} = 0 \ . \end{cases}$$

The formulae of the action are linear differential expressions on g, thus it is easy to obtain all the prolongations.

Choosing the normalization equations

$$\widetilde{a}_J = 0 \ , \forall J \in \mathbb{N}_0^2 \ ,$$

we get equations for every action parameter $(g_y)_J$, while the normalization equations

$$\widetilde{b}_{x\dots x} = 0 \qquad (1)$$

provide us with the equations for all $(g_x)_{x\dots x}$.

The invariantization ι of the coordinate functions on M:

$$I_J^{a_{ij}} = \widetilde{(a_{ij})_J}|_{\text{frame}} \; .$$

The invariants that correspond to the coordinate functions appearing in the chosen normalization equations are constants, and called phantom differential invariants, while the remaining (I_y^b and I^c) form a generating system of differential invariants:

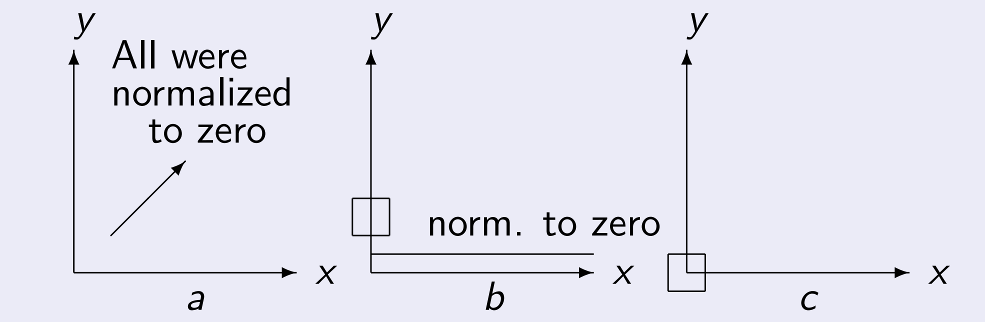

Figure: Three copies of $\mathbb{N}^2$, one for each independent variable a, b, and c showing which derivatives have been normalized. The $\square$ indicates a generating invariant.

Keeping in mind that formulae for every prolongation of the action depends on only FINITE number of group parameters $g_J(x_0, y_0)$, we can prove that finite case theorems valid in the case of our particular Lie pseudo-group action also.
Thus, the recurrence formulae of Olver and Pohjanpelto implies that all the remaining invariants can be ontained by differentiating I_y^b and I^c.

Since the normalization equations imply $g_x = -b$, $g_y = -a$, $g_{xy} = -a_x$, then

$$\begin{aligned} I_y^b &= \widetilde{b}_y|_{\text{frame}} = b_y - a_x \; , \\ I^c &= \widetilde{c}|_{\text{frame}} = c + a(-b) + b(-a) - a_x + ab = c - ab - a_x \; . \end{aligned}$$

Remark. The Laplace's complete generating system is $\{h = c - a_x - ab \; , k = c - b_y - ab\}$.
Their invariants can be expressed in terms of the new ones as

$$h = I^c \; , \quad k = I^c - I_y^b \; .$$

$$L = (p(x,y)D_x + q(x,y)D_y)D_xD_y + \sum_{i+j=0}^{2} a_{ij}(x,y)D_x^iD_y^j$$

Every factorization of L is an extension of a factorization of its symbol $\mathrm{Sym}_L = XY(pX+qY)$. Thus consider <u>NON-commutative</u> factorizations of this polynomial.

12 different factorizations of Sym_L:

1. $(S)(XY)$, properties of the formal adjoints
   $(XY)(S)$, symmetry w.r.t. X, Y :
2. $(X)(YS)$, $(Y)(XS)$, $\leftarrow$ consider only one of the two
   $(YS)(X)$, $(XS)(Y)$, $\leftarrow$ consider only one of the two
3. $(S)(X)(Y)$, $(S)(Y)(X)$, $\leftarrow$ consider only one of the two
4. $(X)(S)(Y)$, $(Y)(S)(X)$, $\leftarrow$ consider only one of the two
   $(X)(Y)(S)$, $(Y)(X)(S)$, $\leftarrow$ consider only one of the two

where $S = (pX + qY)$

Factorization Type (S)(XY)

Theorem

Operators of an equivalent class given by the values of the invariants I_1, I_2, I_3, I_4, I_5 has a factorization of the factorization type $(pX+qY)(XY)$ if and only if

$$\begin{cases} I_3q^3 - I_4p^3 + (pq(q_y - p_x) + 2(p_yq^2 - q_xp^2))I_1 \\ +pq(pI_{1x} - qI_{1y}) - 3pqI_2 = 0 \ , \\ I_sI_2 + I_r + 2pq^2I_{2x} + q^3I_{2y} = 0 \ , \end{cases}$$

where $I_s = \frac{q}{p}(4p(qp_x + pq_x) + 2q(pq_y + qp_y) + I_1)$,
$I_r = \frac{q^3p}{2}I_{1xy} - qp^2(qI_{4y} - pI_{4x}) + \frac{q^3}{p}I_5 + q^2p^2I_{1xx} - \frac{3q^2pq_x}{2}I_{1y} + pI_1I_4 + \Big(-$
$2qp^2q_{xx} + 6q_x^2p^2 + q^2q_xp_y + 4qpq_xp_x - q^2pp_{xx} + q^2p_xq_y - \frac{3q^2pq_{xy}}{2} +$
$5qpq_xq_y + 2p_x^2q^2 - \frac{q^3p_xp_y}{p}\Big)I_1 + 3p^2(qq_y + pq_x)I_4 + \Big(2q_x + \frac{qp_x}{p}\Big)I_1^2 -$
$pq\Big(\frac{3qq_y}{2} + 2qp_x + 4pq_x\Big)I_{1x} - qI_1I_{1x}.$

Analogously other three main types

$$
\begin{aligned}
&2. \quad (X)(YS)\ , \\
&3. \quad (S)(X)(Y)\ , \\
&4. \quad (X)(S)(Y)\ ,
\end{aligned}
$$

where $S = (pX + qY)$.

The obtaining of conditions is not an automatized process.

The formal adjoint for

$$L = \sum_{|J| \le d} a_J D^J \ ,$$

where $a_J \in K$, $J \in \mathbf{N}^n$, $|J|$ is the sum of the components of J, is

$$L^\dagger(f) = \sum_{|J| \le d} (-1)^{|J|} D^J(a_J f)\ ,\ \forall f \in K\ .$$

Properties:

- $(L^\dagger)^\dagger = L$,
- $(L_1 \circ L_2)^\dagger = L_2^\dagger \circ L_1^\dagger$, the type of factorizations changes!
- $\mathrm{Sym}_L = (-1)^{\mathrm{ord}(L)} \mathrm{Sym}_{L^\dagger}$.

Lemma

The operation can be defined on the equivalent classes of LPDOs.

Example: hyperbolic operators of order 2

The family of operators $\{L = D_{xy} + a(x,y)D_x + b(x,y)D_y + c(x,y)\}$ has a complete generating set of invariants $\{h = c - a_x - ab\ ,\, k = c - b_y - ab\}$.

For the formal adjoint

$$L^\dagger = D_{xy} - aD_x - bD_x + c - a_x - b_y$$

we have

$$h^\dagger = c - b_y - ab\ , \quad k^\dagger = c - a_x - ab\ .$$

Thus,

$$\{h\ , k\} \to \{k\ , h\}$$

Theorem (formal adjoint for equivalent classes)

Consider the equivalent classes of L_3 *given by the values of the invariants* I_1, I_2, I_3, I_4, I_5. *Then the operation of taking the formal adjoint is defined by the following formulae*

$$\begin{aligned}
I_1^\dagger &= I_1 - 2q^2 p_y - 2p^2 q_x + 2p_x qp + 2q_y qp\ ,\\
I_2^\dagger &= -I_2 - qp^2 q_{xy} + q_y p^2 q_x + q^2 pp_{xy} - q^2 p_x p_y\ ,\\
I_3^\dagger &= -I_3 + \tfrac{1}{q^2}\Big(2pI_2 - (2p_y q + q_y p)I_1 + qpI_{1y} - 2p_y q_y q^2 p +\\
&\qquad 2q^3 p_y^2 + q_{yy} q^2 p^2 - q^3 pp_{yy}\Big)\ ,\\
I_4^\dagger &= -I_4 + \tfrac{1}{p^2}\Big(-2qI_2 - (p_x q + 2q_x p)I_1 + qpI_{1x} + 2p^3 q_x^2 - 2p^2 q_x qp_x\\
&\qquad +p_{xx} q^2 p^2 - qp^3 q_{xx}\Big)\ ,\\
I_5^\dagger &= I_5 + p_1 I_1 + p_3 I_3 + p_4 I_4 + p_{12} I_{1y} + p_{11} I_{1x} + p^2 I_{1xy} - qpI_{3x} - \tfrac{p^3}{q} I_{4y} +\\
&\quad -pI_{2y} + \tfrac{p^2}{q} I_{2x} + (-2q^2 p^3 q_x + 4p_y q^4 p - q^2 pI_1 - 2q^3 p^2 p_x)/(q^4 p)I_2
\end{aligned} \tag{2}$$

where $p_1 = (4q_x p_y p + p_x q_y p - 2q_{xy} p^2)/q + (4q_x q_y p^2)/q^2 + 3p_x p_y - p_{xy} p$, $p_3 = 2qp_x + pq_x$, $p_4 = (2q_y p^3 + p^2 p_y q)/q^2$, $p_0 = p^3 q_x q_{yy} - 2q^2 p_x p_y^2 -$

Corollary

Consider the equivalent classes of L_3 possessing the properties $p = 1$ and $q = 1$ and which are given by the values of the invariants I_1, I_2, I_3, I_4, I_5. Then the operation of taking of the formal adjoint is defined by the following formulae

$$\left.\begin{array}{lcl} I_1^\dagger & = & I_1 \ , \\ I_2^\dagger & = & -I_2 \ , \\ I_3^\dagger & = & -I_3 + 2I_2 + I_{1y} \ , \\ I_4^\dagger & = & -I_4 - 2I_2 + I_{1x} \ , \\ I_5^\dagger & = & I_5 + I_{1xy} - I_{3x} - I_{4y} - I_{2y} + I_{2x} - I_1 I_2 \ . \end{array}\right\}$$

Formal Adjoints for Computation of Factorization Conditions

L has a factorization of a factorization type $(S_1)(S_2)$.

$\Updownarrow$

$L^\dagger$ has a factorization of the factorization type $(S_2)(S_1)$.

$\Updownarrow$

Conditions in terms of invariants $I_1^\dagger, \dots, I_5^\dagger$.

$\Updownarrow$

These conditions after the substitutions of the expressions in terms of $I_1, \dots, I_5$ for the invariants $I_1^\dagger, \dots, I_5^\dagger$.

Completely automatized process for any number of factors.

LPDOs Whose Symbol Has Constant Coefficients Only

$\Rightarrow$ $\exists$ a normal form with $\mathrm{Sym} = (X+Y)XY$. Symbol is invariant $\Rightarrow$ can consider equivalent classes of L_3 with the property $p = q = 1$. Let such a class be given by the values of the invariants $I_1, \dots, I_5$.

Theorem

Consider equivalent classes possessing the property $p = q = 1$, and given by the values of the invariants I_1, I_2, I_3, I_4, I_5. Operators of the class have a factorization of factorization type

$(S)(XY) \Leftrightarrow$

$$I_3 - I_4 + I_{1x} - I_{1y} - 3I_2 = 0 \quad \& \quad I_1 I_2 + I_r + 2I_{2x} + I_{2y} = 0\,, \tag{3}$$

where $I_r = \frac{1}{2} I_{1xy} - I_{4y} + I_{4x} + I_5 + I_{1xx} + I_1 I_4 - I_1 I_{1x}$;

$(S)(X)(Y) \Leftrightarrow$

$$(3) \quad \& \quad I_2 - I_4 + I_{1x} = 0\,;$$

$(S)(Y)(X) \Leftrightarrow$

$$(3) \quad \& \quad -2I_2 - I_4 + I_{1x} = 0\,;$$

$(X)(SY) \Leftrightarrow$

$$I_4 = 0 \quad \& \quad I_{2x} + I_5 - I_{3x} + I_{1xy}/2 = 0\,; \tag{4}$$

$(X)(S)(Y) \Leftrightarrow$

$$(4). \quad \& \quad I_3 - I_{1y} - 2I_2 = 0\,;$$

$(X)(Y)(S) \Leftrightarrow$

$$(4). \quad \& \quad I_3 = I_2\,;$$

$(XY)(S) \Leftrightarrow$

$$I_4 = I_3 - I_2 \quad \& \quad I_{1xy}/2 + I_1 I_4 + I_5 = 0\,.$$

$(YS)(X) \Leftrightarrow$

$$I_4 = I_{1x} - 2I_2 \quad \& \quad I_5 = I_1 I_2\,.$$

$(XS)(Y) \Leftrightarrow$

$$I_3 - I_{1y} - 2I_2 = 0 \quad \& \quad I_5 = I_{2x} + I_{1xy}/2\,;$$

$(Y)(SX) \Leftrightarrow$

$$l_3 = 0 \quad \& \quad l_5 = (l_4 + l_2)_y + l_1 l_2 - l_{1xy}/2 \; ; \qquad (5)$$

$(Y)(X)(S) \Leftrightarrow$

$$(5) \quad \& \quad l_4 = -l_2 \; ;$$

$(Y)(S)(X) \Leftrightarrow$

$$(5) \quad \& \quad l_4 - l_{1x} = -2l_2 \; ;$$

A Maple-Package for LPDOs with Parametric Coefficients

Description

- The number of variables – Arbitrary.
- The orders of LPDOs – Arbitrary.
- Parameters – Arbitrary.
- Easy access to the coefficients of LPDOs.
- Application to a function $\longrightarrow$ to a standard Maple PDE form.

Basic Procedures

- The basic arithmetic of LPDOs (addition, composition, mult. by a function on the left).
- Transposition and conjugation of LPDOs.
- Application to a function $\longrightarrow$ to a standard Maple PDE form.
- Simplification Tools for coefficients.

More Advanced Possibilities

- Standard Laplace invariants.
- Standard Laplace Transformations.
- Standard Laplace Chain.
- Laplace Invariants for extended Schrödinger operators: $\Delta_2 + aD_x + bD_y + c$.
- Laplace Transformations for those.
- Laplace Chain for those.
- Full System of Invariants for operators $L_3 = D_x D_y (pD_x + qD_y) + \dots$
- Obstacles to factorizations of $2, 3$ orders $\longrightarrow$ Grigoriev-Schwarz Factorization.

Laplace Invariants via Vessiot Equivalence Method

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Outline

1. Introduction
   - Linear Partial Differential Operators
2. Computational Results – Overview
   - Invariants for Third Order LPDOs
   - Invariants for Fourth Order LPDOs
3. Vessiot Equivalence Method for LPDOs
   - Natural Bundles
   - Prolongation and Projection
   - Embedding Theorem
4. Summary

Introduction – Laplace Example

- Linear partial differential operators (LPDOs) of order 2:

$$L = \partial_x \partial_y + a\,\partial_x + b\,\partial_y + c$$

- Gauge transformations:

$$L \mapsto g^{-1} L g, \qquad g = g(x, y).$$

- Laplace invariants:

$$h = c - a_x - ab, \qquad k = c - b_y - ab.$$

- The operator L can be factorised if $h = 0$ or $k = 0$:

$$\begin{aligned} L &= (\partial_x + b)\,(\partial_y + a) + h, \\ &= (\partial_y + a)\,(\partial_x + b) + k. \end{aligned}$$

Introduction – General Situation

- Consider arbitrary LPDOs of order d:

$$L = \sum_{|\mu| \le d} a_\mu(x) \partial^\mu, \qquad \partial^\mu = \partial_{x_1}^{\mu_1} \cdots \partial_{x_n}^{\mu_n}$$

  with symbol $\sum_{|\mu|=d} a_\mu(x) X^\mu$.
- The factorisation of LPDOs is gauge invariant:

$$g^{-1} L g = g^{-1} L_1 L_2\, g = (g^{-1} L_1\, g)(g^{-1} L_2\, g).$$

- Conditions for factorisation $\leftrightarrow$ Laplace invariants.

Methods to compute invariants:

- Partial factorisation (obstacles) [SW07b], [SW07a]
- Moving frames [MS08]
- ...
- Vessiot equivalence method [Ves03]

Third Order LPDOs

The number of invariants in a generating set:

Symbol, order	[MS08] 0	1	2	3	total	Vessiot 0	1	2	total
X^3	2	2	1		5	2	3		5
X^3, (a)	2	0	2		4	2	1	1	4
X^3, (b)	1	1	0	1	3	1	1	1	3
X^3, (c)	0	1	1		2	0	2		2
X^2Y	1	3	1		5	1	5		6
$XY(pX+qY)$	3	3	1		7	3	4	1	8
full						5	4	1	10

- Moving frames: small invariants of higher order,
- Vessiot: large invariants of minimal order.

In future: Combine both methods!

Invariants for $L_{X^3,(c)} = \partial_x^3 + a_{20}\partial_x^2 + a_{10}\partial_x + a_{00}$

- Moving frames [MS08]:

$$\begin{aligned} I^{a_{10}} &= a_{10} - \frac{1}{3}a_{20}^2 - a_{20,x}, \\ I_x^{a_{00}} &= a_{00} - \frac{1}{3}a_{10}a_{20} + \frac{2}{27}a_{20}^3 - \frac{1}{3}a_{20,xx}. \end{aligned}$$

- Vessiot:

$$\begin{aligned} I_1^1 &= -a_{10} + \frac{1}{3}a_{20}^2 + a_{20,x}, \\ I_2^1 &= a_{10,x} - 3a_{00} + a_{20}a_{10} - \frac{2}{3}a_{20}a_{20,x} - \frac{2}{9}a_{20}^3. \end{aligned}$$

- Comparison:

$$I^{a_{10}} = -I_1^1 \qquad I_x^{a_{00}} = -\frac{1}{3}(I_1^2 + I_{1,x}^1)$$

Invariants for Fourth Order LPDOs

Results of the Vessiot equivalence method:

Symbol, order	0	1	2	3	4
X^4	5	5	1		
X^4 (a)	3	6			
X^4 (d)	2	2	1	0	2
$\vdots$					
$X^3 Y$	4	7	1		
$X^2 Y^2$	3	10			
$X^3 (pX + qY)$	5	7	1		
$X^2 Y (pX + qY)$	4	9	1		
$X^2 (pX + qY) (rX + sY)$	5	9	1		
$X\, Y\, (pX + qY) (rX + sY)$	5	9			
$X\, Y\, (pX^2 + qY^2)$	5	6	1		

Natural Bundles

Let X be a manifold, coordinates $(x) = (x^1, \dots, x^n)$.

- $\mathrm{Diff}_{\mathrm{loc}}(X, X)$: local diffeomorphisms $\varphi : X \to X$.
- Pseudogroup $\Theta \subseteq \mathrm{Diff}_{\mathrm{loc}}(X, X)$.
- A natural Θ-bundle is a fibre bundle

$$\pi : \mathcal{F} \to X : (x, v) \to (x)$$

  such that each $\tilde{x}(x) \in \Theta$ lifts to $\Phi : \mathcal{F} \to \mathcal{F}$ as:

$$\tilde{x} = \tilde{x}(x), \qquad v = \Phi_{\tilde{v}}(\tilde{x}, \tilde{x}_q).$$

  In other words: Θ acts on $\mathcal{F}$.
- A section of $\mathcal{F}$ is called geometric object:

$$\omega : X \to \mathcal{F} : (x) \mapsto (x, v = \omega(x)).$$

- $\psi : \mathcal{F} \to \mathbb{R}$ is an invariant if $\psi \circ \Phi = \psi \quad \forall\, \tilde{x}(x) \in \Theta$.

Laplace Example I

- Pseudogroup Θ of gauge transformations:

$$X \to X: \begin{pmatrix} x \\ y \\ u \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} = x \\ \tilde{y} = y \\ \tilde{u} = e^{g(x,y)} u \end{pmatrix}.$$

- The natural Θ-bundle $\mathcal{F}$ for the Laplace operators

$$L = \partial_x \partial_y + a\,\partial_x + b\,\partial_y + c$$

has coordinates $(x, y, u;\ a, b, c)$.
- Each gauge transformation lifts to $\mathcal{F}$ via $\tilde{L} \mapsto e^{-g} \tilde{L} e^{g}$:

$$\begin{aligned} a &= \tilde{a} + g_y \\ b &= \tilde{b} + g_x \\ c &= \tilde{c} + g_{xy} + \tilde{a}\, g_x + \tilde{b}\, g_y. \end{aligned}$$

- A section $a(x, y)$, $b(x, y)$, $c(x, y)$ specifies an LPDO.

Prolongation and Projection

- Choosing $v = v(x)$ and $\tilde{v} = \tilde{v}(\tilde{x})$, the Θ-action on $\mathcal{F}$

$$v = \Phi_{\tilde{v}}(\tilde{x}, \tilde{x}_q)$$

can be seen as a PDE system for $\tilde{x}(x)$ of order q.
- Prolongation $\mathcal{F} \rightsquigarrow J_1(\mathcal{F})$:

$$v_x = D_x \Phi_{(\tilde{v}, \tilde{v}_{\tilde{x}})}(\tilde{x}, \tilde{x}_{q+1}).$$

- Projection $\mathcal{F}_{(1)} = J_1(\mathcal{F})/K_{q+1}$:

$$w = \Psi_{(\tilde{v}, \tilde{w})}(\tilde{x}, \tilde{x}_q)$$

by eliminating derivatives of order $q + 1$.
- Vessiot structure equations: Integrability conditions.

Laplace Example II

- The Θ-action on $\mathcal{F}$ is (with $q = 2$):

$$\begin{aligned} a &= \tilde{a} + g_y \\ b &= \tilde{b} + g_x \\ c &= \tilde{c} + g_{xy} + \tilde{a}\, g_x + \tilde{b}\, g_y + g_x g_y . \end{aligned}$$

- First prolongation to $J_1(\mathcal{F})$:

$$\begin{aligned} a_x &= \tilde{a}_x + g_{xy}, & a_y &= \tilde{a}_y + g_{yy}, \\ b_x &= \tilde{b}_x + g_{xx}, & b_y &= \tilde{b}_y + g_{xy}, \\ c_x &= \tilde{c}_x + g_{xxy} + \dots, & c_y &= \tilde{c}_y + g_{xyy} + \dots . \end{aligned}$$

- Projection: $\mathcal{F}_{(1)}$ has the improved coordinates:

$$h = a_x - c + ab, \quad , a_y, \quad b_x, \quad k = b_y - c + ab.$$

- Invariants: Projection to order zero.

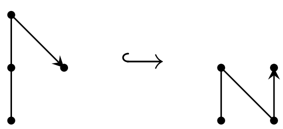
Embedding Theorem

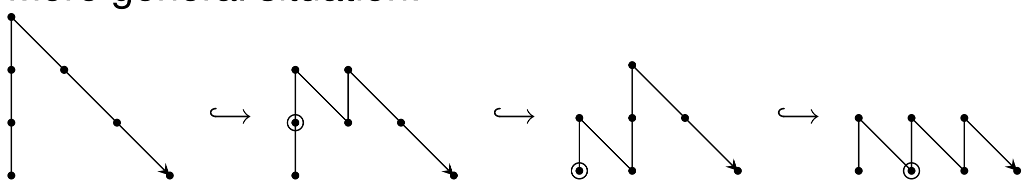
Theorem

If the symbol of $\Phi_{\tilde{v}}(\tilde{x}, \tilde{x}_q) = v$ is 2-acyclic for generic $\tilde{v}(\tilde{x})$, then

$$\iota : J_2(\mathcal{F})/K_{q+2} \to J_1(\mathcal{F}_{(1)})$$

is an embedding.

- Visualisation: 
- Computing $\mathrm{im}(\iota)$ involves only linear algebra.
- The invariants on $J_2(\mathcal{F})$ and on $\mathrm{im}(\iota)$ coincide.
- More general situation:



Laplace Example III

- The bundle $\mathcal{F}_{(1)}$ has the coordinates $(x, y, u;\ a, b, c)$ and

$$h = a_x - c + ab, \quad , d = a_y, \quad e = b_x, \quad k = b_y - c + ab.$$

- Prolongation to $J_1(\mathcal{F}_{(1)})$ and the embedding $\mathrm{im}(\iota)$:

$$\begin{array}{llll} a_x = h + c - ab & a_y = d & b_x = e & b_y = k + c - ab \\ c_x & c_y & & \\ d_x = h_y + \dots & d_y & e_x & e_y = k_x + \dots \\ h_x & h_y & k_x & k_y \end{array}$$

- Projection to $\mathcal{F}_{(2)}$:

$$h_x, \qquad h_y, \qquad k_x, \qquad k_y.$$

- All new coordinates on $\mathcal{F}_{(2)}$ are invariants
$\Rightarrow$ $\{h, k\}$ is a generating set of invariants.

Summary

- Pseudogroup Θ, natural bundle $\mathcal{F}$.
- Prolongation and projection yields natural bundles

$$\mathcal{F}, \quad \mathcal{F}_{(1)}, \quad \mathcal{F}_{(2)}, \quad \dots$$

  using the Embedding Theorem.
- Invariants on $J_i(\mathcal{F})$ = invariants on $\mathcal{F}_{(i)}$.
- All new coordinates of $\mathcal{F}_{(i)}$ are invariants
$\Rightarrow$ generating set of invariants on $\mathcal{F}_{(i)}$.
- Computation of invariants:
  - Moving frames on $\mathcal{F}_{(i)}$ or
  - Linear PDEs on $\mathcal{F}_{(i)}$.
- Sucessfully treated fourth order LPODs.
- Even a fifth order example (X^3Y^2) was computable...

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Computing the Jacobson form of matrices of differential operators in polynomial time

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Outline

1 Definitions
- Differential operators
- Jacobson form
- The main theorem

2 The algorithm
- Overview
- Reduction
- Modular computations

3 Epilogue

Differential fields

Let F be a field.
A derivation is an additive map $\vartheta : F \longrightarrow F$ that fulfils the Leibniz-rule

$$\vartheta(a\,b) = a\,\vartheta(b) + \vartheta(a)\,b \quad \text{for all } a, b \in F.$$

The pair (F, ϑ) is called a differential field.

Examples are

- The rational functions $\mathbb{Q}(x)$ with the usual derivation d/dx.
- The meromorphic functions with the usual derivation.
- Every field with the zero map.

Differential operators

Definition

Let (F, ϑ) be a differential field.
Take a variable ∂ and consider polynomial expressions

$$a_n\partial^n + \ldots + a_1\partial + a_0$$

with coefficients $a_0, \ldots, a_n \in F$.

The ring $R = F[\partial; \mathrm{id}, \vartheta]$ is the set of all these polynomials together with the usual addition and multiplication given by the **Leibniz rule**

$$\partial a = a\partial + \vartheta(a) \quad \text{for all } a \in F.$$

(This mimics composition of linear differential operators in ϑ).

We call R the ring of **differential operators**.

Example & properties

Let $R = F[\partial; \mathrm{id}, \vartheta]$ be a ring of differential operators.

- Differential operators behave *almost* like ordinary polynomials.
- Multiplication is not commutative.
- But for all $f, g \in R$ we have

$$\operatorname{ord}(fg) = \operatorname{ord} f + \operatorname{ord} g \quad \text{and} \quad \operatorname{lc}(fg) = \operatorname{lc}(f)\operatorname{lc}(g).$$

  This makes R a (non-commutative) Euclidean domain.

As **example**, take $F = \mathbb{F}_5(x)$ and $\vartheta = d/dx$. Here,

$$x\partial^2 + 2\partial + x = \partial^2 \cdot x + x$$

Hence, left-division of $x\partial^2 + 2\partial + x$ by ∂^2 yields the remainder x.

Jacobson form

Let R be a ring of differential operators. A matrix $M \in {}^mR^n$ is said to be in **Jacobson form** if

$$M = \left.\overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & 1 & \ddots & & \vdots \\ \vdots & & \ddots & f & \ddots & \vdots \\ \vdots & & & \ddots & 0 & \ddots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}}^{n \text{ columns}}\right\} m \text{ rows}$$

where $f \in R$.

Example for the Jacobson form

The matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial^3 + x\partial^2 - x\partial - x^2 & 0 \end{pmatrix}$$

is in Jacobson form – the matrix

$$\begin{pmatrix} \partial + x & 1 & 2\partial^2 + 2x\partial + x \\ -1 & \partial^2 - x & x\partial^2 - x^2 \end{pmatrix}$$

is not in Jacobson form.

Remarks

- The Jacobson form is a two-sided normal form.
- It is a generalisation of the Smith form for integer matrices.
- A (more general) version of the Jacobson form exists in every (not necessarily commutative) principle ideal domain.
  This was done by Nathan Jacobson, Tadashi Nakayama and Oswald Teichmüller.
- It is *almost* unique.

The main theorem

Let R be a ring of differential operators over a field F.

Theorem

Let $M \in {}^mR^n$. If

$$[F : \mathrm{Const}(F)] \geqslant m \cdot \operatorname{ord} M$$

then we can compute unimodular matrices $S \in ({}^mR^m)^*$ and $T \in ({}^nR^n)^*$ such that

$$SMT$$

is in Jacobson form, using only polynomially (in m, n and $\operatorname{ord} M$) many field operations.

Example

Let $F = \mathbb{F}_5(x)$ and $R = F[\partial; \mathrm{id}, d/dx]$, and let

$$M = \begin{pmatrix} \partial + x & 1 & 2\partial^2 + 2x\partial + x \\ -1 & \partial^2 - x & x\partial^2 - x^2 \end{pmatrix} \in {}^2R^3.$$

Since $[F : \mathrm{Const}(F)] = 5 \geqslant 4 = 2 \cdot 2$ (number of rows times maximal order) the algorithm is applicable.

We will compute

$$\begin{pmatrix} 0 & -1 \\ 1 & \partial + x \end{pmatrix} \cdot M \cdot \begin{pmatrix} 1 & \partial^2 - x & -2\partial \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial^3 + x\partial^2 - x\partial - x^2 & 0 \end{pmatrix}.$$

Outline of the algorithm

Given $M \in {}^mR^n$ we perform the following steps

1. Remove linear dependencies of the rows and columns: Transform

$$M \rightsquigarrow \left(\begin{array}{c:c} M'' & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} \end{array}\right)$$

   by linear transformations where $M'' \in {}^kR^k$ is square and has linearly independent rows.

2. Compute **modularly** in

$$\frac{R^k}{R^k M''}$$

   to transform M'' into Jacobson form.

Removing linear dependencies

Overview

- Removal of linear dependencies is done with so-called row- and column-reduction.
- Row- (column-) reduction removes highest order terms from a row (column) by elementary row (column) operations.
- This is iterated as long as possible.
- The remaining non-zero rows (columns) are linearly independent.
- The number of remaining non-zero rows (columns) after reduction equals the rank of M.
- We will apply first column-reduction then row-reduction, i. e., we transform

$$M \underset{\substack{\text{column-}\\\text{reduction}}}{\rightsquigarrow} \left(\begin{array}{c:c} M' & \mathbf{0} \end{array}\right) \underset{\substack{\text{row-}\\\text{reduction}}}{\rightsquigarrow} \left(\begin{array}{c:c} M'' & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} \end{array}\right)$$

Gröbner bases and the (weak) Popov form

- We can further inter-reduce the rows of M'' such that entries of maximal order do not overlap.
- The result is called the (weak) Popov form of M''.
- The (weak) Popov form is a Gröbner basis of $R^k M''$ with respect to the term over position ordering.
- Since M'' is square and reduction at most reduces the order, that means that
$$\dim_F \frac{R^k}{R^k M''} \leqslant m \cdot \operatorname{ord} M < \infty.$$

Example (Reduction)

Removing linear dependencies

Over $\mathbb{F}_5(x)$ with derivation d/dx, consider again

$$M = \begin{pmatrix} \partial + x & 1 & 2\partial^2 + 2x\partial + x \\ -1 & \partial^2 - x & x\partial^2 - x^2 \end{pmatrix}.$$

We column-wise consider the leading coefficients of M:

$$\mathrm{LC}_{\mathrm{col}}(M) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & x \end{pmatrix}.$$

Since there is a **linear dependency**, we may erase ∂^2 in the last column.

Example (Reduction)

Removing linear dependencies

Over $\mathbb{F}_5(x)$ with derivation d/dx, we compute

$$MQ = \begin{pmatrix} \partial + x & x & 0 \\ -1 & \partial^2 + 1 & 0 \end{pmatrix}.$$

We **row-wise** consider the leading coefficients of MQ:

$$\mathrm{LC}_{\mathrm{row}}(MQ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In fact, MQ is already in **(weak) Popov form**.

Cyclic vectors

Let $\mathfrak{M} = R^k / R^k M''$.

- Since $\dim_F \mathfrak{M} \leqslant m \cdot \operatorname{ord} M \leqslant [F : \operatorname{Const}(F)]$, by the cyclic vector theorem there exists $v \in \mathfrak{M}$ such that

$$Rv = \mathfrak{M}.$$

- Computing the annihilator Rf of v, we obtain an R-isomorphism

$$\varphi \colon \mathfrak{M} \xrightarrow{\sim} \frac{R}{Rf}, \quad v \mapsto \overline{1}.$$

- Defining $g \in {}^kR$ by $\overline{g_j} = \varphi(\overline{\mathfrak{e}_j})$ we have for $w \in R^k$

$$\varphi(\overline{w}) = \overline{wg}$$

where $\mathfrak{e}_1, \ldots, \mathfrak{e}_k$ are the unit vectors in R^k.

Computing S and T

- One may prove that $\operatorname{gcrd}(g_1, \ldots, g_k) = 1$; hence there exists an invertible matrix $T \in ({}^kR^k)^*$ with last column g.
- The last column of $M''T$ is (right-) divisible by f, i. e., we may write

$$M''\,T = W \cdot \operatorname{diag}(1, \ldots, 1, f).$$

- A dimension argument shows that **W is invertible**.

  With $S = W^{-1}$ we have

$$S\,M''\,T = \operatorname{diag}(1, \ldots, 1, f)$$

  which is in **Jacobson form**.

Example (Modular computations)

Computing a cyclic vector

Continuing the example, consider the first two columns of MQ. Let

$$M'' = \begin{pmatrix} \partial + x & 1 \\ -1 & \partial^2 - x \end{pmatrix} \quad \text{and} \quad \mathfrak{M} = \frac{R^2}{R^2 M''}.$$

- Leading monomials are $\partial \mathfrak{e}_1$ and $\partial^2 \mathfrak{e}_2$.
- A basis for $\mathfrak{M}$ is $\overline{\mathfrak{e}_1}, \overline{\mathfrak{e}_2}, \overline{\partial \mathfrak{e}_2}$.
- A cyclic vector is $\overline{\mathfrak{e}_2}$, i. e., $R\overline{\mathfrak{e}_2} = \mathfrak{M}$.

  That means, $\overline{\mathfrak{e}_2}, \partial\overline{\mathfrak{e}_2}, \partial^2\overline{\mathfrak{e}_2}$ is an F-basis of $\mathfrak{M}$.
- We compute

$$0 = \left(\partial^3 + x\partial^2 - x\partial - x^2\right) \cdot \overline{\mathfrak{e}_2} = f \cdot \overline{\mathfrak{e}_2};$$

  hence $\mathfrak{M} \cong R/Rf$.

Example (Modular computations)

Computing S and T

We compute

$$\varphi\colon \mathfrak{M} \to \frac{R}{Rf}, \quad \overline{v} \mapsto \overline{v \cdot \begin{pmatrix} \partial^2 - x \\ 1 \end{pmatrix}} \qquad \text{and} \qquad T = \begin{pmatrix} 1 & \partial^2 - x \\ 0 & 1 \end{pmatrix}.$$

Furthermore we have

$$M'' T = \begin{pmatrix} \partial + x & \partial^3 + x\partial^2 - x\partial - x^2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \partial + x & 1 \\ -1 & 0 \end{pmatrix} \cdot \operatorname{diag}(1, f),$$

where $\operatorname{diag}(1, f)$ is in Jacobson form.

Combining this with the first part of the algorithm we obtain

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & \partial + x \end{pmatrix}}_{=S} \cdot M \cdot \underbrace{\begin{pmatrix} 1 & \partial^2 - x & -2\partial \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}}_{=Q\,\operatorname{diag}(T,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \partial^3 + x\partial^2 - x\partial - x^2 & 0 \end{pmatrix}.$$

Summary

Input A matrix $M \in {}^mR^n$ where $R = F[\partial; \mathrm{id}, \vartheta]$ and $[F : \operatorname{Const}(F)] \geqslant m \cdot \operatorname{ord} M$.

Output Unimodular matrices $S \in ({}^mR^m)^*$ and $T \in ({}^nR^n)^*$ such that SMT is in Jacobson form.

1. Apply column-reduction to M gaining $\left(M' \;\vdots\; \mathbf{0} \right)$.
2. Apply row-reduction to M' gaining $\begin{pmatrix} M'' \\ \hdashline \mathbf{0} \end{pmatrix}$.
3. Transfer M'' into (weak) Popov form and compute a basis for $R^k / R^k M''$ (where $M'' \in {}^kR^k$).
4. Compute a cyclic vector v for $R^k / R^k M''$.
5. Let $\varphi : R^k / R^k M'' \to R/Rf$ where $Rf = \operatorname{Ann}_R v$.
6. Compute $T \in ({}^kR^k)^*$ such that $\varphi(\overline{\mathfrak{e}_j}) = \overline{T_{jk}}$.
7. Let $N = \operatorname{diag}(1, \ldots, 1, f)$ and compute $S = (MT/N)^{-1}$.

Conclusion

- The steps in the algorithm can be done using only polynomial many operations in F.
- Important ingredients of the algorithm were the cyclic vector theorem and row-/column-reduction.
- We may stop after the computation f if only the Jacobson form is needed.
- Other algorithms exist, e. g.:
  - RWTH Aachen (Zerz, Levandovskyy, Schindelar) (can control coefficient growth!).
  - Culianez, G., Quadrat, A., 2005. Formes de hermite et de jacobson: implémentations et applications. Tech. rep., INRIA Sophia Antipolis.

And now lunch. . .

Thank you for your attention!

Differential Invariants of Lie Groups: Generating Sets and Syzygies

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Linz, February 2009

Differential invariants arise in equivalence and classification problems and are used in symmetry reduction techniques

Original motivation: symmetry reduction with a view towards differential elimination.
[Mansfield 01]
Seminal results: the reinterpretation of Cartan's moving frame method.
[Fels & Olver 99]

We introduce today the computationally relevant algebraic structures.
[H05, HK07, H08, H09]

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2 Lie Group Actions and their Invariants **3**

3 Normalized Invariants: Geometric and Algebraic Construction **6**

4 Differential Algebra of invariants **7**

5 Bibliography **10**

1 Differential Algebra

Differential Polynomial Rings

$$\mathbb{F} \text{ a field}$$

$$\mathbb{F} = \mathbb{Q}(x, y)$$

$$\Delta = \{\delta_1, \ldots, \delta_m\} \text{ derivations on } \mathbb{F}$$

$$\delta_1 = \frac{\partial}{\partial x},\ \delta_2 = \frac{\partial}{\partial y}$$

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\mathcal{Y} = \{\phi, \psi\}$$

$$\mathbb{F}[\, y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\,] = \mathbb{F}\,[\![\mathcal{Y}]\!]$$

$$\mathbb{F}\,[\![\phi, \psi]\!] = \mathbb{F}[\phi, \phi_x, \phi_y, \ldots, \psi \ldots]$$

$$\phi_{xxy} \rightsquigarrow \phi_{x^2y} \rightsquigarrow \phi_{(2,1)}$$

$$\delta_i\,(y_\alpha) = y_{\alpha+\epsilon_i}$$

$$\frac{\partial}{\partial x}\,(\phi_{xxy}) = \phi_{xxxy} \rightsquigarrow \delta_1\left(\phi_{(2,1)}\right) = \phi_{(3,1)}$$

$$\epsilon_i = (0, \ldots, \underset{i^{\text{th}}}{1}, \ldots, 0)$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

$$\delta_i\delta_j = \delta_j\delta_i$$

Derivations with nontrivial commutations

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\Delta = \{\delta_1, \ldots, \delta_m\}$$

$$\delta_i\,\delta_j - \delta_j\,\delta_i = \sum_{l=1}^{m} c_{ijl}\,\delta_l$$

$$c_{ijl} \in \mathbb{K}\,[\![\mathcal{Y}]\!]$$

$$\mathbb{K}\,[\![\mathcal{Y}]\!]?$$

Differential polynomial ring $\mathbb{K}\,[\![\mathcal{Y}]\!]$ with non commuting derivations

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\mathrm{D} = \{\delta_1, \ldots, \delta_m\}$$

$$\mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}]$$

$$\delta_i\,(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} \text{ if} \alpha_1 = \ldots = \alpha_{i-1} = 0 \\[1ex] \delta_j\delta_i\left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\,\delta_l\left(y_{\alpha-\epsilon_j}\right) \\ \quad \text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \quad \text{while } \alpha_1 = \ldots = \alpha_{j-1} = 0 \end{cases}$$

If the c_{ijl} satisfy

- $c_{ijl} = -c_{jil}$
- $\delta_k(c_{ijl}) + \delta_i(c_{jkl}) + \delta_j(c_{kil}) = \sum_{\mu=1}^{m} c_{ij\mu}c_{\mu kl} + c_{jk\mu}c_{\mu il} + c_{ki\mu}c_{\mu jl}$

& there exists an *admissible ranking* $\prec$

- $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec y_\beta$,
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} \prec z_{\beta+\gamma}$,
- $\sum_{l \in \mathbb{N}_m} c_{ijl}\,\delta_l(y_\alpha) \prec y_{\alpha+\epsilon_i+\epsilon_j}$

then $\delta_i\delta_j(p) - \delta_j\delta_i(p) = \sum_{l=1}^{m} c_{ijl}\,\delta_l(p) \quad \forall p \in \mathbb{K}[\, y_\alpha \mid \alpha \in \mathbb{N}^m\,] = \mathbb{K}\,[\![\mathcal{Y}]\!]$

[H05]

2 Lie Group Actions and their Invariants

Lie Group $\mathcal{G}$

$\mathcal{G}$ a r-dimensional smooth manifold, locally parameterized by $\mathbb{R}^r$

$$\begin{array}{rccl} m: & \mathcal{G}\times\mathcal{G} & \rightarrow & \mathcal{G} \\ & (\lambda,\mu) & \mapsto & \lambda\cdot\mu \end{array} \quad \text{and} \quad \begin{array}{rccl} i: & \mathcal{G} & \rightarrow & \mathcal{G} \\ & \lambda & \mapsto & \lambda^{-1} \end{array} \quad \text{smooth}$$

$$e\in\mathcal{G} \quad e\cdot\lambda=\lambda\cdot e=\lambda$$

Group action

$\mathcal{G}$ a Lie group $\qquad$ $\mathcal{M}$ an open subset of $\mathbb{R}^n$

$$\text{Action } g: \begin{array}{ccc} \mathcal{G}\times\mathcal{M} & \rightarrow & \mathcal{M} \\ (\lambda,z) & \mapsto & \lambda\star z \end{array} \quad \text{smooth}$$

$$e\star z=z \qquad (\lambda\cdot\mu)\star z=\lambda\star(\mu\star z)$$

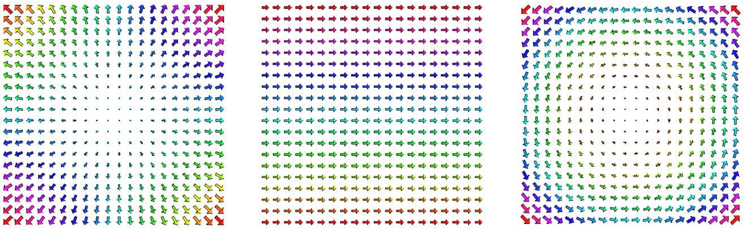
Orbit of z: $\mathcal{O}_z=\{\lambda\star z \mid \lambda\in\mathcal{G}\}\subset\mathcal{M}$

Semi-regular Lie group actions

	scaling	translation+reflection	rotation
$\mathcal{G}$	$\mathbb{R}^*$	$\mathbb{R}\times\{-1,1\}$	$SO(2)$
$\mathcal{M}$	$\mathbb{R}^2\setminus O$	$\mathbb{R}^2$	$\mathbb{R}^2\setminus O$
$\lambda\star z$	$\begin{pmatrix}\lambda z_1\\ \lambda z_2\end{pmatrix}$	$\begin{pmatrix} z_1+\lambda_1\\ \lambda_2 z_2\end{pmatrix}$	$\begin{pmatrix}\cos\lambda & -\sin\lambda\\ \sin\lambda & \cos\lambda\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$

Orbits:

Infinitesimal generator



$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ $\qquad$ $\frac{\partial}{\partial x}$ $\qquad$ $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$

Infinitesimal generators

$$\xi_1\frac{\partial}{\partial z_1}+\ldots+\xi_d\frac{\partial}{\partial z_d}$$

a vector field the flow of which is the action of a one-dimensional (connected) subgroup of $\mathcal{G}$.
$\mathrm{V}_1,\ldots,\mathrm{V}_r$ a basis of infinitesimal generators for the action on $\mathcal{M}$ of the r-dimensional group $\mathcal{G}$.

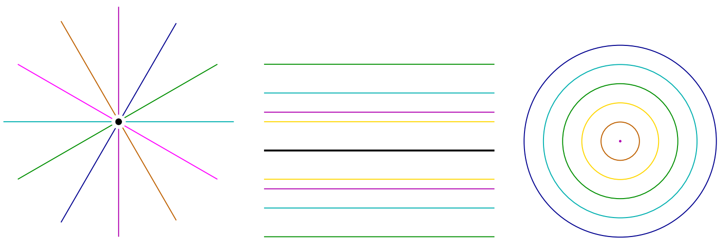
Local Invariants

$f:\mathcal{U}\subset\mathcal{M}\to\mathbb{RK}$ smooth

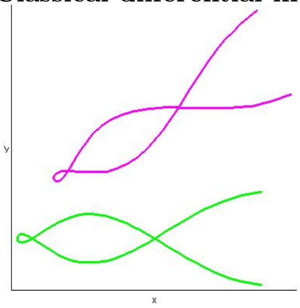
$$f(\lambda\star z)=f(z) \text{ for } \lambda\in\mathcal{G} \text{ close to } e$$
$$\Leftrightarrow$$
$$f \text{ is constant on orbits within } \mathcal{U}$$

$$\Leftrightarrow$$
$$\mathrm{V}_1(f)=0,\ldots,\mathrm{V}_r(f)=0$$

Examples

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K}\times\{-1,1\}$	$SO(2)$
			
rational	$\frac{x}{y}$	y^2	x^2+y^2
local	$\frac{x}{y}$	y	$\sqrt{x^2+y^2}$

Classical differential invariants



$E(2)$ $\qquad \alpha^2+\beta^2=1$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$Y_X = \frac{\beta + \alpha y_x}{\alpha - \beta\, y} \quad Y_{XX} = \frac{y_{xx}}{(\alpha - \beta\, y)^3}$$

Curvature: $\sigma = \sqrt{\frac{y_{xx}^2}{(1+y_x^2)^3}}$ a differential invariant

Arc length: $ds = \sqrt{1+y_x^2}\, dx$

Invariant derivation: $\dfrac{d}{ds} = \dfrac{1}{\sqrt{1+y_x^2}} \dfrac{d}{dx}$

Jets / Differential algebraProlongation

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $\qquad g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ $\qquad \mathrm{V}_1^0, \dots, \mathrm{V}_r^0$

$(x_1, \dots, x_m)$ coordinates on $\mathcal{X} \rightsquigarrow$ independent variables

$(u_1, \dots, u_n)$ coordinates on $\mathcal{U} \rightsquigarrow$ dependent variables

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $\qquad g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$ $\qquad \mathrm{V}_1^k, \dots, \mathrm{V}_r^k$

[DifferentialGeometry]

additional coordinates $u_\alpha = \dfrac{\partial^{|\alpha|} u}{\partial x^\alpha}$, $|\alpha| \leq k$
$\rightsquigarrow$ the derivatives of u w.r.t x up to order k

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_\alpha u_{\alpha+\epsilon_i} \frac{\partial}{\partial u_\alpha}$$

Differential polynomial ring: $\mathbb{K}(x)\, [\![u]\!] = \mathbb{K}(x)\, [u_\alpha \,|\, \alpha \in \mathbb{N}^m]$

$$\mathrm{D}_i u_\alpha = u_{\alpha+\epsilon_i}.$$

[diffalg]

$f : \mathrm{J}^k \to \mathbb{R}$ differential invariant of order k if $\mathrm{V}^k(f) = 0$.

Invariant derivation

$$\mathcal{D} : \mathcal{F}(\mathrm{J}^k) \to \mathcal{F}(\mathrm{J}^{k+1}) \;\text{ s.t }\; \mathcal{D} \circ \mathrm{V} = \mathrm{V} \circ \mathcal{D}$$

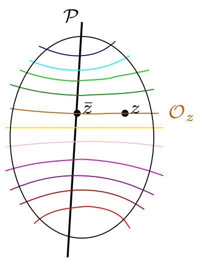
$f : \mathrm{J}^k \to \mathbb{R}$ a differential invariant
$\Rightarrow$ $\mathcal{D}(\mathrm{f})$ a differential invariant of order $k+1$.

What is a computationnally relevant algebraic structure for differential invariants?

$$\mathbb{K}\,[\![y_1, \ldots, y_n]\!] \;/\; [\![S]\!]$$

3 Normalized Invariants: Geometric and Algebraic Construction

Local cross-section $\mathcal{P}$



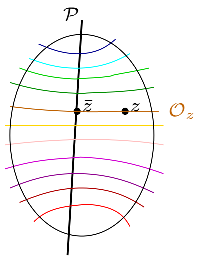
- $\mathcal{P}$ an embedded manifold of dimension $n-d$
  $\mathcal{P} = \{z \in \mathcal{U} \mid p_1(z) = \ldots = p_d(z) = 0\}$
- $\mathcal{P}$ is transverse to $\mathcal{O}_z$ at $z \in \mathcal{P}$.
- $\mathcal{P}$ intersect $\mathcal{O}_z^0$ at a unique point, $\forall z \in \mathcal{U}$.

$\Leftrightarrow$ the matrix $(V_i(p_j))_{1 \le i \le r, 1 \le j \le d}$ has rank d on $\mathcal{P}$.

A local invariant is uniquely determined by a function on $\mathcal{P}$.

[Fels Olver 99, H. Kogan 07b]

Invariantization $\bar{\iota} f$ of a function f



$f : \mathcal{U} \to \mathbb{R}$ smooth

$\bar{\iota} f$ is the unique *local* invariant with $\bar{\iota} f|_{\mathcal{P}} = f|_{\mathcal{P}}$

$$\bar{\iota} f(z) = f(\bar{z})$$

Normalized invariants: $\bar{\iota} z_1, \ldots, \bar{\iota} z_n$. $\qquad \bar{\iota} f(z) = f(\bar{\iota} z)$

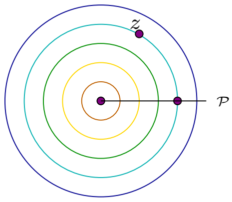
Generation and rewriting:
f local invariant $\Rightarrow$ $f(z_1, \ldots, z_n) = f(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$

Relations: $p_1(\bar{\iota} z_1, \ldots, \bar{\iota} z_n) = 0, \ldots, p_d(\bar{\iota} z_1, \ldots, \bar{\iota} z_n) = 0$

[Fels Olver 99, H. Kogan 07b]

Normalized invariants. Example.

$\mathcal{G} = SO(2)$, $\mathcal{M} = \mathbb{R}^2 \setminus O$



$\mathcal{P} : z_2 = 0,\ z_1 > 0$ $\mathcal{U} = \mathcal{M}$

$$(\bar{\iota} z_1, \bar{\iota} z_2) = \left(\sqrt{z_1^2 + z_2^2}, 0\right)$$

Replacement property:

$$f(z_1, z_2) \text{ invariant } \Rightarrow f(z_1, z_2) = f(\bar{\iota} z_1, 0).$$

Normalized invariants in practice

We mostly do not need $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ explicitly.

We can work formally with $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$, subject to the relationships $p_1(\bar{\iota} z) = 0, \ldots, p_d(\bar{\iota} z) = 0$.

Computing normalized invariants

In the algebraic case, the normalized invariants $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ form a $\overline{\mathbb{K}(z)}^G$-zero of the *graph-section* ideal

$$(G + (Z - \lambda \star z) + P) \cap \mathbb{K}(z)[Z]$$

The coefficients of the reduced Gröbner basis of the graph-section ideal form a generating set for $\mathbb{K}(z)^G$ endowed with a simple rewriting algorithm.

[H. Kogan 07a 07b]

4 Differential Algebra of invariants

Differential invariants

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ $\mathrm{V}_1^0, \ldots, \mathrm{V}_r^0$

$(x_1, \ldots, x_m)$ coordinates on $\mathcal{X}$

$(u_1, \ldots, u_n)$ coordinates on $\mathcal{U}$

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$ $\mathrm{V}_1^k, \ldots, \mathrm{V}_r^k$

additional coordinates $u_\alpha = \dfrac{\partial^{|\alpha|} u}{\partial x^\alpha}$, $|\alpha| \leq k$

Normalized invariants of order k

$$\mathcal{I}^k = \{\bar\iota x_1, \ldots, \bar\iota x_m\} \cup \{\bar\iota u_\alpha \,|\, |\alpha| \le k\}$$

Generation in finite terms

r_k, the dimension of orbits on J^k, stabilizes at order s

$$r_0 \le r_1 \le \ldots \le r_s = r_{s+1} = \ldots = r.$$

$\mathcal{P}^s : p_1 = 0, \ldots, p_r = 0$ defines a cross-section on J^{s+k}

$$\mathcal{I}^{s+k} = \{\bar\iota x_1, \ldots, \bar\iota x_m\} \cup \{\bar\iota u_\alpha \,|\, |\alpha| \le s+k\}$$

Construct: $\mathcal{D}_1, \ldots, \mathcal{D}_m : \mathcal{F}(\mathrm{J}^{s+k}) \to \mathcal{F}(\mathrm{J}^{s+k+1})$ s.t. $\mathcal{D}_i \mathrm{V}_a = \mathrm{V}_a \mathcal{D}_i$

Key Prop: $\bar\iota u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar\iota u_\alpha) + K_{ia}\,\bar\iota\,(\mathrm{V}_a(u_\alpha))$ $\qquad K = \bar\iota\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$

Col: Any differential invariants can be contructively written in terms of $\mathcal{I}^{s+1}$ and their derivatives.

[Fels Olver 99]

Algebra of Differential Invariants

$$\mathbb{K}\,[\![\mathfrak{x}_i, \mathfrak{u}_\alpha \,|\, |\alpha| \le s+1]\!] \;/\; [\![S]\!]$$

$$\left[\mathcal{D}_i, \mathcal{D}_j\right] = \sum_{k=1}^{m} \Lambda_{ijk}\,\mathcal{D}_k$$
where $\Lambda_{ijk} = \sum_{c=1}^{r} K_{ic}\,\bar\iota(\mathrm{D}_j(\mathrm{V}_c(x_k))) - K_{jc}\,\bar\iota(\mathrm{D}_i(\mathrm{V}_c(x_k)))$.

The monotone derivatives of $\mathcal{I}^{s+1}$,

$$\left\{\mathcal{D}_1^{\beta_1} \ldots \mathcal{D}_m^{\beta_m}(\bar\iota x_i)\right\} \cup \left\{\mathcal{D}_1^{\beta_1} \ldots \mathcal{D}_m^{\beta_m}(\bar\iota u_\alpha) \,|\, |\alpha| \le s+1\right\},$$

generate all differential invariants.

[H 05, 08]

Syzygies = Differential relationships

A subset S of the following relationships

$$p_1(\bar\iota x, \bar\iota u_\alpha) = 0, \ \ldots, \ p_r(\bar\iota x, \bar\iota u_\alpha) = 0$$

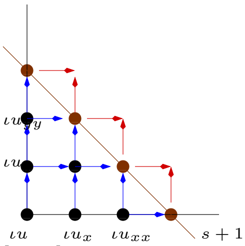
$$\mathcal{D}_i(\bar\iota x_j) = \delta_{ij} - K_{ia}\bar\iota\,(\mathrm{V}(x_j)),$$

$$\mathcal{D}_i(\bar\iota u_\alpha) = \bar\iota u_{\alpha+\epsilon_i} - K_{ia}\bar\iota\,(\mathrm{V}(u_\alpha)),\ |\alpha| \leq s$$

$$\mathcal{D}_i(\bar\iota u_\alpha) - \mathcal{D}_j(\bar\iota u_\beta) = K_{ja}\bar\iota\,(\mathrm{V}(u_\beta)) - K_{ia}\bar\iota\,(\mathrm{V}(u_\alpha)),$$

$$\alpha + \epsilon_i = \beta + \epsilon_j,\ |\alpha| = |\beta| = s+1.$$

form a *complete set of differential syzygies.*



[H08]

$$\bar\iota u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar\iota u_\alpha) + K_{ia}\,\bar\iota\,(\mathrm{V}_a(u_\alpha))$$

$$K = \bar\iota\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$$

Representations

Generators + Syzygies + Rewriting

- Differential elimination on a complete set of syzygies allows to reduce the number of differential invariants.
- We can reduce substantially the number of generating invariants by differential elimination on the syzygies.
  - Euclidean and affine surfaces [Olver 07]
  - Conformal and projective surfaces [H. Olver 07]
  - Special orthogonal 3-dimensional manifolds [H09]

[AIDA], [*diffalg*]

Edge and Maurer-Cartan invariants

We can always restrict to $m\,r + d_0$ generating invariants

$$\bar\iota u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar\iota u_\alpha) + K_{ia}\,\bar\iota\,(\mathrm{V}_a(u_\alpha)) \qquad K = \bar\iota\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$$

Thm: The *edge invariants* $\mathcal{E} = \{\bar\iota(\mathcal{D}_i(p_a))\} \cup \mathcal{I}^0$ form a generating set when the the cross-section is of minimal order.

We can obtain their syzygies by elimination.

Thm: The *Maurer-Cartan invariants* $\{K_{ia}\} \cup \mathcal{I}^0$ form a generating set of differential invariants.

We can obtain their syzygies from the structure equations.

[H08, H09]

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Origins of the project: Symmetry reduction

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Computing (algebraic) invariants

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Differential Algebraic Structure of Invariants

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Computing in the Algebra of Differential Invariants

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Software

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Ideal Intersections in Rings of Partial Differential Operators

Fritz Schwarz

Outline of the Talk

(1) Motivation & Examples.

(2) Basic Concepts from Differential Algebra.

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Motivation & Examples 1

Goal of Complete Theory

- ▷ Describe all possible types of closed form solutions of linear pde's.
- ▷ Which cases may be solved algorithmically? Design and implement algorithms for them.
- ▷ A. D. Polyanin, *Handbook of Linear Partial Differential Equations, Chapman & Hall/CRC, 2002.*

EXAMPLE 1. *(Forsyth 1906)*

$$Lz \equiv z_{xy} + \frac{2}{x-y}z_x - \frac{2}{x-y}z_y - \frac{4}{(x-y)^2}z = 0.$$

General Solution: F, G undetermined functions.

$$z = 2(x-y)F(y) + (x-y)^2F'(y) - 2(x-y)G(x) + (x-y)^2G'(x).$$

Loewy Decomposition:

$$L = Lclm\Big(\Big\langle \partial_{xx} - \frac{2}{x-y}\partial_x + \frac{2}{(x-y)^2}, \partial_{xy} + \frac{2}{x-y}\partial_x - \frac{2}{x-y}\partial_y - \frac{4}{(x-y)^2}\Big\rangle,$$

$$\Big\langle \partial_{xy} + \frac{2}{x-y}\partial_x - \frac{2}{x-y}\partial_y - \frac{4}{(x-y)^2}, \partial_{yy} + \frac{2}{x-y}\partial_y + \frac{2}{(x-y)^2}\Big\rangle\Big).$$

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Motivation & Examples 2

EXAMPLE 2.

$$Lz = z_{xxx} + (y+1)z_{xxy} + \big(1 - \tfrac{1}{x}\big)z_{xx}$$

$$+\big(1 - \tfrac{1}{x}\big)(y+1)z_{xy} - \tfrac{1}{x}z_x - \tfrac{1}{x}(y+1)z_y = 0.$$

General Solution. F, G, H undetermined functions.

$$z = F(y)e^{-x} + G\big((y+1)e^{-x}\big) + (x+1)e^{-x}\int H(y)(y+1)dy.$$

Loewy Decomposition.

$$L = \Big(\partial_x - \frac{1}{x}\Big)(\partial_{xx} + (y+1)\partial_{xy} + \partial_x + (y+1)\partial_y$$

$$= \Big(\partial_x - \frac{1}{x}\Big)Lclm(\partial_x + 1, \partial_x + (y+1)\partial_y).$$

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Motivation & Examples 3

Example 3. *(Blumberg 1912).*

$$Lz = z_{xxx} + xz_{xxy} + 2z_{xx} + 2(x+1)z_{xy} + z_x + (x+2)z_y = 0.$$

General Solution. F, G, H undetermined functions.

$$z = F(y - \tfrac{1}{2}x^2) + G(y)e^{-x} + \int H\big(\bar{y} + \tfrac{1}{2}x^2\big)e^{-x}dx\Big|_{\bar{y}=y-\frac{1}{2}x^2}.$$

Factorizations.

$$L = \begin{cases} \big(\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y\big)(\partial_x + 1) \\ Lclm\big(\partial_x + 1, \partial_x + 1 - \dfrac{1}{x}\big)(\partial_x + x\partial_y). \end{cases}$$

$$\begin{aligned} &Lclm(\partial_x + 1, \partial_x + x\partial_y) = \\ &\quad \langle L_1 \equiv \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y, \\ &\quad\quad L_2 \equiv \partial_{xxy} + x\partial_{xyy} - \frac{1}{x}\partial_{xx} - \frac{1}{x}\partial_{xy} + x\partial_{yy} - \frac{1}{x}\partial_x - \big(1 + \frac{1}{x}\big)\partial_y\rangle \end{aligned}$$

Loewy Decomposition.

$$L = \begin{pmatrix} (1, x) \\ \big(0, \partial_x + 1 + \dfrac{1}{x}\big) \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

===

Basic Concepts from Differential Algebra

Rings of partial differential operators:

$$\mathcal{D} \equiv \mathbb{Q}(x, y)[\partial_x, \partial_y] \text{ and } \mathcal{D} \equiv \mathbb{Q}(x, y, z)[\partial_x, \partial_y, \partial_z]$$

Left Ideal: $I = \langle l_1, l_2, \ldots \rangle$, $l_i \in \mathcal{D}$, form Janet basis.

Hilbert-Kolchin Polynomial: $H_I(n) \equiv \binom{n+k}{n} - dim\, I_n$; $k = 1, 2$.

Gauge: $g_I \equiv (deg\, H_I, lcoef\; H_I)$

$= (Differential\ type, typical\ differential\ dimension) \simeq size\ of\ solutions.$

Least common left multiple: $Lclm(I, J)$.

Greatest common right divisor: $Gcrd(I, J)$.

Leading terms of an ideal: $I = \langle ... \rangle_{LT}$.

Terms not higher than a given term: $O(\tau)$

General Reference: E. Kolchin, Differential Algebra and Algebraic Groups, Academic Press, 1973.

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$\boxed{\text{Ideal Intersections in the Plane 1}}$

THEOREM 1. *Let the ideals* $I_i = \langle \partial_x + a_i\partial_y + b_i \rangle$ *for* $i = 1, 2$ *with* $I_1 \neq I_2$ *be given. Both ideals have gauge* $(1, 1)$. *There are three different cases for their intersection* $I_1 \cap I_2$, *all are of gauge* $(1, 2)$.

i) If $a_1 \neq a_2$ *and* $\left(\dfrac{b_1 - b_2}{a_1 - a_2} \right)_x = \left(\dfrac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \right)_y$ *there holds*

$$I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT} \text{ and } I_1 + I_2 = \langle \partial_x, \partial_y \rangle_{LT}.$$

ii) If $a_1 \neq a_2$ *and* $\left(\dfrac{b_1 - b_2}{a_1 - a_2} \right)_x \neq \left(\dfrac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \right)_y$ *there holds*

$$I_1 \cap I_2 = \langle \partial_{xxx}, \partial_{xxy} \rangle_{LT} \text{ and } I_1 + I_2 = \langle 1 \rangle.$$

iii) If $a_1 = a_2 = a$ *and* $b_1 \neq b_2$ *there holds*

$$I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT} \text{ and } I_1 + I_2 = \langle 1 \rangle.$$

==

Ideal Intersections in the Plane 2

PROOF. Auxiliary parameter u, define

$$u(\partial_x + a_1\partial_y + b_1) \text{ and } (1-u)(\partial_x + a_2\partial_y + b_2)$$

New indeterminate $w = uz$, lexicographic term ordering $w > z$ yields

$$w_x + a_1 w_y + b_1 w \text{ and } w_x + a_2 w_y + b_2 w - z_x - a_2 z_y - b_2 z. \tag{1}$$

If $a_1 \neq a_2$ autoreduction leads to

$$\begin{aligned} &w_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} w - \frac{a_1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z), \\ &w_y + \frac{b_1 - b_2}{a_1 - a_2} w + \frac{1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z). \end{aligned} \tag{2}$$

Defining $U \equiv z_x + a_2 z_y + b_2 z$, integrability condition is

$$\begin{aligned} &\left[\left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)_y - \left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x\right] w - \frac{1}{a_1 - a_2} U_x - \frac{a_1}{a_1 - a_2} U_y \\ &\qquad - \left[\left(\frac{1}{a_1 - a_2}\right)_x + \left(\frac{a_1}{a_1 - a_2}\right)_y + \frac{b_1}{a_1 - a_2}\right] U = 0. \end{aligned} \tag{3}$$

If coefficient of w vanishes, (2) and (3) are Janet basis. This is case i).

If coefficient of w does not vanish, use (3) to eliminate w in (2). Result has leading derivatives u_{xxx} and u_{xxy}. This is case ii).

If $a_1 = a_2 = a$ autoreduction of (1) yields two expressions of the type $w + O(z_x)$ and $O(z_{xx})$. This is case iii). □

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Ideal Intersections in the Plane 3

EXAMPLE 4. Consider the two gauge $(1,1)$ ideals

$$I_1 = \langle \partial_x + 1 \rangle \text{ and } I_2 = \langle \partial_x + (y+1)\partial_y \rangle.$$

Condition for case $i)$ of Theorem 1 is satisfied. Consequently

$$Lclm(I_1, I_2) = \langle \partial_{xx} + (y+1)\partial_{xy} + \partial_x + (y+1)\partial_y \rangle,$$

$$Gcrd(I_1, I_2) = \langle \partial_x + 1, \partial_y - \frac{1}{y+1} \rangle$$

of gauge $(1,2)$ and $(0,1)$ respectively. $\square$

EXAMPLE 5. The two ideals $I_1 = \langle \partial_x + 1 \rangle$ and $I_2 = \langle \partial_x + x\partial_y \rangle$, both of gauge $(1,1)$, do not satisfy the condition of case $i)$; furthermore there holds $a_1 \neq a_2$. Therefore by case $ii)$ the intersection ideal is

$$Lclm(I_1, I_2) = \langle \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy}$$
$$+2\partial_x + (2x+3)\partial_y, \partial_{xxy} + x\partial_{xyy} - \frac{1}{x}\partial_{xy} + x\partial_{yy} - \frac{1}{x}\partial_x - \big(1 + \frac{1}{x}\big)\partial_y \rangle$$

of gauge $(1,2)$; $Gcrd(I_1, I_2) = \langle 1 \rangle$. $\square$

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Ideal Intersections in Three-Space 1

THEOREM 2. *Let the ideals $I_i = \langle \partial_x + a_i\partial_y + b_i\partial_z + c_i \rangle$ for $i = 1, 2$ with $I_1 \neq I_2$ be given; 9 cases for their intersection ideal $I_1 \cap I_2$ have to be distinguished; their gauge is always $(2, 2)$. The expressions P, Q, R, S_1, S_2, T_1 and T_2 involving only the coefficients of the I_i, and U, V and W involving also the indeterminate u are defined below in the proof.*

i) *If $a_1 \neq a_2$ and $P = Q = 0$, or $a_1 = a_2 = a$ and $b_1 \neq b_2$, there holds* $I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT}$ *and* $I_1 + I_2 = \langle \partial_x, \partial_y \rangle_{LT}$.

ii) *If $a_1 = a_2 = a$, $b_1 = b_2 = b$, $c_1 \neq c_2$ and $P = 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

iii) *If $a_1 \neq a_2$, $P = 0$ and $Q \neq 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxx}, \partial_{xxy} \rangle_{LT}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

iv) *If $a_1 \neq a_2$, $P \neq 0$ and $S_1 = S_2 = 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxx}, \partial_{xxy} \rangle_{LT}$ *and* $I_1 + I_2 = \langle \partial_x, \partial_y, \partial_z \rangle_{LT}$.

v) *If $a_1 = a_2 = a$, $a_z \neq 0$ and $T_1 = T_2 = 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxx}, \partial_{xxz} \rangle_{LT}$ *and* $I_1 + I_2 = \langle \partial_x, \partial_y, \partial_z \rangle_{LT}$.

vi) *If $a_1 = a_2 = a$, $a_z = 0$ and $R \neq 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxx}, \partial_{xxz} \rangle_{LT}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

vii) *If $a_1 \neq a_2 = a$, $P \neq 0$ and $S_2 \neq 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxyy}, \partial_{xxyz}, \partial_{xxx} \rangle_{LT}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

viii) *If $a_1 \neq a_2$, $P \neq b_2$, $S_1 \neq 0$ and $S_2 = 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxxx}, \partial_{xxxz}, \partial_{xxy} \rangle_{LT}$ *and* $I_1 + I_1 = \langle 1 \rangle$.

ix) *If $a_1 = a_2 = a$, $b_1 \neq b_2$, $a_z \neq 0$ and $T_2 \neq 0$ there holds* $I_1 \cap I_2 = \langle \partial_{xxyz}, \partial_{xxzz}, \partial_{xxx} \rangle_{LT}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

==

Ideal Intersections in Three-Space 2

Proof. Define differential polynomials

$$
(4)\qquad \begin{gathered} w_x + a_1 w_y + b_1 w_z + c_1 w, \\ w_x + a_2 w_y + b_2 w_z + c_2 w - u_x - a_2 u_y - b_2 u_z - c_2 u. \end{gathered}
$$

Term order lex, $w > u$ and $x > y > z$. If $a_1 \neq a_2$ define

$$
(5)\qquad U \equiv u_x + a_2 u_y + b_2 u_z + c_2 u = O(u_x)
$$

Autoreduction of (4)

$$
(6)\qquad w_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} w_z + \frac{a_1 c_2 - a_2 c_1}{a_1 - a_2} w - \frac{a_1}{a_1 - a_2} U,
$$

$$
(7)\qquad w_y + \frac{b_1 - b_2}{a_1 - a_2} w_z + \frac{c_1 - c_2}{a_1 - a_2} w + \frac{1}{a_1 - a_2} U.
$$

Single integrability condition between (6) and (7).

$$
P w_z + Q w + V = 0
$$

where

$$
(8)\qquad \begin{aligned} P \equiv & \left(\frac{b_1 - b_2}{a_1 - a_2} \right)_x - \left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \right)_y \\ & + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \left(\frac{b_1 - b_2}{a_1 - a_2} \right)_z - \frac{b_1 - b_2}{a_1 - a_2} \left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \right)_z, \end{aligned}
$$

$$
(9)\qquad \begin{aligned} Q \equiv & \left(\frac{c_1 - c_2}{a_1 - a_2} \right)_x - \left(\frac{a_1 c_2 - a_2 c_1}{a_1 - a_2} \right)_y \\ & + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \left(\frac{c_1 - c_2}{a_1 - a_2} \right)_z - \frac{b_1 - b_2}{a_1 - a_2} \left(\frac{a_1 c_2 - a_2 c_1}{a_1 - a_2} \right)_z, \end{aligned}
$$

$$
(10)\qquad \begin{aligned} V \equiv & \frac{1}{a_1 - a_2} (U_x + a_1 U_y + b_1 U_z + c_1 U) \\ & - \frac{1}{(a_1 - a_2)^2} [(a_1 - a_2)_x + a_{1,z} b_2 - a_{2,z} b_1 + a_{1,y} a_2 - a_{2,y} a_1] U \end{aligned}
$$

If $P = Q = 0 \longrightarrow case\, i)$. If $P = 0$, $Q \neq 0 \longrightarrow case\, iii)$.

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Ideal Intersections in Three-Space 3

EXAMPLE 6. Let $d_1 \equiv \partial_x - \partial_y$ and $d_2 \equiv \partial_x - \partial_z$.
By Theorem 2, case i), there holds

$$Lclm(d_1, d_2) = \partial_{xx} - \partial_{xy} - \partial_{xz} + \partial_{yz}$$

$$Gcrd(d_1, d_2) = \langle \partial_x - \partial_z, \partial_y - \partial_z \rangle$$

EXAMPLE 7. Let $d_1 \equiv \partial_x - \partial_y + z$ and $d_2 \equiv \partial_x - \partial_z$.
By Theorem 2, case ii), there holds

$$\begin{aligned} Lclm(d_1, d_2) = \langle & \partial_{xxx} - 3\partial_{xxz} - \partial_{xyy} + 2(\partial_{xyz} + \partial_{yzz}) \\ & -2z(\partial_{xz} + \partial_{yz} - \partial_{zz}) - (z^2 - 4)(\partial_x - \partial_z), \\ & \partial_{xxy} - \partial_{xxx} - +\partial_{yyz} - \partial_{yzz} \\ & -z(\partial_{xx} - \partial_{zz}) + 2z(\partial_{xy} - \partial_{yz}) - (z^2 + 2)(\partial_x + \partial_z) \rangle \end{aligned}$$

==

Basic Concepts from Differential Algebra

THEOREM 3. *The left ideals in the rings* $\mathbb{Q}(x,y)[\partial_x,\partial_y]$ *and* $\mathbb{Q}(x,y,z)[\partial_x,\partial_y,\partial_z]$ *have the following properties.*

i) The ideals form a lattice w.r.t. Gcrd and Lclm.

ii) The ideals of differential type zero form a sublattice.

iii) The principal ideals do <u>*not*</u> *form a sublattice.*

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Summary & Further Work

→ Sum and intersection of first-order left operators in the rings $\mathbb{Q}(x,y)[\partial_x,\partial_y]$ and $\mathbb{Q}(x,y,z)[\partial_x,\partial_y,\partial_z]$ are completely classified. This is the foundation for a theory of decomposing second- and third-order operators.

→ Determine the possible right factors of higher-order operators and the resulting structure of the solutions of the corresponding equations.

→ The what extent can the factorization be performed algorithmically? Is the existence of first-order right factors in general decidable? If not, what is the boarderline for decidability? (Darboux polynomials, Laplace divisors).

→ Is it possible the generalize these algebraic methods to certain classes of non-linear equations?

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A Skew Polynomial Approach to Integro-Differential Operators*

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* Joint work with Johannes Middeke

What are Integro-Differential Operators?

Definition

For an integro-differential algebra $\mathcal{F}$ over a field K, we construct

$$\mathcal{F}[\partial, \textstyle\int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\textstyle\int] \dotplus \mathcal{F}[\mathrm{E}],$$

as the K-algebra of *integro-differential operators*. Structure follows!

Construct summands as left $\mathcal{F}$-submodules of $K\langle \mathcal{B}, \partial, \int, \mathrm{E}\rangle$.

- First summand $\mathcal{F}[\partial]$, generated over $\mathcal{F}$ by $(\partial^i \mid i \geq 0)$.
- Second summand $\mathcal{F}[\int]$, generated over $\mathcal{F}$ by $(\int b \mid b \in \mathcal{B})$.
- Third summand $\mathcal{F}[\mathrm{E}]$, generated over $\mathcal{F}$ by $(\mathrm{E}\partial^i \mid i \geq 0)$.

We have an action $\bullet \colon \mathcal{F}[\partial, \int] \times \mathcal{F} \to \mathcal{F}$ given by:

$$\partial \colon \mathcal{F} \to \mathcal{F} \qquad \textstyle\int \colon \mathcal{F} \to \mathcal{F} \qquad \mathrm{E} = 1 - \textstyle\int \circ \partial$$

What is an Integro-Differential Algebra?

Definition

Let $\mathcal{F}$ be an algebra over a field K. If $(\mathcal{F}, \partial)$ is a differential algebra, $\int : \mathcal{F} \to \mathcal{F}$ a K-linear section of the derivation (meaning $\partial \int = 1$), and the *differential Baxter axiom*

$$(\textstyle\int f')(\int g') = (\int f')g + f(\int g') - \int (fg)'$$

is satisfied, we call $(\mathcal{F}, \partial, \int)$ an *integro-differential algebra*. We require $(\mathcal{F}, \partial)$ *ordinary* in the sense that $\dim \operatorname{Ker}(\partial) = 1$.

Immediate consequences:

- Plain Baxter axiom $(\int f)(\int g) = \int (f \int g) + \int (g \int f)$
- Evaluation $\mathtt{E} = 1 - \int \partial$ is a character.

Algebra Structure

Think of the prototype model $\mathcal{F} = C^\infty(\mathbb{R})$ or $\mathcal{F} = K[x]$.

$$\partial \bullet f = \frac{df}{dx} \qquad \textstyle\int \bullet f = \displaystyle\int_0^x f(x)\,dx \qquad \mathtt{E} \bullet f = f(0)$$

Multiplication table:

gf	$=$	$g \bullet f$	∂f	$=$	$f \partial + \partial \bullet f$
$\mathtt{E}^2$	$=$	$\mathtt{E}$	$\partial \int$	$=$	1
$\mathtt{E} f$	$=$	$(\mathtt{E} \bullet f)\,\mathtt{E}$	$\partial \mathtt{E}$, $\mathtt{E} \int$	$=$	0
$\int f \int$	$=$	$(\int \bullet f) \int - \int (\int \bullet f)$			
$\int f \partial$	$=$	$f - \int (\partial \bullet f) - (\mathtt{E} \bullet f)\,\mathtt{E}$			
$\int f \mathtt{E}$	$=$	$(\int \bullet f)\,\mathtt{E}$			

Remarks on Integro-Differential Operators

Possible but cumbersome to define multiplication on K-basis.
Subalgebras $\mathcal{F}[\partial], \mathcal{F}[\int], \mathcal{F}[\mathrm{E}] \leq \mathcal{F}[\partial, \int]$.
Unlike $\mathcal{F}[\partial]$, the algebras $\mathcal{F}[\int]$ and $\mathcal{F}[\mathrm{E}]$ have no unit.

Third summand $\mathcal{F}[\mathrm{E}]$ coincides with *evaluation ideal* (E):

$$\boxed{\mathcal{F}[\partial, \textstyle\int] = \mathcal{F}[\partial] \dotplus \mathcal{F}[\textstyle\int] \dotplus (\mathrm{E})}$$

Connection to boundary problems (see next Talk):

$\partial^{-1} \neq \int$ since $\partial \int = 1$ but $\int \partial = 1 - \mathrm{E} \neq 1$. Relation to localization: See later!

$(\partial, [\mathrm{E}])^{-1} = \int$ in the monoid of boundary problems

$(\partial^2 - x\partial + 1, [2\mathrm{E}_0\partial - \mathrm{E}_1, \mathrm{E}_0\partial + \mathrm{E}_1])^{-1} =$ Green's Operator

Boundary problems: More characters E_p, analogous construction.
Green's Operator: Combination of $\int$ and E_p, algorithmic.

An Integro-Weyl Algebra?

From now on, we specialize to $\mathcal{F} = K[x]$. Hence—skew polynomials!

Leibniz rule: $[\partial, x] = 1$ Baxter rule: $[x, \ell] = \ell^2$

Definition

The skew polynomial ring $A[\xi; \sigma, \delta]$ consists of the elements $a_0 + a_1\xi + \cdots + a_n\xi^n$ with $a_0, \ldots, a_n \in A$. Addition is termwise, multiplication via $\xi a = \sigma(a)\,\xi + \delta(a)$. We use $A[\xi; \delta] \equiv A[\xi; 1, \delta]$.

$A = K[x], \xi = \partial$	$A = K[x], \xi = \ell$
$\partial x = x\partial + \boxed{1}\ \}\delta(x)$ ☺	$\ell x = x\ell + \boxed{(-\ell^2)}\ \}\delta(x)$ ☹
$A = K[\partial], \xi = x$	$A = K[\ell], \xi = x$
$x\partial = \partial x + \boxed{(-1)}\ \}\delta(\partial)$ ☺	$x\ell = \ell x + \boxed{\ell^2}\ \}\delta(\ell)$ ☺

An Integro-Weyl Algebra!

Definition

We write $\mathrm{A}_1(\ell)$ for the *integro Weyl algebra* $K[\ell][x;\delta]$ with $\delta(\ell) = \ell^2$. Analogously, we denote the *differential Weyl algebra* $K[\partial][x;\delta]$ with $\delta(\partial) = -1$ by $\mathrm{A}_1(\partial)$.

Similarities/Differences between $\mathrm{A}_1(\ell)$ and $\mathrm{A}_1(\partial)$:

- Both are Noetherian integral domains, but only $\mathrm{A}_1(\partial)$ is simple.
- While $\mathrm{A}_1(\partial)$ acts canonically on $K[x]$, what is $\ell \bullet 1$?
- Unlike in $\mathrm{A}_1(\partial)$, there is a natural grading in $\mathrm{A}_1(\ell)$.
- Similar to $\mathrm{A}_1(\partial)$, also $\mathrm{A}_1(\ell)$ has K-bases $(\ell^i x^j)$ and $(x^j \ell^i)$.
- But $\mathrm{A}_1(\ell)$ additionally has the mid basis $(x^m, x^m \ell x^n)$.

$$\longrightarrow K[x][\textstyle\int] \cong \mathrm{A}_1(\ell).$$

Ideals of the Integro-Weyl Algebra

Proposition (Lam '91, Thm. 3.15)

For a $\mathbb{Q}$-algebra A, the ring $A[\xi;\delta]$ is simple iff δ is not an inner derivation and A does not have a nontrivial δ-ideal I. Otherwise, the skew polynomials with coefficients in I form an ideal of $A[\xi;\delta]$.

Now this reveals $\mathrm{A}_1(\ell)$ to be non-simple:

Lemma

An ideal I of $K[\ell]$ is a nontrivial δ-ideal iff $I = (\ell^n)$ with $n > 0$.

Going Integro-Differential

- Up to now, only integro or differential Weyl algebra.
- Also combined algebra representable as skew polynomial ring.
- First construct appropriate coefficient ring with ∂ and ℓ.
- Combined algebra is "almost" $K[x][\partial, \int]$. What's missing?
- Integral operators from localization? A more severe mutilation.

Coefficient Ring

Choose coefficient ring A and derivation δ for $A[x;\delta]$ such that:

$\partial, \ell \in A$	Derivation δ
$\partial\ell = 1$	$\partial x - x\partial = 1$ and $x\ell - \ell x = \ell^2$

Definition

"Constant coefficient integro-differential operators"

$$K\langle\partial,\ell\rangle = K\langle D, L\rangle/(DL-1)$$

with derivation $\delta(\partial) = -1$ and $\delta(\ell) = \ell^2$.

Zero divisors: $\partial\,(1-\ell\partial) = \partial - \partial\ell\partial = \partial - \partial = 0$

Jacobson '50, Gerritzen '00
Right inverses in rings, approach based on representation theory
$K\langle\partial,\ell\rangle$ is not Noetherian!

Normal Forms

K-basis of $K\langle\partial,\ell\rangle$: $\quad \ell^i\partial^j$

(Normal forms modulo Gröbner basis $DL-1$)

Define

$$\mathrm{E} = 1-\ell\partial \quad \text{and} \quad e_{ij} = \ell^i \mathrm{E} \partial^j$$

Another K-basis of $K\langle\partial,\ell\rangle$: $\quad \partial^j, \quad \ell^i, \quad e_{ij}$

K-vector space generated by e_{ij} is the *evaluation ideal* (E):

$$\ell e_{ij} = e_{i+1,j} \qquad \text{and} \qquad \partial e_{ij} = e_{i-1,j}, \quad \partial e_{0j} = 0$$

Decomposition

$$\boxed{K\langle\partial,\ell\rangle = K[\partial] \dotplus K[\ell]\backslash K \dotplus (\mathrm{E})}$$

differential subrings (without unit), δ-ideal

Ideal Structure

Proposition

Every nonzero ideal in $K\langle\partial,\ell\rangle$ contains the evaluation ideal (E). Moreover, (E) is the only proper δ-ideal.

By our construction, ℓ is a right inverse of ∂.
Making it also a left inverse: $\mathrm{E} = 1-\ell\partial = 0$
Laurent polynomials $K[\partial,\partial^{-1}]$: Making ∂ invertible in $K[\partial]$

Proposition

The map with $\partial + (\mathrm{E}) \mapsto \partial$ and $\ell + (\mathrm{E}) \mapsto \partial^{-1}$

$$\varphi\colon K\langle\partial,\ell\rangle/(\mathrm{E}) \to K[\partial,\partial^{-1}]$$

is a differential isomorphism.

Ideals in $K\langle\partial,\ell\rangle$ correspond to ideals in $K[\partial,\partial^{-1}]$, which is a PID.

Integro-Differential Weyl algebra

Definition

The *integro-differential Weyl algebra* is the skew polynomial ring

$$K\langle \partial, \ell \rangle [x; \delta]$$

denoted by $\mathrm{A}_1(\partial, \ell)$.

Skew polynomial construction works over arbitrary rings
Normal forms as before but $\deg fg \leq \deg f + \deg g$

$K\langle \partial, \ell \rangle$ is not Noetherian $\Rightarrow \mathrm{A}_1(\partial, \ell)$ is not Noetherian

$(\mathtt{E})$ is a non-trivial δ-ideal in $K\langle \partial, \ell \rangle \Rightarrow$

Proposition

$\mathrm{A}_1(\partial, \ell)$ is not simple.

Decomposition

Decomposition of coefficient ring

$$\boxed{K\langle \partial, \ell \rangle = K[\partial] \dotplus K[\ell]\backslash K \dotplus (\mathtt{E})}$$

gives

$$\boxed{\mathrm{A}_1(\partial, \ell) = \mathrm{A}_1(\partial) \dotplus \mathrm{A}_1(\ell)\backslash K[x] \dotplus (\mathtt{E})}$$

where $(\mathtt{E})$ is the *evaluation ideal* in $\mathrm{A}_1(\partial, \ell)$:

$$\begin{aligned}(\mathtt{E}) &\subset \mathrm{A}_1(\partial, \ell)\\ &= \text{skew polynomials with coefficients in } (\mathtt{E}) \subset K\langle \partial, \ell \rangle\end{aligned}$$

Localization versus Evaluation

Integro-differential Weyl algebra

$$\mathrm{A}_1(\partial, \ell)$$

ℓ is *some* right inverse of ∂

$\swarrow$ $\searrow$

ℓ should also be a left inverse (two-sided inverse)

Localization

$K[\partial, \partial^{-1}][x; \delta]$

ℓ should be an integral with integration constant $c \in K$ (evaluation $x \mapsto c$)

Integro-differential operators

$K[x][\partial, \textstyle\int]$

Localization

For coefficients

$$K\langle \partial, \ell \rangle / (\mathrm{E}) \cong K[\partial, \partial^{-1}]$$

lifting to skew polynomials (universal property) $\Rightarrow$

Theorem

We have

$$\mathrm{A}_1(\partial, \ell)/(\mathrm{E}) \cong K[\partial, \partial^{-1}][x; \delta]$$

as a differential isomorphism.

Analogously for general $\mathcal{F}$,

$$\mathcal{F}[\partial, \textstyle\int]/(\mathrm{E}) \cong \mathcal{F}[\partial] \dotplus \mathcal{F}[\textstyle\int]$$

Fixing the Integration Constant

Want to fix the integration constant $c \in K$ meaning $\mathtt{E} \bullet x = c$ in $K[x]$

Construct a decomposition of the evaluation ideal $(\mathtt{E})$

In analogy to $K[x][\mathtt{E}]$, consider

$$B \leq \mathrm{A}_1(\partial, \ell)$$

K-vector space with basis

$$(x^k \mathtt{E} \partial^j)$$

Lemma

In $\mathrm{A}_1(\partial, \ell)$, we have

$$(\mathtt{E}) = B \dotplus (\mathtt{E}\, x - c\, \mathtt{E})$$

for every $c \in K$.

Back to Integro-Differential Operators

From

$$(\mathtt{E}) = B \dotplus (\mathtt{E}\, x - c\, \mathtt{E}),$$

$$\mathrm{A}_1(\partial, \ell) = \mathrm{A}_1(\partial) \dotplus \mathrm{A}_1(\ell) \backslash K[x] \dotplus (\mathtt{E})$$

we see that as K-vector spaces

$$\mathrm{A}_1(\partial, \ell)/(\mathtt{E}\, x - c\, \mathtt{E}) = \mathrm{A}_1(\partial) \dotplus \mathrm{A}_1(\ell) \backslash K[x] \dotplus B \cong K[x][\partial, \textstyle\int]$$

Since this holds also as K-algebras:

Theorem

If $\int$ is an integral operator for the standard derivation ∂ on $K[x]$, then

$$\mathrm{A}_1(\partial, \ell)/(\mathtt{E}\, x - c\, \mathtt{E}) \cong K[x][\partial, \textstyle\int]$$

with $c = \mathtt{E} \bullet x \in K$ as the constant of integration.

Conclusion and Outlook

- Constructed integro-differential operators as skew polynomials.
- Integro-differential Weyl algebra: rich structure, first steps.
- Useful for algorithmic treatment.
- Compute Green's operators, factor into lower order problems.
- Extension to more evaluations $\longrightarrow$ boundary problems.
- From ordinary to partial differential equations.

Implementation of Integro-Differential Operators in TH∃OREM∀

Loredana Tec
(joint work with: Bruno Buchberger, Georg Regensburger and Markus Rosenkranz)

Hagenberg, February 6, 2009

Goals

- Build up domains for various types of polynomials.
- Functor based approach in TH∃OREM∀.
- Implement polynomial reduction for systems with infinitely many generators.
- Implementation of integro-differential operators.
- Applications for solving/manipulating boundary problems.

The TH∃OREM∀ System

The TH∃OREM∀ System is an integrated environment for

- proving
- computing
- solving

in various domains of mathematics.

A unified logical frame for

- Generic programming by functors.
- Proving (correctness proofs for algorithms).

An Example of a Functor

The following functor takes an "alphabet domain" (ordered set of "letters") and builds the corresponding domain of "words" over it.

Definition["Word Monoid", **any**[L],

 LexWords[L] = **Functor**[W, **any**[v, w, ξ, η, $\bar{\xi}$, $\bar{\eta}$],

$$\begin{array}{l}
\mathbb{s} = \langle\rangle \\
\hline
\underset{W}{\in}[w] \Longleftrightarrow \bigwedge \begin{cases} \text{is-tuple}[w] \\ \underset{i=1,\ldots,|w|}{\forall} \underset{L}{\in}[w_i] \end{cases} \\
\underset{W}{\square} = \langle\rangle \\
v \underset{W}{*} w = v \asymp w \\
\left(\langle\eta, \bar{\eta}\rangle \underset{W}{>} \langle\rangle\right) \Leftrightarrow \text{True} \\
\left(\langle\rangle \underset{W}{>} \langle\bar{\eta}\rangle\right) \Leftrightarrow \text{False} \\
\left(\langle\eta, \bar{\eta}\rangle \underset{W}{>} \langle\xi, \bar{\xi}\rangle\right) \Leftrightarrow \bigvee \begin{cases} \eta \underset{L}{>} \xi \\ (\eta = \xi) \bigwedge \langle\bar{\eta}\rangle \underset{W}{>} \langle\bar{\xi}\rangle \end{cases}
\end{array}$$

]]

The Construction of the Monoid Algebra

MonoidAlgebra[K, W], leading to:

- Standard commutative polynomials: $\mathbb{W} = \mathbb{N}^n$.
- Noncommutative polynomials: $\mathbb{W} = \{x_1, \ldots, x_n\}^*$.
- Exponential polynomials: $\mathbb{W} = \mathbb{N} \times \mathbb{C}$.

The MonoidAlgebra Functor

$$
\begin{array}{l}
\textbf{Definition}\Big[\text{"Monoid Algebra"},\ \textbf{any}[K, W],\\
\quad \textbf{MonoidAlgebra}[K, W] = \textbf{Functor}\Big[P,\ \textbf{any}[c, d, f, g, \ldots],\\
\qquad \dfrac{\mathbb{s} = \langle\rangle}{\underset{P}{\in}[f] \Leftrightarrow \bigwedge \begin{cases} \text{is-tuple}[f] \\ \underset{i=1,\ldots,|f|}{\forall} \bigwedge \begin{cases} \text{is-tuple}[f_i] \\ |f_i| = 2 \\ \underset{K}{\in}[(f_i)_1] \\ \underset{W}{\in}[(f_i)_2] \\ (f_i)_1 \neq \underset{K}{0} \end{cases} \\ \underset{i=1,\ldots,|f|-1}{\forall} (f_i)_2 \underset{W}{>} (f_{i+1})_2 \end{cases}} \\
\qquad \underset{P}{1} = \langle\langle \underset{K}{1}, \underset{W}{\square} \rangle\rangle \\
\qquad \underset{P}{0} = \langle\rangle \\
\qquad \langle\rangle \underset{P}{*} g = \langle\rangle \\
\qquad f \underset{P}{*} \langle\rangle = \langle\rangle \\
\qquad \langle\langle c, \xi\rangle, \bar{m}\rangle \underset{P}{*} \langle\langle d, \eta\rangle, \bar{n}\rangle = \\
\qquad\quad \left(\langle\langle c \underset{K}{*} d,\ \xi \underset{W}{*} \eta\rangle\rangle \underset{P}{+} \langle c, \xi\rangle \underset{P}{*} \langle\bar{n}\rangle\right) \underset{P}{+} \langle\bar{m}\rangle \underset{P}{*} \langle\langle d, \eta\rangle, \bar{n}\rangle \\
\qquad f \underset{P}{+} g = \ldots \quad \Big]\Big]
\end{array}
$$

Integro-Differential Operators

$\mathcal{F}[\partial, \int]$: free K-algebra generated by the symbols ∂ and $\int$, the functions $f \in \mathcal{F}$ and the characters φ, modulo the equations:

$$
\begin{array}{|lcl|lcl|}
\hline
fg & = & f \bullet g & \partial f & = & \partial \bullet f + f\partial \\
\varphi\psi & = & \psi & \partial\varphi & = & 0 \\
\varphi f & = & (\varphi \bullet f)\,\varphi & \partial \int & = & 1 \\
\hline
\int f \int & = & \multicolumn{4}{l|}{(\int \bullet f) \int - \int (\int \bullet f)} \\
\int f \partial & = & \multicolumn{4}{l|}{f - \int (\partial \bullet f) - (E \bullet f)\, E} \\
\int f \varphi & = & \multicolumn{4}{l|}{(\int \bullet f)\, \varphi} \\
\hline
\end{array}
$$

where $f, g \in \mathcal{F}$ functions, φ, ψ characters.

- It is an infinite parametrized noncommutative Groebner Basis.
- Arithmetic is done by computing with normal forms.

Int-Diff Op Computations

Example1: Baxter Rule: ∫∫ = -∫ x +∫ x

```
Compute[AsGreen_g[⟨⟨1, ⟨"∫"⟩⟩⟩ *_I ⟨⟨1, ⟨"∫"⟩⟩⟩]]
```

$-1\,A\,\lceil x\rceil + \lceil x\rceil\,A$

Example2: $\left(x^3\,\partial+e^{-x}\int e^{2\,x}\,x\right)\bullet x^2\,e^x$

```
Compute[AsGreen_g[⟨⟨1, ⟨⟨"⌈⌉", ⟨3, 0⟩⟩, "∂"⟩⟩,
    ⟨1, ⟨⟨"⌈⌉", ⟨0, -1⟩⟩, "∫", ⟨"⌈⌉", ⟨0, 2⟩⟩, ⟨"⌈⌉", ⟨1, 0⟩⟩⟩⟩⟩ ⊙_F ⟨⟨1, ⟨2, 1⟩⟩⟩]]
```

$\frac{-1}{3}\left\lceil e^{2*x}*x^2\right\rceil + \frac{-2}{27}\left\lceil e^{2*x}\right\rceil + \frac{2}{9}\left\lceil e^{2*x}*x\right\rceil + \frac{1}{3}\left\lceil e^{2*x}*x^3\right\rceil + 2\left\lceil e^{x}*x^4\right\rceil + \left\lceil e^{x}*x^5\right\rceil$

Examples of Boundary Problems

Solution Method for Two-Point Boundary Problems

Given $f\in C^\infty[a,b]$, find $u\in C^\infty[a,b]$ s.t. $\begin{cases}\mathcal{D}\,u=f\\ \mathcal{B}_1\,u=\ldots=\mathcal{B}_n\,u=0\end{cases}$.

We want to find an operator G : f ↦ u.

The Green's operator can be computed as:

```
GreensOp_P[𝒟, ℬ] = where[𝒻 = FundSys[𝒟],
    (1_𝒜 -_𝒜 Proj_P[ℬ, 𝒻]) *_𝒜 RightInv_P[𝒟]]
```

where the projector is computed as follows:

```
Proj_P[ℬ, ℱ] = ℱ *_MatOps[𝒜] (∇_MatOps[K] (EvalMat_𝒜[ℬ, ℱ]) ·_MatOps[𝒜] ℬ)
```

Currently, $\mathcal{D}$ is assumed to have constant coefficients, so the fundamental right inverse is computed by:

```
RightInv_P[𝒟] = where[n = deg[𝒟], λ = CharRoots[𝒟],
    (1_K /_K lc[𝒟]) ∏_{i=1,..,n} ⌈e^{λi x}⌉ ∫ ⌈e^{-λi x}⌉]
```

Example 1

Given $f\in C^\infty[0,1]$, find $u\in C^\infty[0,1]$ s.t. $\begin{cases}D^2\,u=f\\ E_0\,u=E_1\,u=0\end{cases}$.

```
Compute[AsGreen_𝔤[GreensOp_𝔹[D², 〈〈〈1, 〈〈"⌊⌋", 0〉〉〉〉, 〈〈1, 〈〈"⌊⌋", 1〉〉〉〉〉]]] // Timing
```

```
{1.89006, -1 A ⌈x⌉ + -1 ⌈x⌉ B + ⌈x⌉ A ⌈x⌉ + ⌈x⌉ B ⌈x⌉}
```

So the Green's function: g(x,ξ) = $\begin{cases} (x-1)\,\xi & \Leftarrow \quad 0 \le \xi \le x \le 1 \\ x(\xi-1) & \Leftarrow \quad 0 \le x \le \xi \le 1 \end{cases}$.

- **Example 2**

Given $f \in C^\infty[0,1]$, find $u \in C^\infty[0,1]$ s.t. $\begin{cases} D^4\,u = f \\ E_0\,u = E_1\,u = E_0\,D^2\,u = E_1\,D^2\,u = 0 \end{cases}$.

```
Compute[AsGreen_𝔤[GreensOp_𝔹[D⁴, 〈〈〈1, 〈〈"⌊⌋", 0〉〉〉〉, 〈〈1, 〈〈"⌊⌋", 1〉〉〉〉,
    〈〈1, 〈〈"⌊⌋", 0〉, "∂", "∂"〉〉〉, 〈〈1, 〈〈"⌊⌋", 1〉, "∂", "∂"〉〉〉〉]]] // Timing
```

$$\left\{8.59186,\ \frac{-1}{2}\lceil x\rceil\,B\,\lceil x^2\rceil + \frac{-1}{2}\lceil x^2\rceil\,A\,\lceil x\rceil + \frac{-1}{6}\,A\,\lceil x^3\rceil + \frac{-1}{6}\lceil x^3\rceil\,B + \frac{1}{6}\lceil x\rceil\,A\,\lceil x^3\rceil + \frac{1}{6}\lceil x\rceil\,B\,\lceil x^3\rceil + \frac{1}{6}\lceil x^3\rceil\,A\,\lceil x\rceil + \frac{1}{6}\lceil x^3\rceil\,B\,\lceil x\rceil + \frac{1}{3}\lceil x\rceil\,A\,\lceil x\rceil + \frac{1}{3}\lceil x\rceil\,B\,\lceil x\rceil\right\}$$

So the Green's function: g(x,ξ) = $\begin{cases} -\frac{1}{2}\,x^2\,\xi - \frac{1}{6}\,\xi^3 + \frac{1}{6}\,x\,\xi^3 + \frac{1}{6}\,x^3\,\xi + \frac{1}{3}\,x\,\xi & \Leftarrow \quad 0 \le \xi \le x \le 1 \\ -\frac{1}{2}\,x\,\xi^2 - \frac{1}{6}\,x^3 + \frac{1}{6}\,x\,\xi^3 + \frac{1}{6}\,x^3\,\xi + \frac{1}{3}\,x\,\xi & \Leftarrow \quad 0 \le x \le \xi \le 1 \end{cases}$.

- **Example 3**

Given $f \in C^\infty[0,\pi]$, find $u \in C^\infty[0,\pi]$ s.t. $\begin{cases} (D^2 + D + 1)\,u = f \\ E_0\,u = E_\pi\,u = 0 \end{cases}$.

```
Compute[AsGreen_GreensAlg[𝔼xp,𝕂][GreensOp_𝔹[D² + 2 D + 1, 〈〈〈1, 〈〈"⌊⌋", 0〉〉〉〉, 〈〈1, 〈〈"⌊⌋", π〉〉〉〉〉]]] //
 Timing
```

$$\left\{2.95596,\ -1\,\lceil e^{-1*x}\rceil\,A\,\lceil e^{x}*x\rceil + -1\,\lceil e^{-1*x}*x\rceil\,B\,\lceil e^{x}\rceil + \pi^{-1}\,\lceil e^{-1*x}*x\rceil\,A\,\lceil e^{x}*x\rceil + \pi^{-1}\,\lceil e^{-1*x}*x\rceil\,B\,\lceil e^{x}*x\rceil\right\}$$

So the Green's function: g(x,ξ) = $\begin{cases} \frac{1}{\pi}\,(x-\pi)\,\xi\,e^{\xi-x} & \Leftarrow \quad 0 \le \xi \le x \le \pi \\ \frac{1}{\pi}\,(\xi-\pi)\,x\,e^{\xi-x} & \Leftarrow \quad 0 \le x \le \xi \le \pi \end{cases}$.

Popov Forms of Matrices of Differential Polynomials

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Feb 7, 2009

Example

Consider system of differential equations

$$\begin{array}{rcrcrcl} y_1''(t)+(t+2)y_1(t) & + & t^2y_2''(t)+y_2(t) & + & y_3'(t)+y_3(t) & = & 0 \\ y_1'(t)+3y_1(t) & + & y_2'''(t)+2y_2'(t)-y_2(t) & + & y_3'''(t)-2t^2y_3(t) & = & 0 \\ y_1'(t)+y_1(t) & + & y_2''(t)+2ty_2'(t)-y_2(t) & + & y_3''''(t) & = & 0. \end{array}$$

We usually deal with such systems by first converting them to first order systems

$$A(t)Y'(t) = B(t)Y(t) + C(t)$$

and then using various techniques to build various solutions or solution types (e.g. existence of rational function or exponential solutions).

Example : Matrix Form

Our original example can be represented by a differential matrix equation

$$\begin{bmatrix} D^2+(t+2) & t^2D^2+1 & D+1 \\ D+3 & D^3+2D-1 & D^3-2t^2 \\ D+1 & D^2+2tD+1 & D^4 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \mathbf{0}.$$

In general, systems that we are looking at are of the form

$$A(D)Y(t) = B(t).$$

Question : What form does $A(D)$ need to be in order that one can convert easily to a first order system?

Example (cont.)

Let D be the differentiation operator on t. If the system of equations is represented by:

$$\begin{bmatrix} D^2+(t+2) & t^2D^2+1 & D+1 \\ D+3 & D^3+2D-1 & D^3-2t^2 \\ D+1 & D^2+2tD+1 & D^4 \end{bmatrix} \cdot \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \mathbf{0},$$

then we can rewrite

$$y_1''(t) = -(t+2)y_1(t) - t^2y_2''(t) - y_2(t) - y_3'(t) - y_3(t)$$
$$y_2'''(t) = -y_1'(t) - 3y_1(t) - 2y_2'(t) + y_2(t) - y_3'''(t) + 2t^2y_3(t)$$
$$y_3''''(t) = -y_1'(t) - y_1(t) - y_2''(t) - 2ty_2'(t) - y_2(t)$$

Example (cont.)

- For systems not having this ‘special form’ one can always do row operations, derivations and eliminations to put a matrix of differential operators into the correct form.
- Basically given $A(D)$ one looks for an invertible $U(D)$ such that

$$U(D) \cdot A(D) = P(D) = \text{ matrix in special form}$$

- Special form needs to have columns of highest order in each row and one row cannot ‘interfere’ with columns of higest order in other rows.

Questions

- What are these special normal forms?
- How to compute such normal forms?
- Where does one go for ideas for these normal forms?

WARNING : this is only a preliminary report on this topic.

Outline

1. Motivation
2. Matrix Normal Forms
   - Introduction
   - Examples
3. Popov Normal Form
   - Basic Popov Facts
4. Computation of Popov Forms
   - History
   - Popov Form via Matrix GCLD
   - Method of Mulders-Strojohann
   - Fraction-Free Popov Computation

Todays Topic

Given : $\mathbf{A}(D) \in \mathbb{K}^{m \times n}[D]$.

Do row operations $\mathbf{U}(D)$

$$\mathbf{U}(D)\mathbf{A}(D) = \text{ easier}$$

(easier $= \mathbf{B}(D) \in \mathbb{K}^{m \times n}[D]$ in some sort of normal form)

$\mathbf{U}(D) \in \mathbb{K}^{m \times m}[D]$ invertible

Also wish to do this with matrices of Ore operators

Useful to see how one does these with matrices of polynomials

Why useful for Matrix Polynomials? : Matrix GCD

Given $B(z), C(z) \in \mathbb{K}^{m\times m}[z]$:

Find **Greatest Right Common Divisor** (gcrd) $D(z) \in \mathbb{K}^{m\times m}[z]$.

$$\begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} B(z) \\ C(z) \end{bmatrix} = \begin{bmatrix} D(z) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

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Why useful for Matrix Polynomials? : Matrix GCD

Given $B(z), C(z) \in \mathbb{K}^{m\times m}[z]$:

Find **Greatest Right Common Divisor** (gcrd) $D(z) \in \mathbb{K}^{m\times m}[z]$.

$$\begin{bmatrix} B(z) \\ C(z) \end{bmatrix} = \begin{bmatrix} V_{11}(z)D(z) \\ V_{21}(z)D(z) \end{bmatrix}$$

$$\begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}$$

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Popov Forms of Matrices of Differential Polynomials

Why useful for Matrix Polynomials? : Matrix GCD

Given $B(z), C(z) \in \mathbb{K}^{m\times m}[z]$:

Find **Greatest Right Common Divisor** (gcrd) $D(z) \in \mathbb{K}^{m\times m}[z]$.

$$\begin{bmatrix} B(z) \\ C(z) \end{bmatrix} = \begin{bmatrix} V_{11}(z)D(z) \\ V_{21}(z)D(z) \end{bmatrix}$$

$$U_{11}(z)V_{11}(z) + U_{12}(z)V_{21}(z) = I_m$$

- Matrix polynomials (in fact rational expressions of form $A(z) = U(z) \cdot V(z)^{-1}$) used in linear control theory

$$v \longrightarrow \blacksquare \longrightarrow Av$$

- Matrix GCDs needed for minimal rational matrix expressions
- Builds input-output model for control system
- Concept of Transfer functions also seems to exist for nonlinear control (Ziming Li [FoCM'08])

Example : Hermite Normal Form

$$\mathbf{H}(z) = \begin{bmatrix} h_{1,1}(z) & h_{1,2}(z) & \cdots & h_{1,m}(z) \\ 0 & h_{2,2}(z) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & & h_{m-1,m}(z) \\ 0 & \cdots & 0 & h_{m,m}(z) \end{bmatrix}$$

is in Hermite Normal Form if:

- Upper triangular
- diagonal entries monic
- degrees of diagonal entries max in columns
- any zero rows at bottom

Useful in solving linear system $\mathbf{H}(z)\vec{x}(z) = \vec{b}(z)$

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Example

$$\text{Input} : A(z) = \begin{bmatrix} z^2+1 & z & z^3 \\ z & 0 & z \\ z & z & z^3-1 \end{bmatrix}$$

$$\text{Output} : B(z) = \begin{bmatrix} 1 & 0 & -z^2+z+1 \\ 0 & z & z^2-z-1 \\ 0 & 0 & z^3-z^2 \end{bmatrix}$$

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Some Additional Remarks

- Also have **Smith Normal Form** for row and column equivalence.

$$\mathbf{U}(z) \cdot \mathbf{A}(z) \cdot \mathbf{V}(z) = diag(s_1(z), \cdots, s_m(z))$$

  where $s_i(z) | s_{i+1}(z)$ for all i. Determinantal divisors. Invariant factors. Useful for solving

$$\mathbf{A}(z)\vec{x}(z) = \vec{b}(z).$$

- Also have noncommutative versions of these normal forms
  - e.g. for matrices $\mathbf{A}(D)$ of differential operators
  - again useful for solving systems, but now of the form

$$\mathbf{A}(D)\vec{x}(z) = \vec{b}(z).$$

  - e.g. used by Singer [1985] for LODE decision procedures for systems

This talk : Popov Form

- Hermite Normal Form does not have controlled degrees
  - e.g. degrees of HNF can be larger than input degree
- Popov's form (1969) : purpose was to allow for simple conversion of state space to transfer functions in linear systems theory.
- Villard (1996) introduced Popov form to computer algebra community
- Popov form related to Gröbner bases
- Can extend to noncommutative domains (e.g. Ore domains)
- Question : How to compute (effectively)?

Definition : Row Popov Form

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{1,2} & f_{1,3} & \cdots & f_{1,n-1} & f_{1,n} \\ f_{21} & f_{2,2} & f_{2,3} & \cdots & f_{2,n-1} & f_{2,n} \\ f_{31} & f_{3,2} & f_{3,3} & \cdots & f_{3,n-1} & f_{3,n} \\ \vdots & & & & & \\ f_{n-1,1} & f_{n-1,,2} & f_{n-1,3} & \cdots & f_{n-1,n-1} & f_{n-1,n} \\ f_{n,1} & f_{n,,2} & \cdots & \cdots & f_{n,n-1} & f_{n,n} \end{bmatrix}$$

- Diagonal entries monic and of row degree
- deg $f_{j,i} <$ deg $f_{i,i}$ for $j \neq i$
- deg $f_{i,j} <$ deg $f_{i,i}$ for $j < i$
- deg $f_{i,j} \leq$ deg $f_{i,i}$ for $j > i$
- zero rows at bottom

Lots of variations (via reordering).

Example

E.g. : Input degree bounds

$$\begin{bmatrix} 3 & 3 & 2 & 3 \\ 3 & 4 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 6 & 7 & 6 & 7 \end{bmatrix}$$

Output degree bounds for Popov form

$$\begin{bmatrix} 3 & 3 & 2 & 3 \\ 2 & 4 & 3 & 3 \\ 2 & 3 & 4 & 4 \\ 2 & 3 & 3 & 7 \end{bmatrix}$$

Alternatively

An polynomial matrix $\mathbf{A}(z)$ is in Popov Form if:

1. it has rank $\mathbf{A}(z)$ non-zero rows;
2. the leading row coefficient is triangular, with monic leading entries;
3. the leading entry of each row has the highest degree in its columns.

Also called a Polynomial Echelon Form (Kailath book [1980]).

Any input matrix $\mathbf{A}(z)$ can be transformed into a unique Popov form by row operations.

Popov form as Gröbner Bases

Monomials on vectors $\mathbb{K}^{1\times n}[z]$:

$$z^\alpha e_j = [0, \ldots, 0, z^\alpha, 0, \ldots, 0]$$

Ordering on monomials of $\mathbb{K}^{1\times n}[z]$:

- Position over Term (POT):

$$z^\alpha e_i < z^\beta e_j \quad \Longleftrightarrow \quad i < j \quad \text{or } i = j \text{ and } \alpha < \beta$$

- Term over Position (TOP):

$$z^\alpha e_i < z^\beta e_j \quad \Longleftrightarrow \quad \alpha < \beta \quad \text{or } \alpha = \beta \text{ and } i < j.$$

If M is a submodule of $\mathbb{K}^{1\times n}[z]$ then we can now speak of Gröbner bases for the module M.

Popov form as Gröbner Bases

(Kojima, Rapisarda, Takaba [System & Control Letters 2007])

Let M be a submodule of $\mathbb{K}^{1\times m}[z]$ with a *term over position* ordering. Then

$\{f_i\}_{i=1,..,s}$ is a reduced Gröbner basis for the module $M \iff$:

(a) $M = \langle f_1, \ldots, f_s \rangle$;

(b) The matrix row$(f_1, \ldots, f_s)$ is in Popov form.

If TOP is replaced by *position over term* ordering then Popov form in (b) is replaced by Hermite form.

Previous Works

- Popov form algorithm for polynomial matrices:
  - Villard
  - Mulders and Storjohann
  - Beckermann, Labahn, Villard
  - ...
- A number of other algorithms for row/column-reduced form of polynomial matrices:
  - Beelen, van den Hurk, Praagman
  - Neven and Praagman
  - ...

Previous Works (cont.)

- Elimination-based approaches for Ore Popov form (Giesbrecht, Labahn, Zhang).
- EG elimination and variants (Abramov, Abramov and Bronstein).
- The FFreduce algorithm (Beckermann, Cheng, Labahn) computes:
  - a minimal polynomial basis for the left nullspace (in Popov form);
  - GCRD and LCLM (special cases only)
- The FFreduce algorithm is fraction-free. i.e. No fractions are introduced while controlling coefficient growth.
- A modular algorithm (Cheng, Labahn) for the same

Method of G. Villard (1996)

- $\mathbf{A}(z)^{-1} = \Delta(z)^{-1}\mathbf{A}^*(z)$ where:
  - $\mathbf{A}^*(z)$ is adjoint of $\mathbf{A}(z)$
  - $\Delta(z)$ is diagonal matrix with $\det \mathbf{A}(z)$ on diagonals.
- $\mathbf{A}^*(z)\mathbf{A}(z) = \Delta(z)$ and $\mathbf{A}^*(z) \cdot I = \mathbf{A}^*(z)$ so :
  - $\mathbf{A}^*(z)$ is a gcld of $\Delta(z)$ and $\mathbf{A}^*(z)$.
  - All other gcld's $\mathbf{G}(z)$ are then multiples, i.e.

$$\mathbf{G}(z) = \mathbf{A}^*(z)\mathbf{V}(z) \text{ with } \mathbf{V}(z) \text{ unimodular}$$

Method of G. Villard (1996)

- $\mathbf{A}(z)^{-1} = \Delta(z)^{-1}\mathbf{A}^*(z)$
- If $\mathbf{A}(z)^{-1} = D(z)^{-1}N(z)$ with $D(z)$ of minimal determinant degree in Popov form then

$$D(z) = \mathbf{G}(z)^{-1}\Delta(z) = \mathbf{V}(z)^{-1}\mathbf{A}^*(z)^{-1}\Delta(z) = \mathbf{U}(z)\mathbf{A}(z)$$

with $\mathbf{U}(z)$ unimodular.
- Therefore find a minimal realization of $\mathbf{A}(z)^{-1}$ having a denominator in Popov form.
- Algorithm exists for the above computation.
- Good for parallel computation

Mulders-Storjohann Procedure

First transform $\mathbf{A}(z)$ to *Weak Popov Form* - basically where pivots are on seperate rows but nothing more. Then convert to Popov Form
E.g. : degree bounds

$$\begin{bmatrix} 3 & 3 & 2 & 3 \\ 3 & 4 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 6 & 7 & 6 & 7 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 6 & 6 & 7 & 7 \end{bmatrix}$$

Mulders-Storjohann Procedure

First transform $\mathbf{A}(z)$ to *Weak Popov Form* - basically where pivots are on seperate rows but nothing more. Then convert to Popov Form
E.g. : degree bounds

$$\begin{bmatrix} 3 & 3 & 2 & 3 \\ 2 & 4 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 6 & 7 & 6 & 7 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 3 & 3 & 3 \\ 3 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 6 & 6 & 7 & 7 \end{bmatrix}$$

Motivation
Matrix Normal Forms
Popov Normal Form
Computation of Popov Forms
History
Popov Form via Matrix GCLD
Method of Mulders-Strojohann
Fraction-Free Popov Computation

Mulders-Storjohann Procedure

First transform $\mathbf{A}(z)$ to *Weak Popov Form* - basically where pivots are on seperate rows but nothing more. Then convert to Popov Form
E.g. : degree bounds (and so on ..)

$$\begin{bmatrix} 3 & 3 & 2 & 3 \\ 2 & 4 & 3 & 3 \\ 2 & 3 & 4 & 4 \\ 2 & 3 & 3 & 7 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 3 & 3 & 3 \\ 3 & 2 & 3 & 4 \\ 4 & 2 & 4 & 4 \\ 3 & 2 & 7 & 3 \end{bmatrix}$$

Symbolic Domains

- Basic coefficient domain: Quotient field: $\mathbb{F}(\alpha_1, \ldots, \alpha_k)$
  - symbols are first class objects in CA environments.
- Polynomial arithmetic easier than arithmetic with rational functions

$$\frac{a(x)}{b(x)} + \frac{c(x)}{d(x)} = \frac{a(x) \cdot d(x) + b(x) \cdot c(x)}{b(x) \cdot d(x)}$$

  Need to rcognize 0 : need to normalize out gcd's at every step
- Basic goal:

  To work with polynomial arithmetic in integral domain (e.g. in $\mathbb{F}[\alpha_1, \ldots, \alpha_k]$) rather than in quotient field.
- Want to do our arithmetic **fraction-free** but at the same time to minimize growth of intermediate computation.

Symbolic Domains

$$A = \begin{bmatrix} a & b & c & \cdots & \cdots \\ d & e & f & \cdots & \cdots \\ g & h & i & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix} \approx \begin{bmatrix} a & b & c & \cdots & \cdots \\ 0 & \tilde{e} & \tilde{f} & \cdots & \cdots \\ 0 & \tilde{h} & \tilde{i} & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

- Cross multiplication gives exponential growth of coeffs
- Fraction-free Gaussian elimination (FFGE)

$$A \approx \begin{bmatrix} a & b & c & \cdots & \cdots \\ 0 & \tilde{e} & \tilde{f} & \cdots & \cdots \\ 0 & 0 & a(..) & \cdots & a(...) \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & a(..) & \cdots & a(...) \end{bmatrix}.$$

  Allows for linear growth of coefficient size.
- Important : computes Cramer solution of linear problem.

Popov Form via Order Basis

- $\mathbf{U}(z)\mathbf{A}(z) = \mathbf{T}(z)$ same as $[\mathbf{U}(z), \mathbf{T}(z)] \begin{bmatrix} \mathbf{A}(z) \\ -I_n \end{bmatrix} = 0$
- $\mathbf{U}(z)\mathbf{A}(z) = \mathbf{T}(z)$ same as $[\mathbf{U}(z), \mathbf{T}(z)z^{\vec{r}}] \begin{bmatrix} \mathbf{A}(z)z^{\vec{r}} \\ -I_n \end{bmatrix} = 0$

  for any vector $\vec{r}$.
- Choose $\vec{r}$ intelligently so that $[\mathbf{U}(z), \mathbf{T}(z)z^{\vec{r}}]$ has leading coefficient the same as leading coefficient of $[0, \mathbf{T}(z)]$.
- Find Popov form for $[\mathbf{U}(z), \mathbf{T}(z)z^{\vec{r}}]$

Works because we can use *order bases* to solve last problem.

Good because order basis computation can be done via fraction-free methods (FFGE method of Beckermann-Labahn)

Popov Form via Order Basis (cont.)

- Order basis finds a module basis for problem:

$$f_1(z)m_1(z) + \cdots + f_n(z)m_n(z) = O(z^{\sigma})$$

- Order basis is form of an $n \times n$ matrix polynomial
- FFGE computes order basis in a *shifted* Popov Form using fraction-free arithmetic
- Choose vector $\vec{r}$ intelligently (use adjoint of $\mathbf{A}(z)$) so that one can embed Popov computational inside

$$\begin{bmatrix} \mathbf{M}_{11}(z) & \mathbf{M}_{12}(z) \\ \mathbf{M}_{21}(z) & \mathbf{M}_{22}(z) \end{bmatrix} \begin{bmatrix} \mathbf{A}(z)z^{\vec{r}} \\ -I_n \end{bmatrix} = \begin{bmatrix} \mathbf{R}(z)z^{\vec{\sigma}} \\ 0 \end{bmatrix}$$

Future Topics

1. Want fraction-free reduction procedure
2. Relationship of Popov Form (and its computation) to work of Pryce [2001] with Taylor series for numerical solution of DAEs
3. Higher order methods for systems of linear odes without conversion to first order systems
4. Involve adjoint calculation in process.
   - Did this in case of Order Basis (B & L, submitted to ISSAC 2009)

Moving Frames and Noether's Theorem

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Joint work with Tania Gonçalves

University of Kent

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Running Example: projective $SL(2)$ action

$$g \cdot x = x, \qquad g \cdot t = t, \qquad g \cdot u = \frac{au+b}{cu+d}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1$$

Via the chain rule, induce an action on u_x etc:

$$g \cdot u_x = \frac{\partial(g \cdot u)}{\partial(g \cdot x)} = \frac{u_x}{(cu+d)^2}$$

Lowest order invariants are

$$W = \frac{u_t}{u_x}, \qquad V = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} := \{u; x\}.$$

V and W are functionally independent, but there is a differential identity or syzygy, in this case

$$\frac{\partial}{\partial t} V = \underbrace{\left(\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + V_x\right)}_{\text{KdV operator}} W.$$

Interesting note: If $W = V$, then $V(x,t)$ satisfies the Korteweg de Vries equation. That is, if $u_t = u_x\{u; x\}$, then $\{u; x\}$ satisfies KdV.

Many examples like this. Gloria Mari Beffa has papers exploring moving frames and Poisson structures for Hamiltonian PDE.

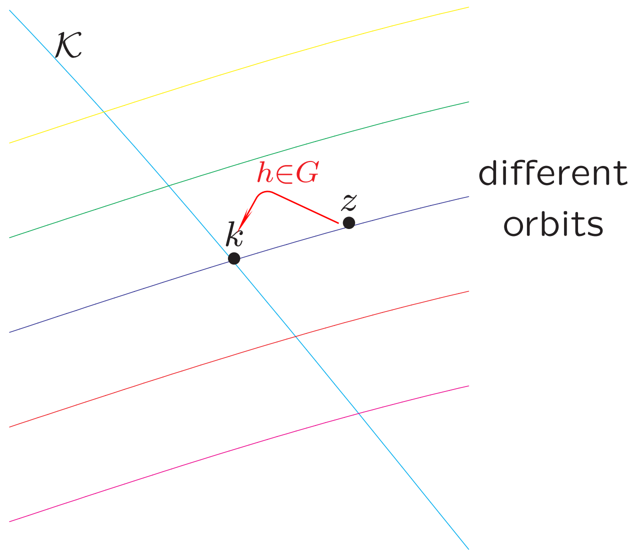
For many applications, want:
Given the Lie group action, derive the invariants and their syzygies algorithmically, that is, without prior knowledge of 100 years of differential geometry, and with minimal effort.

Major progress: Fels and Olver's* reformulation of Cartan's moving frame, and recent preprints by Hubert.

Why: ease of calculations, from variational calculus, solution of DEs via symmetries, . . . numerics, computer vision . . .

*Acta App. math **51** (1998) and **55** (1999)

Moving Frame if $G \times M \to M$ is a regular, free action



$\rho : M \to G$ $\rho(z) = h$ is equivariant

Calculation of a moving frame

Specify $\mathcal{K}$, the cross-section, as the locus of $\Phi(z) = 0$. Then solve $\Phi(g \cdot z) = 0$ for g. In practice, solve

$$\phi_j(g \cdot z) = 0, \qquad j = 1, \ldots, r = \dim(G)$$

for the r independent parameters describing g. Call the solution $\rho(z)$. Invoke IFT. Unique solution yields

$$\rho(g \cdot z) = \rho(z) \cdot g^{-1}.$$

• local solutions only this way: but see Hubert and Kogan, FoCM **7** (2007) and J. Symb. Comp., **42** (2007).

Recall running example:

$$g \cdot u = \frac{au+b}{cu+d}, \qquad g \cdot u_x = \frac{u_x}{(cu+d)^2}, \qquad g \cdot u_{xx} = \frac{\partial}{\partial x}(g \cdot u_x)$$

We have $z = (u, u_x, u_{xx})$ and we take

$$\Phi(g \cdot z) = 0 : \quad g \cdot u = 0, \quad g \cdot u_x = 1, \quad g \cdot u_{xx} = 0$$

to get

$$a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2u_x^{3/2}}.$$

Hence in matrix form,

$$\rho(u, u_x, u_{xx}) = \begin{pmatrix} \frac{1}{\sqrt{u_x}} & -\frac{u}{\sqrt{u_x}} \\ \frac{u_{xx}}{2u_x^{3/2}} & \frac{2u_x^2 - uu_{xx}}{2u_x^{3/2}} \end{pmatrix}.$$

Seeing is believing! The equivariance looks like

$$\begin{aligned} \rho(g \cdot z) &= \begin{pmatrix} \dfrac{1}{\sqrt{g \cdot u_x}} & -\dfrac{g \cdot u}{\sqrt{g \cdot u_x}} \\ \dfrac{g \cdot u_{xx}}{2(g \cdot u_x)^{3/2}} & \dfrac{2(g \cdot u_x)^2 - (g \cdot u)(g \cdot u_{xx})}{2(g \cdot u_x)^{3/2}} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1}{\sqrt{u_x}} & -\dfrac{u}{\sqrt{u_x}} \\ \dfrac{u_{xx}}{2u_x^{3/2}} & \dfrac{2u_x^2 - uu_{xx}}{2u_x^{3/2}} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \rho(z) g^{-1} \end{aligned}$$

Recall $g \cdot u_x = u_x/(cu+d)^2$ ...

Invariants: The components of $I(z) = \rho(z) \cdot z$ are invariant.

$$I(g \cdot z) = \rho(g \cdot z) \cdot (g \cdot z) = \rho(z) g^{-1} g \cdot z = \rho(z) \cdot z.$$

In practice for our running example

$$V = g \cdot u_{xxx}|_{\text{frame}}, \qquad W = g \cdot u_t|_{\text{frame}}$$

Various notations exist in the literature:

$$g \cdot u^\alpha_K|_{\text{frame}} = I^\alpha_K = \iota(u^\alpha_K) = \bar{\iota}(u^\alpha_K)$$

The same construction yields invariant differential operators,

$$\mathcal{D}_j = \frac{\partial}{\partial g \cdot x_j}\Big|_{\text{frame}},$$

also exterior forms, integral moments, difference expressions, and so on.

All differential invariants are functions of the I^α_K by the Replacement Theorem:

If $F(x, u, u_x, u_{xx} \ldots)$ is an invariant, then

$$\begin{aligned} F(x, u, u_x, u_{xx} \ldots) &= F(g \cdot x, g \cdot u, g \cdot u_x, g \cdot u_{xx} \ldots) \\ &= F(g \cdot x, g \cdot u, g \cdot u_x, g \cdot u_{xx} \ldots)|_{\text{frame}} \\ &= F(\iota(x), I^u, I^u_1, I^u_{11}, \ldots) \end{aligned}$$

$$\frac{\partial}{\partial x_j} u^\alpha_K = u^\alpha_{Kj}$$

$$\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & & \\
u^\alpha_{yy} & u^\alpha_{xyy} & \bullet & \bullet & & \\
u^\alpha_y & u^\alpha_{xy} & u^\alpha_{xxy} & \bullet & \bullet & \\
\partial_y \big\uparrow & & & & & \\
u^\alpha \xrightarrow[\partial_x]{} & u^\alpha_x & u^\alpha_{xx} & u^\alpha_{xxx} & \bullet & \bullet
\end{array}$$

- ♣ Generated by u^α
- ♣ The lattice has no "holes".
- ♣ The operators commute.

$$\mathcal{D}_j I^\alpha_K = I^\alpha_{Kj} + M^\alpha_{Kj}$$

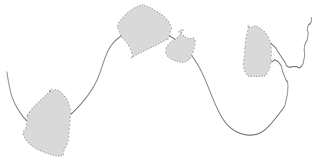
$$\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & & \\
I_{22} & I_{122} & \bullet & \bullet & & \\
I_2 \xrightarrow{\mathcal{D}_x} & I_{12} & I_{112} & I_{1112} & \bullet & \\
 & & & \big\uparrow \mathcal{D}_t & & \\
0 & 1 & 0 & I_{111} & \bullet & \bullet
\end{array}$$

Picture for running $SL(2)$ example

- ♣ More than one generator
- ♣ Differential syzygies.
- ♢ Symbolic formulae for the M^α_{Kj}.

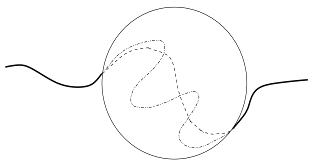
Variational problems with Symmetry

Motivating problem

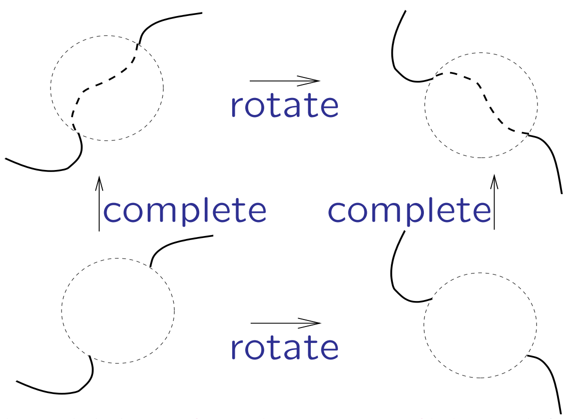


Given an occluded curve, want to fill in the missing bits, but how?

Possible infillings:



The solution should be equivariant with respect to translation and rotation in the plane:



The solution should be simplest possible while still fooling the human eye.

The actual problem is "solved", actually, set up to be solved, by taking the solution to minimize an integral of the form

$$\mathcal{L}[u] = \int L(\kappa, \kappa_s, \ldots)\, \mathrm{d}s$$

where κ is the Euclidean curvature and s the Euclidean arclength. That is, a variational problem with the relevant Lie group invariance.

So many applications of variational problems with Lie group symmetry!

- Find the Euler Lagrange equations directly in terms of the invariants†

- Minimal effort and required prior information

- Noether's theorem in these variables

- Obtain extremals in the original variables

†Kogan and Olver, Acta Appl. Math **76** (2003)

Recall how to calculate Euler Lagrange equations:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\mathcal{L}[u+\epsilon v] \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\int_a^b L(x, u+\epsilon v, u_x+\epsilon v_x, u_{xx}+\epsilon v_{xx}, \dots)\,\mathrm{d}x \\
&= \int_a^b \left(\frac{\partial L}{\partial u}v + \frac{\partial L}{\partial u_x}v_x + \frac{\partial L}{\partial u_{xx}}v_{xx} + \dots\right)\,\mathrm{d}x \\
&= \int_a^b \left[\left(\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u_{xx}} + \dots\right)v \right. \\
&\qquad \left. + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial u_x}v + \frac{\partial L}{\partial u_{xx}}v_{xx} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_{xx}}\right)v + \dots\right)\right]\,\mathrm{d}x \\
&= \int E(L)v\,\mathrm{d}x + \left[\frac{\partial L}{\partial u_x}v + \dots\right]_a^b
\end{aligned}$$

In other words:

Step 1: a derivative wrt u and its derivatives

Step 2: integration by parts

Note: the variation is with respect to u, and not the invariant.
The EL equation for $\int \kappa^2\,\mathrm{d}s$ is

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0$$

Since κ is a second order invariant, not hard to see will get a fourth order equation.

But where does the κ^3 come from?

Answer: a syzygy!

Trick one To get

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\mathcal{L}[u^\alpha + \epsilon v^\alpha]$$

where u^α is implicit, set

$$u^\alpha = u^\alpha(x,t)$$

where t is a dummy variable, both x and t are invariant and

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}.$$

Setting

$$v^\alpha_K \leftrightarrow u^\alpha_{Kt}$$

yields the same symbolic result.

Looking at our running example, suppose we have a Lagrangian of the form

$$\mathcal{L}[u] = \int \underbrace{L(V, V_x, V_{xx}, \ldots)\,\mathrm{d}x}_{\text{involves } x \text{ only}}$$

Introduce the dummy variable t to effect the derivative wrt u. Hence, we obtain a new invariant $I_2 = \iota(u_t) = u_t/u_x$, with the syzygy we saw already,

$$\frac{\partial}{\partial t}V = \mathcal{H}I_2 = \left(\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + V_x\right) I_2.$$

$$\frac{\partial}{\partial t}\int L(x,V,V_x,V_{xx},\ldots)\,\mathrm{d}x$$

$$= \int \left(\frac{\partial L}{\partial V} + \frac{\partial L}{\partial V_x}\frac{\partial}{\partial x} + \cdots\right)\frac{\partial}{\partial t}V\,\mathrm{d}x$$

$$= \int \underbrace{\left(\frac{\partial L}{\partial V} - \frac{\partial}{\partial x}\frac{\partial L}{\partial V_x} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial V_{xx}} + \cdots\right)}_{E^V(L)}\mathcal{H}(I_2)\,\mathrm{d}x + \text{ B.T's}$$

$$= \int \mathcal{H}^*\left(E^V(L)\right) I_2\,\mathrm{d}x + \text{ more B.T's}$$

where $\mathcal{H}^*$ is the adjoint of $\mathcal{H}$. Thus in this case,

$$E^u(L) = \mathcal{H}^*\left(E^V(L)\right)$$

Examples Since in this case $\mathcal{H}^* = -\mathcal{H}$:

1. for $\mathcal{L} = \int V\,\mathrm{d}x = \int\{u;x\}\,\mathrm{d}x$ we obtain

$$E^u(L) = -(\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + V_x)(1) = -V_x$$

2. for $\mathcal{L} = \int \frac{1}{2}V^2\,\mathrm{d}x = \int\{u;x\}^2\,\mathrm{d}x$ we obtain

$$E^u(L) = -(\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + V_x)(V) = -(V_{xxx} + 3VV_x)$$

To handle $\int \kappa^2 \, \mathrm{d}s$, where s is arclength, note s is such that $u_s^2 + x_s^2 = 1$, that is, there is an arclength constraint.

• Main trick: reparameterize to $u = u(s)$, $x = x(s)$, so that there are 2 dependent variables.
• For frame given by $g \cdot x = g \cdot u = g \cdot u_x = 0$, the syzygies are

$$\frac{\partial}{\partial t} \begin{pmatrix} I^u_{11} \\ I^x_1 \end{pmatrix} = \mathcal{H} \begin{pmatrix} I^u_2 \\ I^x_2 \end{pmatrix}$$

where $\mathcal{H}$ is a matrix of operators. Apply method to

$$\int \left[L(I^u_{11}, \ldots) - \lambda(s) \left(I^x_1 - 1 \right) \right] \mathrm{d}s$$

Carefully eliminating λ from the EL system yields the result.

Noether's Theorem

provides, for one dimensional problems, first integrals of $E(L) = 0$ in the case $L \, \mathrm{d}x$ is invariant under a Lie group action.

The formula for the integrals, one for each group parameter, is obtained by careful collection of the boundary terms in the integration by parts process we've just seen.

The formula is well known and can be calculated symbolically in Maple.

I decided "to see what I could see" by obtaining the first integrals in the original variables, and writing what I could in terms of the invariants. The result was beyond my wildest dreams.

For Lagrangians of the form $\int L(V, V_x, \ldots)\,\mathrm{d}x$ where $V = \{u; x\}$, I obtained

$$\mathbf{c} = \underbrace{\begin{pmatrix} d^2 & 2bd & -b^2 \\ cd & ad+bc & -ab \\ -c^2 & -2ac & a^2 \end{pmatrix}}_{R(g)}\Bigg|_{\text{frame}} \begin{pmatrix} \frac{\partial^2}{\partial x^2}E^V(L) + VE^V(L) \\ -2\frac{\partial}{\partial x}E^V(L) \\ -2E^V(L) \end{pmatrix}$$

Recall the frame is

$$a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2(u_x)^{3/2}}, \qquad ad - bc = 1.$$

- $R(gh) = R(h)R(g)$, and so $R(\rho(z))$ is equivariant

Which representation yields $R(g)$? And how to calculate the vector of invariants directly?

The Adjoint representation of G on infinitesimal vector fields

Infinitesimal vector fields For

$$g \cdot u = \frac{au + b}{cu + (1 + bc)/a}$$

the identity element e is given by $a = 1$, $b = c = 0$. Then

$$\frac{\partial}{\partial a}\Big|_e\, g \cdot u = 2u, \qquad \frac{\partial}{\partial b}\Big|_e\, g \cdot u = 1, \qquad \frac{\partial}{\partial c}\Big|_e\, g \cdot u = -u^2$$

If $g(t)$ is a path in $G = SL(2)$ with $g(0) = e$, then we have the vector field

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\, g(t) \cdot u = \left(2\alpha u + \beta - \gamma u^2\right)\partial_u$$

for constants α, β, γ. These comprise the set $\mathcal{X}_{SL(2)}(M)$ which is a three dimensional subspace of $\mathcal{X}(M)$.

Given $G \times M \to M$, the induced Adjoint action on $\mathcal{X}(M)$ is

$$g \cdot (f(u)\partial_u) = f(g \cdot u)\, \partial_{g \cdot u} = f(g \cdot u) \left(\frac{\partial g \cdot u}{\partial u}\right)^{-1} \partial_u$$

Theorem

$$g \cdot \mathcal{X}_G(M) \in \mathcal{X}_G(M)$$

Can be easier to calculate induced action on the arbitrary constants α, β, γ (the coAdjoint action)

Calculating

$$\begin{aligned} & \left(2\alpha(g \cdot u) + \beta - \gamma(g \cdot u)^2\right) \left(\tfrac{\partial g \cdot u}{\partial u}\right)^{-1} \partial_u \\ = \;& \left(2\alpha(au+b)(cu+d) + \beta(cu+d)^2 - \gamma(au+b)^2\right) \partial_u \\ = \;& \left(2(g \cdot \alpha)u + (g \cdot \beta) - (g \cdot \gamma)u^2\right) \partial_u \end{aligned}$$

yields

$$\begin{pmatrix} g \cdot \beta \\ g \cdot \alpha \\ g \cdot \gamma \end{pmatrix} = \underbrace{\begin{pmatrix} d^2 & 2bd & -b^2 \\ cd & ad+bc & -ab \\ -c^2 & -2ac & a^2 \end{pmatrix}}_{\mathcal{A}d(g)^T} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}$$

Noether's Theorem via Moving frames (Gonçalves, ELM)

Let $\int L(\kappa^\alpha, \kappa^\alpha_x, \ldots)\,\mathrm{d}x$ be invariant under $G \times M \to M$, $M = J^N((x, u^\alpha))$ with generating invariants κ^α and $g \cdot x = x$. Introduce the dummy variable t to effect the variation, and suppose that

$$\int \frac{\partial}{\partial t} L \,\mathrm{d}t = \int \sum E^\alpha(L) I^\alpha_2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{\alpha, J=1\cdots 1} I^\alpha_{2J} C^\alpha_J\right]$$

where this defines the vector $\mathcal{C}^\alpha = (C^\alpha_J)$. Let $(a_1, a_2, \ldots a_r)$ be coordinates of G about e. Define the matrix of invariantized infinitesimals

$$\Omega^\alpha = \begin{array}{c c} & \begin{array}{ccc} \cdots & u^\alpha_J & \cdots \end{array} \\ \begin{array}{c} \vdots \\ a_j \\ \vdots \end{array} & \left(\begin{array}{ccc} \cdots & \vdots & \cdots \\ & g \cdot \left(\frac{\partial}{\partial a_j} u^\alpha_J|_e\right)\Big|_{\text{frame}} & \\ \cdots & \vdots & \cdots \end{array}\right) \end{array}$$

Then the r first integrals obtained via Noether's theorem can be written in the form

$$\underset{r\times r}{\mathcal{A}d(\rho)^{-1}} \sum_\alpha \underset{r\times N}{\Omega^\alpha} \; \underset{N\times 1}{\mathcal{C}^\alpha} \; = \; \underset{r\times 1}{\mathbf{c}}$$

• Can see straightaway what the induced action on these first integrals is, since $\mathcal{A}d(\rho)$ is equivariant. The infinitesimal form of this action was well known.

• An invariant ODE $\Delta = 0$ can be converted to the triangular system $I(\Delta) = 0$, $\rho_x = A(I)\rho$, where $A(I)$ is known for any representation (ELM). Under favourable conditions, need only to solve the invariantized EL equations for the invariants as functions of x: no further integration is needed!!!

Open problems

• Know in principle, but have not calculated, how to do higher dimensional problems

• Method relies on having a dummy variable t, invariants κ^α, $I_2^\alpha = g \cdot u_t^\alpha|_{\text{frame}}$ and a syzygy of the form

$$\frac{\partial}{\partial t}\left(\kappa^\alpha \right) = \mathcal{H}\left(I_2^\alpha \right)$$

Hubert's recent papers on syzygies inform this part

• Can the method be adopted to finite difference variational problems

Counting solutions of differential and polynomial systems

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Linz 2009

Introduction

Counting polynomials (algebraic case)

Differential Thomas Algorithm

Linear differential equations: M. Janet (1920's)

Given a linear system of pdes. Find free Taylor coefficients!

Example: $u = u(x,y)$

- System 1:

$$\begin{aligned} u + x\boxed{u_x} &= 0 \\ \boxed{u_{y,y}} - u_y &= 0 \end{aligned}$$

  Parametric derivatives: u, u_y

- System 2:

$$u + x\boxed{u_x} = 0$$

  Parametric derivatives:$u, u_y, u_{y,y}, \ldots$
  Enumeration: $\frac{1}{1-y}$

Algebraic systems: J. M. Thomas (1930s)

Decompose system into pairwise disjoint simple systems.
by using equations and **inequations**

Example: $x^3 - y^2 - 1 = 0$ decomposes into

$$\begin{aligned} \boxed{x^3} - y^2 - 1 &= 0 \\ \boxed{y^2} + 1 &\neq 0, \end{aligned} \quad \text{and} \quad \begin{aligned} \boxed{x} &= 0, \\ \boxed{y^2} + 1 &= 0 \end{aligned}$$

Number of solutions:

$3 * (\infty - 2)$ $\qquad$ $1 * 2$

adding up to

$$3\infty - 4 \text{ solutions.}$$

People involved:

Present project:

- V. Gerdt (insisted on importance of Thomas' work)
- T. Bächler (implements algebraic Thomas decomposition)
- M. Lange-Hegermann (implements differential Thomas decomposition)
- D. Robertz (packages Involutive, Janet)
- myself (defined counting polynomials)

Other work on triangular systems:

- (with some reference to Thomas): Wu, D. Wang
- F. Boulier, E. Hubert, . . .
- (disjoint decompositions) Moreno Maza e. a.

Plan of this talk

<u>Aim</u>: Connect

- enumeration of Taylor coefficients for linear pdes
- the count of the solutions of algebraic systems
- to obtain a
- count of holomorphic solutions for polynomial pde systems.

<u>Outline</u>:

- The counting polynomial (algebraic case)
- Thomas algorithm (algebraic case)
- Thomas algorithm (differential case)
- Ideas for the counting polynomial in the differential case

Introduction

Counting polynomials (algebraic case)

Differential Thomas Algorithm

$x^3 - y^2 - 1 = 0$ revisited: bad news

- 1st decomposition ($x > y$):

$$\begin{array}{rclcrcl} \boxed{x^3} - y^2 - 1 & = & 0 & \text{and} & \boxed{x} & = & 0, \\ \boxed{y^2} + 1 & \neq & 0, & & \boxed{y^2} + 1 & = & 0 \end{array}$$

\# $\quad 3 * (\infty - 2) \quad\quad 1 * 2$

adding up to $\quad 3\infty - 4 \quad$ solutions.

- 2nd decomposition ($y > x$):

$$\begin{array}{rclcrcl} x^3 - \boxed{y^2} - 1 & = & 0 & \text{and} & \boxed{y} & = & 0, \\ \boxed{x^3} - 1 & \neq & 0, & & \boxed{x^3} - 1 & = & 0 \end{array}$$

\# $\quad 2 * (\infty - 3) \quad\quad 1 * 3$

adding up to $\quad 2\infty - 3 \quad$ solutions.

Two basic ideas:

K algebraically closed field, characteristic zero.

1. $p(x) \in K[x]$ squarefree of degree $n > 0$.
   Number of solutions $a \in K$ of
   - $p(a) = 0$ is n.
   - $p(a) \neq 0$ is $\infty - n$.
2. $q(x, y) \in K[x, y]$ squarefree of degree $m > 0$.
   Assume $q(a, y) \in K[y]$ squarefree of degree $m > 0$ for all $a \in K$ with $p(a) = 0$. Then
   Number of solutions $(a, b) \in K^2$ of
   - $p(a) = 0, q(a, y) = 0$ is $n * m$.
   - $p(a) = 0, q(a, y) \neq 0$ is $n * (\infty - m)$.

   Assume $q(a, y) \in K[y]$ squarefree of degree $m > 0$ for all $a \in K$ with $p(a) \neq 0$. Then
   Number of solutions $(a, b) \in K^2$ of
   - $p(a) \neq 0, q(a, y) = 0$ is $(\infty - n) * m$.
   - $p(a) \neq 0, q(a, y) \neq 0$ is $(\infty - n) * (\infty - m)$.

Geometric view: iterated fibrations

Notation:

- Projections $\pi_i : K^i \to K^{i-1} : (a_1, \ldots, a_i) \mapsto (a_1, \ldots, a_{i-1})$.
- Equations $E \subseteq K[x_1, \ldots, x_n]$ finite.
- Inequations:: $U \subseteq K[x_1, \ldots, x_n]$ finite.
- Set of solutions
  $V(E, U) := \{a \in K^n | p(a) = 0, q(a) \neq 0 \text{ for } p \in E, q \in U\}$.
- Truncated sets of solutions:

$$
\begin{aligned}
V_n(E, U) &:= V(E, U) \\
V_{n-1}(E, U) &:= \pi_n(V_n(E, U)) \\
&\cdots \\
V_1(E, U) &:= \pi_2(V_2(E, U))
\end{aligned}
$$

<u>Note</u>: Each fibre of $\pi_i : V_i(E, U) \to V_{i-1}(E, U)$ either finite or cofinite in K.

Simple systems and their counting polynomial

(E,U) is a **simple system** if for each i:

- all fibres of $\pi_i : V_i(E,U) \to V_{i-1}(E,U)$ have the same cardinality in K,
- namely f_i in case of finiteness
- and $\overline{f}_i$ for their complements in K otherwise.

Counting polynomial for simple system (E,U):

$$c(E,U) := \prod_j f_j * \prod_j (\infty - \overline{f}_j) \in \mathbb{Z}[\infty]$$

- f_j defined $\iff E \cap (K[x_1,\ldots x_j] - K[x_1,\ldots x_{j-1}]) \neq \emptyset$.
- $\overline{f}_j > 0 \iff U \cap (K[x_1,\ldots x_j] - K[x_1,\ldots x_{j-1}]) \neq \emptyset$.

Describable sets and their counting polynomials

- Def.: $M \subseteq K^n$ **describable** if $M = \biguplus_i V(E_i,U_i)$ with each (E_i,U_i) simple
- Thomas: $V(E,U)$ describable
- Counting polynomial $c_M := \sum_i c(E_i,U_i)$ (independent of decomposition)
- $M,N \subseteq K^n$ describable, then also $M \cap N, M \cup N, K^n - M$.
- $c_M + c_N = c_{M\cup N} + c_{M\cap N}$.
- $M \subseteq K^m, N \subseteq K^n$ decribable, then also $M \times N \subset K^{m+n}$ and $c_{M\times N} = c_M * c_N$.

Example: Projective describable sets

All elements of $E \cup U$ homogeneous. Then:

- K^* acts on $V(E,U) - \{0\}$ by multiplication
- $(\infty - 1)|(c(E,U) - 1)$
- $c_p(E,U) := \frac{c(E,U)-1}{(\infty-1}$ projective counting polynomial of $(V(E,U) - \{0\})/K^* \subseteq P^{n-1}(K)$.
- $c_p(E,U) = c_p(E \cup \{x_n\}, U) + c(E \cup \{x_n - 1\}, U)$

Example: Generic counting polynomials

Counting polynomials depend on coordinate system x! To make it independent :

- Apply substitution $\underline{x} = A\underline{y}$ to E and U
- where A is $n \times n$-matrix of indeterminates.
- generic counting polynomial $c_g(E,U)$ computed over algebraic closure of $K(A_{ij})$.

Thomas' Algorithm

Use Euclidean algorithm iteratively:

- Write equations and inequations as polynomials in x_n with coefficients in $K(x_1, \ldots, x_{n-1})$ by dealing only with numerators. (Denominators go into subsystem as inequations)

1. Leading coefficient splitting
2. Resultant splitting
3. Minimize $|E_n \cup U_n|$
4. Avoid multiple roots for equations (discriminants)
5. Remove roots of inequations from equations (resultants)
6. Avoid multiple roots of inequations (discriminants)

Introduction

Counting polynomials (algebraic case)

Differential Thomas Algorithm

Differential Thomas' Algorithm

Basic ideas:

- Use jet coordinates to obtain algebraic equations in these
- Go only for analytic solutions
- Repeat the follwing two steps
  - Apply algebraic Thomas algorithm to split into (algebraically) simple systems
  - Differentiate equations in the spirit of Janet to move towards passivity

  until one has only differentially simple systems

Difficulty: Differential Inequations

Example: x dependent variable, t is independent variable
System: $(\{x'' - x = 0\}, \{x' \neq 0\})$

Equivalent system: $(\{x'' - x = 0\}, \{x \neq 0\})$

Not reachable by the above steps!

Some Ideas

1.) Count Taylor coefficients
2.) only for a restricted class of systems (e. g. orthonomic)
3.) Investigate effect of variable transformation of independent variables

Example:u dependent, x, y is independent variable, $x > y$
System: $(\{u_x u_{xx} + u_{yy} = 0\})$
splits into two systems:

g) $u_x u_{xx} + u_{yy} = 0, u_x \neq 0$
s) $u_x = 0, u_y = 0$

Clearly, there are solutions with $u_x(0,0) = 0, u_x \neq 0$! Not clear how to deal with this in general!

Differential reduction“s” for differential characteristic set computations

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Outline of the talk

- Introduction
- Differential characteristic sets
- Differential reduction
- The Coherent-Autoreduced program
- The Rosenfeld-Gröbner program
- Conclusion

Differential characteristic sets by example

Consider the system

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} + f_1 &= 0 \\ \frac{\partial f_2}{\partial x_2} &= 0 \\ f_2 f_3 &= 0 \end{aligned}$$

where $f_1, f_2, f_3 \in \mathbb{R}(x_1, x_2)$. Characteristic set computations yield that each solution satisfies either

$$\begin{aligned} f_1 &= 0 \\ f_2 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{\partial f_2}{\partial x_1} + f_1 &= 0 \\ \frac{\partial f_2}{\partial x_2} &= 0 \\ f_3 &= 0 \end{aligned}$$

Setting for characteristic set computations

We fix F, Δ, and I such that

- F is a field of characteristic 0,
- Δ is a finite set of commuting derivations on F, and
- I is a finite index set.

Therewith we define

- Y as the family $(y_i)_{i \in I}$.

We model differential equations by elements in $F\{Y\}$.

Differential characteristic set computation strategy by example

- $f_1, f_2, f_3 \in \mathbb{R}(x_1, x_2)$ with
$$\frac{\partial f_2}{\partial x_1} + f_1 = 0, \quad \frac{\partial f_2}{\partial x_2} = 0, \quad f_2 f_3 = 0$$
- In $F\{Y\}$
$$y_{2,\delta_1} + y_1 = 0, \quad y_{2,\delta_2} = 0, \quad y_2 y_3 = 0$$
- Characteristic decomposition in $F\{Y\}$
$$\sqrt{[\ y_{2,\delta_1} + y_1,\ y_{2,\delta_2},\ y_2 y_3\ ]} = [C_1] : H_{C_1}^{\infty} \cap [C_2] : H_{C_2}^{\infty}, \text{ where}$$
$$C_1 = \{\ y_1,\ y_2\ \} \qquad C_2 = \{\ y_{2,\delta_1} + y_1,\ y_{2,\delta_2},\ y_3\ \}.$$
- $f_1, f_2, f_3 \in \mathbb{R}(x_1, x_2)$ with either
$$\begin{aligned} f_1 &= 0 \\ f_2 &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{\partial f_2}{\partial x_1} + f_1 &= 0 \\ \frac{\partial f_2}{\partial x_2} &= 0 \\ f_3 &= 0 \end{aligned}$$

Strategy for characteristic decomposition

$$\sqrt{[P]} = \bigcap_{i \in \{1,2,\ldots,r\}} [A_i] : H_{A_i}^{\infty} = \bigcap_{i \in \{1,2,\ldots,r\}} \bigcap_{j \in \{1,2,\ldots,m_i\}} [C_{i,j}] : H_{C_{i,j}}^{\infty}$$

- First decomposition $\sqrt{[P]} = \bigcap_{i \in \{1,2,\ldots,r\}} [A_i] : H_{A_i}^{\infty}$
  - is performed in $F\{Y\}$
  - each A_i is a coherent autoreduced set
  - each $[A_i] : H_{A_i}^{\infty}$ is radical
- Second decomposition $[A_i] : H_{A_i}^{\infty} = \bigcap_{j \in \{1,2,\ldots,m_i\}} [C_{i,j}] : H_{C_{i,j}}^{\infty}$
  - is performed over *algebraic* polynomial rings
  - each $C_{i,j}$ is a coherent autoreduced set
  - each $[C_{i,j}] : H_{C_{i,j}}^{\infty}$ is radical
  - each $C_{i,j}$ is a characteristic set for $[C_{i,j}] : H_{C_{i,j}}^{\infty}$

Properties of classical differential reduction (for non-contants)

$$\text{cdremas}(p, A) = q,$$

where $p \in F\{Y\} \setminus F$, A is an autoreduced set of $F\{Y\}$, and $q \in F\{Y\}$.

- $\text{dredas}(q, A)$
- $\exists h \in {H_M}^\infty : \ hp \equiv q \pmod{J}$,
  where J is an ideal and $M \subseteq A$
- cdremas is a function
- One reduction function per paper

Differential reduction by predicates (for non-constants)

	$\text{dredas}(q, A)$	M	J	dremas	dremndias
dremdias(p, A, q)	yes	A	$[A]$	yes	no
dremaias(p, A, q)	yes	A	$\langle \overline{A}^{\leq \text{lead}(p)} \rangle$	yes	yes
dremraias(p, A, q)	yes	${}^{p}\overline{A}$	$\langle {}^{p}\overline{A}^{\leq \text{lead}(p)} \rangle$	yes	yes

The congruence relation is

$$\exists h \in {H_M}^\infty : \quad hp \equiv q \pmod{J}.$$

The classic Coherent-Autoreduced program

Input: P: a finite set of elements in $F\{Y\}$

Output: A: s.t. $A \subseteq \sqrt{[P]} \subseteq [A] : H_A^\infty$; A is coherent

$S \leftarrow \emptyset$
$A \leftarrow \emptyset$
$R \leftarrow P$
$D \leftarrow \emptyset$
while $(R \cup D) \neq \emptyset$ **do**
  $S \leftarrow S \cup R \cup D$
  $A \leftarrow$ "lowest ranking" autoreduced set of S
  $R \leftarrow \{$cdremas$(s, A) \mid s \in S\} \setminus \{0\}$
  $D \leftarrow \{$cdremas$(\Delta(a, a'), A) \mid a, a' \in A\} \setminus \{0\}$
end while

Remainder sets

Let $S \subseteq F\{Y\}$,
dremas be any of the given specifications of reduction,
A be an autoreduced subset of $F\{Y\}$, and
$R \subseteq F\{Y\}$.

Then R is called a *remainder set* of S with respect to dremas and A (or RemainderSet(S, dremas, A, R)) if and only if

$$\begin{aligned} & \forall s \in S : (\text{dremas}(s, A, 0) \vee \exists r \in R : \text{dremas}(s, A, r)) \\ \wedge \quad & \forall r \in R \, \exists s \in S : \text{dremas}(s, A, r) \\ \wedge \quad & 0 \notin R. \end{aligned}$$

The Coherent-Autoreduced program using abstracted reduction

Input: P: a finite set of elements in $F\{Y\}$

Output: A: s.t. $A \subseteq \sqrt{[P]} \subseteq [A] : H_A^{\infty}$; A coherent

$S \leftarrow \emptyset$
$A \leftarrow \emptyset$
$R \leftarrow P$
$D \leftarrow \emptyset$
while $(R \cup D) \neq \emptyset$ **do**
  $S \leftarrow S \cup R \cup D$
  $A \leftarrow$ "lowest ranking" autoreduced set of S
  $R \leftarrow$ a $R' \subseteq F\{Y\}$ such that
    RemainderSet(S , dremas , A, R')
  $D \leftarrow$ a $D' \subseteq F\{Y\}$ such that
    RemainderSet($\{\Delta(a, a') \mid a, a' \in A\}$, dremndias, A, D')
end while

The classical Rosenfeld-Gröbner program

Input: P: a finite set of elements in $F\{Y\}$

Output: $\mathcal{A}$: s.t. $\sqrt{[P]} = \bigcap_{i \in \{1,2,\ldots,|\mathcal{A}|\}} [A_i] : H_{A_i}^{\infty}$; each A_i coherent

$\mathcal{S} \leftarrow \{(P, \emptyset, \emptyset, \emptyset)\}$; $\mathcal{A} \leftarrow \emptyset$
while $\mathcal{S} \neq \emptyset$ **do**
  $(G, D, A, H) \leftarrow$ an element of $\mathcal{S}$; $\mathcal{S} \leftarrow \mathcal{S} \setminus \{(G, D, A, H)\}$
  if $G \cup D = \emptyset$ **then**
    $\mathcal{A} \leftarrow \mathcal{A} \cup \text{auto-partial-reduce}(A, H)$
  else
    $p \leftarrow$ an element of $G \cup D$
    $q \leftarrow$ cdremas(p, A)
    $G \leftarrow G \setminus \{p\}$; $D \leftarrow D \setminus \{p\}$
    if $q = 0$ **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{(G, D, A, H)\}$
    else if $q \notin F$ **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \text{splittings}(G, D, A, H, q)$
    end if
  end if
end while

The Rosenfeld-Gröbner program using abstracted reductions

Input: P: a finite set of elements in $F\{Y\}$

Output: $\mathcal{A}$: s.t. $\sqrt{[P]} = \bigcap_{i \in \{1,2,\ldots,|\mathcal{A}|\}} [A_i] : H_{A_i}^{\infty}$; each A_i coherent

$\mathcal{S} \leftarrow \{(P, \emptyset, \emptyset, \emptyset)\}$; $\mathcal{A} \leftarrow \emptyset$
while $\mathcal{S} \neq \emptyset$ **do**
  $(G, D, A, H) \leftarrow$ an element of $\mathcal{S}$; $\mathcal{S} \leftarrow \mathcal{S} \setminus \{(G, D, A, H)\}$
  if $G \cup D = \emptyset$ **then**
    $\mathcal{A} \leftarrow \mathcal{A} \cup \text{auto-partial-reduce}(A, H)$
  else
    $p \leftarrow$ an element of $G \cup D$
    if $p \notin D$ **then**
      $q \leftarrow$ a $q' \in F\{Y\}$ such that dremas$\big(p, A, q'\big)$
    else
      $q \leftarrow$ a $q' \in F\{Y\}$ such that dremraias$\big(p, A, q'\big)$
    end if
    $G \leftarrow G \setminus \{p\}$; $D \leftarrow D \setminus \{p\}$
    if $q = 0$ **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{(G, D, A, H)\}$
    else if $q \notin F$ **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \text{splittings}(G, D, A, H, q)$
    end if
  end if
end while

Conclusion

- Differential reductions can be modelled via predicates.
- Differential reductions need not be deterministic.
- To closely model requirements of programs, we need *more than one* reduction per program.

Rational general solutions of first order non-autonomous parametric ODEs

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February 9, 2009

1 Motivation

2 Constructions and proofs

3 Examples

Introduction

Let $\mathcal{K} = \mathbb{Q}(x)$, y be an indeterminate over $\mathcal{K}$. $y' := \frac{dy}{dx}$.

Autonomous ODEs

$$y'^3 + 4y'^2 + (-27y^2 + 4)y' + 27y^4 - 4y^2 = 0.$$

$$F(y, y') = 0,$$

where $F \in \mathbb{Q}[y, y']$.
Feng and Gao:

$$F(y, y') = 0 \text{ has a nontrivial rational solution} \Rightarrow F(y, z) = 0$$

$$\text{is a rational curve.}$$

It is enough to find a nontrivial rational solution of $F(y, y') = 0$.

Non-autonomous ODEs

$$\begin{aligned}&2xy'^2 + (4y^2x - 8yx + 4x + 2y)y' +\\ &2y^4x - 8y^3x + 12y^2x - y^4 + 4y^3 - 8yx - 5y^2 + 2x + 2y = 0\end{aligned}$$

$$F(x, y, y') = 0,$$

where $F \in \mathbb{Q}[x, y, z]$.
A rational solution $y = f(x)$ defines a rational space curve

$$\gamma(x) = (x, f(x), f'(x))$$

on the surface defined by $F(x, y, z) = 0$.

Construction

Let's assume that the surface $F(x, y, z) = 0$ can be parametrized by a birational map

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)).$$

Suppose that the invert map is

$$\mathcal{P}^{-1}(x, y, z) = (s(x, y, z), t(x, y, z)).$$

In particular, the parametric curve

$$\mathcal{P}^{-1}(x, f(x), f'(x)) = (s(x), t(x))$$

is a rational plane curve and satisfies the relation

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x)) = f(x) \\ \chi_3(s(x), t(x)) = f'(x). \end{cases}$$

Hence

$$\begin{cases} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \cdot t'(x) = 1 \\[2ex] \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} \cdot s'(x) + \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \cdot t'(x) = \chi_3(s(x), t(x)). \end{cases}$$

Let

$$f_1(s, t) = \frac{\partial \chi_2(s, t)}{\partial t} - \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial t},$$

$$f_2(s, t) = \frac{\partial \chi_2(s, t)}{\partial s} - \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial s}$$

and

$$g(s, t) = \frac{\partial \chi_1(s, t)}{\partial s} \cdot \frac{\partial \chi_2(s, t)}{\partial t} - \frac{\partial \chi_1(s, t)}{\partial t} \cdot \frac{\partial \chi_2(s, t)}{\partial s}.$$

Either

$$\begin{cases} g(s(x), t(x)) = 0 \\ f_1(s(x), t(x)) = 0 \\ f_2(s(x), t(x)) = 0 \end{cases}$$

or

$$\begin{cases} s'(x) = \dfrac{f_1(s(x), t(x))}{g(s(x), t(x))} \\ \\ t'(x) = -\dfrac{f_2(s(x), t(x))}{g(s(x), t(x))}. \end{cases}$$

Conversely, suppose that $(s(x), t(x))$ is such that

$$g(s(x), t(x))) = f_1(s(x), t(x)) = f_2(s(x), t(x)) = 0.$$

If

$$\frac{\partial \chi_1(s(x), t(x))}{\partial s} \cdot s'(x) + \frac{\partial \chi_1(s(x), t(x))}{\partial t} \cdot t'(x) = 1$$

then there is a constant c such that

$$f(x) = \chi_2(s(x-c), t(x-c))$$

is a rational solution of $F(x, y, y') = 0$.

Suppose that $(s(x), t(x))$ is a rational solution of the system

$$\begin{cases} s'(x) = \dfrac{f_1(s,t)}{g(s,t)} \\ \\ t'(x) = -\dfrac{f_2(s,t)}{g(s,t)}. \end{cases} \tag{1}$$

Then

$$\chi_1(s(x), t(x)) = x + c$$

for some constant c. Hence

$$y = \chi_2(s(x-c), t(x-c))$$

is a rational solution of $F(x, y, y') = 0$.

Definition

A rational solution $y = f(x)$ of $F(x, y, y') = 0$ is called a rational general solution if for any differential polynomial $G \in \mathcal{K}\{y\}$ we have

$$G(y) = 0 \Leftrightarrow \text{prem}(G, F) = 0.$$

Definition

A rational solution $(s(x), t(x))$ of the system

$$\begin{cases} s'(x) = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t'(x) = \dfrac{N_2(s,t)}{M_2(s,t)}. \end{cases}$$

is called a rational general solution if for any $G \in \mathcal{K}\{s, t\}$ we have

$$G(s(x), t(x)) = 0 \Leftrightarrow \text{prem}(G, \{M_1 s' - N_1, M_2 t' - N_2\}) = 0.$$

Theorem

Let $\bar{y} = f(x)$ be a rational general solution of $F(x, y, y') = 0$. Let

$$(\bar{s}(x), \bar{t}(x)) = \mathcal{P}^{-1}(x, f(x), f'(x)).$$

If $g(\bar{s}(x), \bar{t}(x)) \neq 0$, then $(\bar{s}(x), \bar{t}(x))$ is a rational general solution of (1).

Proof of the theorem 1

It turns out that $(\bar{s}(x), \bar{t}(x))$ is a solution of (1). Suppose that $P \in \mathcal{K}\{s, t\}$ is a differential polynomial such that $P(\bar{s}(x), \bar{t}(x)) = 0$. Let

$$R = \text{prem}(P, \{s' M_1(s, t) - N_1(s, t), t' M_2(s, t) + N_2(s, t)\}).$$

Then

$$R \in \mathcal{K}[s, t].$$

We have to prove that $R = 0$. We know that

$$R(\bar{s}(x), \bar{t}(x)) = R(\mathcal{P}^{-1}(x, f(x), f'(x))) = 0.$$

Let's consider the rational function $R(\mathcal{P}^{-1}(x, y, z)) = \dfrac{U(x, y, z)}{V(x, y, z)}$. Then $U(x, y, y')$ is a differential polynomial satisfying the condition

$$U(x, f(x), f'(x)) = 0.$$

Since $f(x)$ is a rational general solution of $F = 0$ and both F and U are differential polynomials of order 1, we have

$$U(x, y, y') = Q_0 F,$$

where Q_0 is a differential polynomial of order 1 in $\mathcal{K}\{y\}$. Therefore,

$$R(s,t) = R(\mathcal{P}^{-1}(\mathcal{P}(s,t))) = \frac{U(\mathcal{P}(s,t))}{V(\mathcal{P}(s,t))} = \frac{Q_0(\mathcal{P}(s,t))F(\mathcal{P}(s,t))}{V(\mathcal{P}(s,t))} = 0.$$

Thus $(\bar{s}(x), \bar{t}(x))$ is a rational general solution of (1).

Theorem

If the associated system (1) *has a rational general solution, then there exists a constant c such that*

$$\bar{y} = \chi_2(\bar{s}(x-c), \bar{t}(x-c))$$

is a rational general solution of $F(x, y, y') = 0$.

Proof of the theorem 2

Assume that $(\bar{s}(x), \bar{t}(x))$ is a rational general solution of the system (1). Then there exists a constant c such that

$$\bar{y} = \chi_2(\bar{s}(x-c), \bar{t}(x-c))$$

is a rational solution of $F(x, y, y') = 0$. Let G be an arbitrary differential polynomial in $\mathcal{K}\{y\}$ such that $G(\bar{y}) = 0$. Let

$$R = \text{prem}(G, F)$$

be the differential pseudo-remainder of G with respect to F. It follows that

$$R(\bar{y}) = 0.$$

We have to prove that $R = 0$.

Assume that $R \neq 0$. Then

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)} \in \overline{\mathbb{Q}}(s,t).$$

Since

$$R(\chi_1(\bar{s}(x-c), \bar{t}(x-c)), \chi_2(\bar{s}(x-c), \bar{t}(x-c)), \chi_3(\bar{s}(x-c), \bar{t}(x-c))) = 0.$$

we have

$$W(\bar{s}(x-c), \bar{t}(x-c)) = 0.$$

On the other hand, $(\bar{s}(x-c), \bar{t}(x-c))$ is also a rational general solution of (1), it follows that $W(s,t) = 0$. Thus

$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0.$$

Since F is irreducible and $\deg_{y'} R < \deg_{y'} F$, we have $R = 0$. Therefore, $\bar{y}$ is a rational general solution of $F(x, y, y') = 0$.

Example

$$y'x^2 + xy^2 - 2xy - y^2 = 0.$$

A rational parametrization is

$$\mathcal{P}(s,t) = \left(t, \frac{t^2}{s+1}, \frac{-t(-2s-2+t^2-t)}{(s+1)^2} \right).$$

The associated system is

$$\begin{cases} s'(x) = t - 1 \\ t'(x) = 1. \end{cases}$$

Solving this system we obtain

$$\bar{s}(x) = \frac{x^2}{2} + (c_1 - 1)x + c_2, \bar{t}(x) = x + c_1.$$

Therefore,

$$\bar{y} = \frac{2x^2}{x^2 - 2x + 2C}$$

is a rational general solution, where $C = c_2 + c_1 - \frac{c_1^2}{2}$ is an arbitrary constant.

(In this example $g(s,t) = \frac{t^2}{(s+1)^2}$. It gives us a solution $y = 0$.)

Example

$$y'^3 - 4xyy' + 8y^2 = 0.$$

A rational parametrization is

$$\mathcal{P}(s,t) = (t, -4s^2(2s-t), -4s(2s-t)).$$

The associated system is

$$\begin{cases} s'(x) = \dfrac{1}{2} \\ t'(x) = 1. \end{cases}$$

Hence $\bar{s}(x) = \dfrac{x}{2} + c_2, \bar{t} = x + c_1$ for arbitrary constants c_1, c_2. The general solution is

$$\bar{y} = -C(x+C)^2$$

where $C = 2c_2 - c_1$.
Note that in this example $g(s,t) = -8s(t-3s)$. Let $g(s,t) = 0$ we get $s = 0$, or $t = 3s$. This gives us $y = 0$, or $y = \frac{4}{27}x^3$.

Minutes of:	Differential Equations by Algebraic Methods (DEAM) workshop meeting
Date:	8th February 2009
Place:	Johannes-Kepler-University, Linz, Austria

Participants Evelyne Hubert, George Labahn, Arne Lorenz, Elizabeth Mansfield, Johannes Middeke, Ngô Lâm Xuân Châu, Franz Pauer, Wilhelm Plesken, Markus Rosenkranz, Fritz Schwarz, Ekaterina Shemyakova, Franz Winkler

Overview We met at the end of the DEAM workshop to discuss further cooperations and continuation of the workshop.

Research topics The aim of the workshop and the basis for further cooperation were to *solve* differential equations and to *extract and understand the structure* of differential equations. There were four main topics and several subtopics.

1. **Linear algebra for differential operators.**

   Possible directions of research are *normal forms* of matrices of partial differential equations and partial differential equations, their connection to Gröbner bases, and possible applications.

   It was suggested to make a connection to the work of *F. Nataf* and *V. Dolean*.

2. **Factorisation/(Loewy) Decomposition/Symmetry**

   One goal is to use *invariants* to classify and solve differential equations. Invariants should be collected in a *database* (proposed by Elizabeth Mansfield). Connected to invariants is the questions about the *geometry* of differential equations in analogy to algebraic geometry for algebraic equations.

   An application could be provided by analysing *nonlinear control systems*. Furthermore, Fritz Schwarz suggested to study the connection to similarity solutions.

3. **Invariants**

   A possible project is to compute differential operators for *higher orders*. The question arose, what the *connection between the different methods* of invariant computation presented during the talks was. Wilhelm Plesken proposed to study other applications of the *Vessiot method*. Additionally, Evelyne Hubert suggested to relate this to the *Cartan equivalence* and the work of *S. Neut*.

4. **Integral operators and Boundary conditions**

   George Labahn suggested to study the *classical analytic methods for solving boundary problems* in the context of integro-differential operators. Further research should be done on their *representation* and on *integral transforms*.

   Elizabeth Mansfield put it to use this to study *moments* and similar concepts.

Next DEAM workshop We agreed that there should be a next meeting. Proposed times were September 2010 (proposed by Fritz Schwarz) or February 2011 (Franz Winkler/George Labahn). No decision was made on the exact date.

There was also a proposal of inviting more people to the next workshop in dependence of the research topics treated until then.

Student exchange We discussed the possibility of exchanging students between the different groups. We had some discussions of financing these visits. Franz Winkler explained the conditions of the SCIEnce project: This makes exchange inside Europe possible, but cannot be used for exchange with Canada. A proposal was to link research projects between different countries to get extra travel money.

Further topics We might try to draw a connection to *D-module theory* or *differential Galois theory*.

A possible application would be the integration of *Hamiltonian systems*.

Markus Rosenkranz proposed to write a *survey* of differential computer algebra in order to attract people to this area. We recommended taking this as a possible project for the next DEAM workshop. The suggestion was to reserve a session for this.